MATHEMATICS OF COMPUTATION
VOLUME 43, NUMBER 168
OCTOBER 1984, PAGES 447-453

Spectral Properties for the Magnetization Integral Operator

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Abstract. We analyze the spectrum of a certain singular integral operator on the space $(L^2(\Omega))^3$ where Ω is contained in three dimensional Euclidean space and has a Lipschitz continuous boundary. This operator arises in the integral formulation of the magnetostatic field problem. We decompose $(L^2(\Omega))^3$ into invariant subspaces: in one where the operator is the zero map; in one, the identity map; and in one where the operator is positive definite and bounded. These results give rise to the formulation of new efficient numerical techniques for approximating nonlinear magnetostatic field problems [5], [6], [12].

1. Introduction. In this paper we analyze the spectrum of the singular integral operator

$$Aw = \nabla Tw, \tag{1.1}$$

where

$$(Tw)(x) = \frac{1}{4\pi}\int_\Omega w(y)\cdot\nabla_y\left(\frac{1}{r}\right)dy \quad \text{and} \quad r = |x-y| \tag{1.2}$$

defined on $w \in (L^2(\Omega))^3$ for bounded domains Ω contained in R^3. The operator A is used in integral formulations of the magnetostatic field problem and their discretization [1], [3], [11]. Applications of the results given in this paper lead to new efficient numerical procedures [5], [6], [12] for approximating nonlinear magnetostatic field problems.

The spectrum of the operator (1.1) was first analyzed in [4] for smooth simply connected domains by the methods of classical potential theory. In this paper we extend these results to the case of domains with Lipschitz continuous boundaries. In contrast, our analysis is based on deriving an equivalent formulation of the operator as an elliptic boundary value problem. The desired results are obtained by developing the appropriate properties of the boundary value problem.

The outline of the remainder of the paper is as follows. In Section 2, we introduce some notation and state some preliminary results. In Section 3, we state and prove

Received March 17, 1983; revised January 12, 1984.

1980 *Mathematics Subject Classification*. Primary 45E10; Secondary 35J60.

* Work supported by Cornell University and Iowa State University.

** The submitted manuscript has been authored under contract DE-AC02-76CH00016 with the U. S. Department of Energy. Accordingly, the U. S. government retains a nonexclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U. S. Government purposes.

the main results of the paper (Theorem 1 and Corollary 1) which describe the spectral decomposition of the operator A.

2. Preliminaries and Notation. Let Ω denote a bounded domain in three dimensional Euclidean space R^3 with Lipschitz continuous boundary Γ. Sobolev spaces on Ω and Γ of order s will be denoted $H^s(\Omega)$ and $H^s(\Gamma)$ with corresponding norms $\|\cdot\|_{H^s(\Omega)}$ and $|\cdot|_{H^s(\Gamma)}$, respectively, [7], [9]. For negative s, the Sobolev spaces are defined by duality. Let $\mathscr{D}(\Omega)$ be the space of infinitely differentiable functions with support contained in Ω and $\mathscr{D}'(\Omega)$ denote the space of Schwartz distributions on Ω [14]. $C^\infty(\overline{\Omega})$ denotes the space of infinitely differentiable functions on $\overline{\Omega}$ and is dense in $H^s(\Omega)$ for any s. $H_0^s(\Omega)$ is defined to be the completion of $\mathscr{D}(\Omega)$ in the $H^s(\Omega)$ norm.

The notation $\underline{H}$ will denote the product space H^3 which has components in a space H. When H is a Hilbert space, $\underline{H}$ inherits the obvious norms and inner products.

Let $(\cdot,\cdot)_\Omega$ denote the L^2 inner product on Ω given by

$$(u, v)_\Omega \equiv \int_\Omega uv\,dx. \tag{2.1}$$

For vector-valued functions $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, (2.1) will be replaced by

$$(u, v)_\Omega = \int_\Omega u \cdot v\,dx.$$

The Dirichlet inner product on Ω is given by

$$D_\Omega(u, v) \equiv (\nabla u, \nabla v)_\Omega.$$

The Lipschitz continuity assumption on Γ implies that the exterior normal $n = (n_1, n_2, n_3)$ on Γ exists almost everywhere [9]. In addition there are trace and extension operators as described by the following lemma, which may be found in [9].

