In memoriam YUDELL L. LUKE June 26, 1918–May 6, 1983

Yudell Luke was born in Kansas City, Mo., on June 26, 1918, the son of David and Sarah Luke. The household was very traditionally Jewish; in fact, his father served as a sexton in a synagogue. In 1937 Yudell graduated from Kansas City Missouri Junior College and two years later received a B. S. degree with honors from the University of Illinois, followed by an M. S. degree in 1940. While there, he met Laverne Podoll from Chicago who was to become his wife. He taught briefly at the University of Illinois, then served as a full lieutenant in the U.S. Navy from 1942 to 1946, stationed in Hawaii. Upon being discharged, he returned to Kansas City and was immediately hired as the head of the Mathematical Analysis Section of the newly formed Midwest Research Institute. There, Yudell was able to attract and keep together a group of young mathematicians, some of whom later on became researchers in their own right. He advanced to Senior Advisor for Mathematics in 1961, and to Principal Advisor in 1967. After the mathematics group at MRI was dissolved abruptly in 1971, he was appointed to a professorship at the University of Missouri in Kansas City and, in 1978, was given the distinction of Curator's Professor, a position he held until his untimely death.

At the beginning of his career, Yudell was heavily involved in problems of applied mechanics: stress, beam vibrations, aerodynamic lag, supersonic flutter. It was during this early preoccupation with applied problems that Yudell saw the potential usefulness of special function theory and the pressing need to make advanced special functions—integrals of Bessel functions at that time—accessible to the practicing scientist. He began to study these functions in the framework of generalized hypergeometric functions, and his involvement with the latter soon turned into a love affair that was to last throughout his life.

Yudell's main concern was approximation. Foremost in his mind were rational approximations, and he developed a great number of them, not only for specific functions, such as the gamma and incomplete gamma function, elliptic integrals, and elementary functions, but also for general hypergeometric and confluent hypergeometric functions. He used a variety of techniques, most notably his own extension of Lanczos' τ-method, where as forcing term in the differential equation he took not only a multiple of a Chebyshev polynomial, as did Lanczos, but also multiples of more general Jacobi polynomials. He was able to show that in many cases there result approximations of Padé type, specifically those on the main diagonal of the Padé table. A distinguished feature of Yudell's work in this area is his persistent effort of providing detailed information about the error term, either in the form of analytic representations or asymptotic estimates for large degrees. He did not hesitate to develop the necessary asymptotics himself if it was not available in the literature. His results permitted him not only to give unusually sharp a priori error

0025-5718/84 $1.00 + $.25 per page

estimates, valid in large domains of the complex plane, but also, on several occasions, to obtain important convergence statements for Padé approximants along all columns, rows, and diagonals of the Padé table. More recently, his work assumed a computer-oriented flavor, for example, when he developed FORTRAN routines for generating $[n, n]$-type rational approximations to hypergeometric and confluent hypergeometric functions. These either furnish approximations at fixed (complex) argument z and for $n = 1, 2, 3, \ldots$ (until the error is sufficiently small), or yield for given n the coefficients of the desired numerator and denominator polynomials. In a similar vein, he looked at various approximation schemes that have the same complexity (defined by Yudell in his own pragmatic way) and compared them with regard to accuracy attained. Is it better, for example, to apply Padé approximation to convergence factors, rather than to the whole series? Are there any advantages to be gained from using Kummer's transformation for hypergeometric functions prior to the application of the approximation process? These and other questions are answered by meticulous analysis.

Having developed a great deal of expertise in practical rational approximation, it was only natural for Yudell to look around for interesting applications. He was led, in this way, to contribute to questions of univalence for Gauss' error function, to the accurate computation of a technical constant in the theory of trigonometric series, to rational predictor-corrector formulae for nonlinear ordinary differential equations, and eventually was able to interpret many of his rational approximations in terms of summability processes. More importantly, perhaps, he got involved in Padé approximation of the exponential function, a problem of considerable interest in the numerical solution of differential equations, both ordinary and partial. Yudell's contribution, characteristically, consists in providing representations of the error in the approximants on the main and first two subdiagonals in terms of modified Bessel functions as well as related asymptotics. Earlier, already, he obtained asymptotic error bounds for Padé approximants to $\exp(A)$, where A is a bounded linear operator in Banach space. In joint work with G. P. Barker this eventually led to interesting remarks on asymptotic series for matrix functions, in particular, to an extension of Watson's lemma from functions of a complex variable to functions in a matrix argument. By studying the sign of the error term in Padé approximants, and by a variety of other techniques, Yudell also enriched the field of analytic inequalities, deriving a large number of rational inequalities on various intervals of the real line for many of the important special functions. As he rightly points out, such inequalities appear infrequently in the literature.