LEMMA 1. *The trace operator (denoted by T_Ω) extends continuously from $\mathscr{D}(\overline{\Omega})$ to an operator from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$.*

LEMMA 2. *There exists a bounded extension operator E_Ω from $H^{1/2}(\Gamma)$ into $H^1(\Omega)$ satisfying*

$$T_\Omega \circ E_\Omega = I \quad \text{on } H^{1/2}(\Gamma).$$

Our results for the singular integral operator will be stated for a fixed bounded domain Ω with Lipschitz continuous boundary Γ. We shall denote the L^2 inner product on Γ by $\langle\cdot,\cdot\rangle$. In addition $\langle\cdot,\cdot\rangle$ shall be used to denote the duality between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$.

We shall need some auxiliary subspaces of $\underline{L^2(\Omega)}$. Let

$$K \equiv \{\nabla\phi \mid \phi \in H^1(\Omega)\}.$$

Define

$$\mathscr{N}_\Omega \equiv \{\phi \in H^1(\Omega) \mid \nabla\phi = 0 \text{ in } \Omega\}.$$

Note that the functions in $\mathscr{N}_\Omega$ are constant on the components of Ω and Ω has only a finite number of components. Set

$$\mathring{H}^1(\Omega) \equiv \left\{ \phi \in H^1(\Omega) | (\phi, \psi)_\Omega = 0 \text{ for all } \psi \in \mathscr{N}_\Omega \right\}.$$

$\mathring{H}^1(\Omega)$ is obviously a closed subspace of $H^1(\Omega)$. A standard argument used in the proof of the Poincaré-Friedrichs inequality [2] gives that for some $c > 0$

$$\|\phi\|^2_{H^1(\Omega)} \leqslant c D_\Omega(\phi, \phi) \quad \text{for } \phi \in \mathring{H}^1(\Omega) \tag{2.2}$$

and hence $\{ D_\Omega(\cdot, \cdot)\}^{1/2}$ is a norm on $\mathring{H}^1(\Omega)$ which is equivalent to the usual Sobolev norm.

Note that K obviously coincides with

$$\left\{ \nabla\phi | \phi \in \mathring{H}^1(\Omega) \right\}$$

and it is a straightforward consequence of (2.2) that K is closed in $L^2(\Omega)$.

Let

$$K_0 \equiv \left\{ \nabla\phi | \phi \in H^1_0(\Omega) \right\}.$$

Then, arguments similar to those given above imply that K_0 is a closed subspace of $L^2(\Omega)$. The orthogonal complement of K_0 in K will be denoted K_H and the orthogonal complement of K in $L^2(\Omega)$ will be denoted by N. The following lemma was given by Temam [13].

LEMMA 3. *N is the completion in $L^2(\Omega)$ of*

$$\left\{ u \in \mathscr{D}(\Omega) \,|\, \operatorname{div} u = 0 \right\}.$$

3. The Spectral Properties of A. Let w be a Lipschitz continuous vector field defined on Ω. The kernel of the integral operator T is weakly singular and it is shown in [8] that the partial derivatives of Tw exist. Furthermore, the map w to $\partial Tw/\partial x_i$ extends continuously to a bounded operator from $L^p(\Omega)$ into $L^p(\Omega)$ for $p > 1$. The map T is thus a bounded map from $L^2(\Omega)$ to $H^1(\Omega)$.

The goal of this paper is to prove the following theorem and its corollary.

THEOREM 1. *The operator A is a bounded selfadjoint map on $L^2(\Omega)$ and satisfies*

(i) Ker $A = N$.

(ii) *A is the identity when restricted to K_0.*

(iii) *K_H is an invariant subspace of A.*

(iv) *The spectrum of A on K_H is contained in the interval* $[\lambda_0, \Lambda_0]$ *where* $0 < \lambda_0 \leqslant \Lambda_0 \leqslant 1$.