Series expansion is another important source of approximations in which Yudell took an active interest. One owes to him expansions of the confluent hypergeometric function, and of many of its special cases, in series of Bessel functions, and more generally expansions of hypergeometric functions in other hypergeometric functions. Still more general are the expansions of Meijer's G-function in other G-functions which he developed together with Jet Wimp. These contain, among others, expansions of hypergeometric functions in Jacobi, Laguerre and Hermite polynomials and expansions of the sine and cosine integrals in squares of Bessel functions. Of considerable practical interest are expansions in Chebyshev polynomials, for which he gave many examples, both numerically and in analytic form. There are peculiar computational difficulties associated with the generation of the desired expansion coefficients, owing to the fact that they often represent solutions of second and higher order linear difference equations possessing minimum or intermediate growth properties. The development of stable algorithms for computing such solutions is an interesting branch of computational mathematics, to which Yudell contributed a useful variant of J. C. P. Miller's backward recurrence algorithm, partly in collaboration with Jet Wimp. Functions of several variables can be expanded in multiple Chebyshev series. Yudell was one of the first to provide numerical tables of associated coefficients in the case of Bessel functions of a real argument and real order between 0 and 1.

In addition to rational approximation and series expansion, there is a third large area—numerical quadrature—that has attracted Yudell's interest. Already in the early 50's he was developing interpolatory quadrature rules with equally spaced nodes for integrals and iterated integrals exhibiting singular or oscillatory weight functions. He provided not only relevant numerical data, but also discussed the error terms, in particular their Peano kernels, in his characteristically pragmatic, but effective way. The work on Filon-type formulae for oscillatory integrals is perhaps his best-known contribution from that period. Furthermore, following Poisson, Turing, Goodwin and others, he helped popularizing the exceptional qualities of the composite trapezoidal and midpoint rules as a means of evaluating many special functions, such as Bessel functions, complete elliptic integrals, the error function, and others. More recently, he turned to general quadrature rules of Gaussian type and related interpolation processes. Among other things, he discovered a novel expansion of the error term, in which intervene the coefficients in the expansion of the integrand (or, rather, the smooth factor multiplying the weight function in the integrand) in the appropriate system of orthogonal polynomials and certain quantities depending only on these orthogonal polynomials. He applied this to Stieltjes-type integrals and generalizations thereof, and in particular to integral representations of the hypergeometric function, thereby opening up yet another approach for its numerical evaluation.

Yudell's early work on special functions is summarized in his book "Integrals of Bessel Functions", published in 1962 by McGraw-Hill. Rather modestly titled, this is actually a comprehensive compendium not only of the functions in the title, but also of Bessel functions themselves and of generalized hypergeometric functions, of which Bessel functions and their integrals are, or can be expressed in terms of, special cases. Yudell also contributed a chapter of the same title to the famous "Handbook of Mathematical Functions" edited by M. Abramowitz and I. A. Stegun. His life work, however, culminated in the two volumes of "The Special Functions and their Approximations", published in 1969 by Academic Press, and the follow-up volumes "Mathematical Functions and their Approximations" of 1975 and "Algorithms for the Computation of Mathematical Functions" of 1977, both also with Academic Press. These works contain an amazing wealth of information, theoretical as well as practical, pertaining to special functions, summarizing and systematizing to a large extent Yudell's own research and that of his collaborators, without neglecting, however, relevant work of others. The "Mathematical Functions" is currently being translated into Russian.

His intellectual interests were never limited to mathematics alone. He loved opera, philosophy, baseball, among other things. While at MRI he gave an extensive series of lectures on the history of philosophy, focusing especially on Spinoza, whose work, he believed, contains the most meaningful elements of those ethical and intellectual ideals which alone can provide a personal bedrock in an uncertain, frenetically changing world. He ended the last lecture with a quotation from Spinoza's book of Ethics, "That which is noble is as difficult as it is rare". It had the force of a personal credo.

Yudell's 1975 book was dedicated to his daughters Molly, Janis, Linda, and Debra and their husbands, and his book of 1977 to his present and future grandchildren. Undoubtedly, the loving support of his family greatly fostered his mathematical growth, and it is natural and indicative of Yudell's personality that he wished the fruits of his life-long research dedicated to them. In his surviving family, and in his grateful students and colleagues, his values will endure.

WALTER GAUTSCHI & JET WIMP