COROLLARY 1. *Λ_0 in Theorem 1 can be taken less than 1 if and only if the complement of Ω has no bounded components.*

We shall need additional notation and lemmas for the proof of Theorem 1. Let Ω' be an arbitrary domain in R^3. As in [10], we consider the spaces $W^1_0(\Omega')$ defined by

$$W^1_0(\Omega') \equiv \left\{ \phi \in \mathscr{D}'(\Omega') | \nabla\phi \in L^2(\Omega') \text{ and } \phi/(1 + r) \in L^2(\Omega') \right\}$$

where r measures the distance to the origin. Then $W^1_0(\Omega')$ has the natural norm

$$\|\phi\|_{W^1_0(\Omega')} = \left\{ \|\nabla\phi\|^2_{L^2(\Omega')} + \|\phi/(1 + r)\|^2_{L^2(\Omega')} \right\}^{1/2}. \tag{3.1}$$

The following lemma is given in [10].

LEMMA 4. *If Ω' has no bounded components then* (3.1) *is equivalent to the norm* $\{D_\Omega(u,u)\}^{1/2}$ *for u in* $W_0^1(\Omega')$.

Let Ω_c denote the interior of the complement of the region Ω. For any bounded domain Ω' contained in Ω_c, any function in $W_0^1(\Omega_c)$, when restricted to Ω', is in $H^1(\Omega')$ and thus it is a straightforward consequence of Lemmas 1 and 2 that there exist appropriately bounded trace and extension operators T_{Ω_c} and E_{Ω_c} between $W_0^1(\Omega_c)$ and $H^{1/2}(\Gamma)$. Let

$$\mathcal{N}_{\Omega_c} \equiv \left\{\phi \in W_0^1(\Omega_c) \,|\, \nabla\phi = 0 \text{ on } \Omega_c\right\}.$$

Note that functions in $\mathcal{N}_{\Omega_c}$ are constant on the bounded components of Ω_c and zero on the unbounded component of Ω_c. We define

$$\mathring{W}_0^1(\Omega_c) \equiv \left\{\phi \in W_0^1(\Omega_c) \,|\, (\phi,\psi)_{\Omega_c} = 0 \text{ for all } \psi \in \mathcal{N}_{\Omega_c}\right\}.$$

Then Lemma 4 and (2.2) imply that $\{D_{\Omega_c}(\cdot,\cdot)\}^{1/2}$ is a norm equivalent to (3.1) on $\mathring{W}_0^1(\Omega_c)$. Let $W_{0,0}^1(\Omega_c)$ be defined by

$$W_{0,0}^1(\Omega_c) \equiv \left\{\psi \in W_0^1(\Omega_c) \,|\, T_{\Omega_c}(\psi) = 0\right\},$$

and

$$K_H(\Omega_c) = \left\{\nabla\psi \,|\, \psi \in W_0^1(\Omega_c) \text{ and } D_{\Omega_c}(\psi,\phi) = 0 \text{ for all } \phi \in W_{0,0}^1(\Omega_c)\right\}.$$

We shall use the following lemmas in the proof of Theorem 1.

LEMMA 5. *For $\phi \in H^{1/2}(\Gamma)$ there exists a unique extension h_ϕ^Ω in $H^1(\Omega)$ satisfying*

$$D_\Omega\left(h_\phi^\Omega, \psi\right) = 0 \quad \text{for all } \psi \in H_0^1(\Omega).$$

The above statement holds with Ω, $H^1(\Omega)$ and $H_0^1(\Omega)$ replaced by Ω_c, $W_0^1(\Omega_c)$ and $W_{0,0}^1(\Omega_c)$.

Remark. h_ϕ^Ω (resp. $h_\phi^{\Omega_c}$) is just the harmonic extension of ϕ into Ω (resp. Ω_c).

Let $\mathcal{N}_\Gamma \equiv \{T_\Omega(\psi) \,|\, \psi \in \mathcal{N}_\Omega\}$ and define

$$H^{-1/2}(\Gamma)/\mathcal{N}_\Gamma \equiv \left\{\phi \in H^{-1/2}(\Gamma) \,|\, \langle\phi,\psi\rangle = 0 \text{ for all } \psi \in \mathcal{N}_\Gamma\right\}.$$

LEMMA 6. *For a function σ in $H^{-1/2}(\Gamma)/\mathcal{N}_\Gamma$ there exists a unique function ψ in $\mathring{H}^1(\Omega)$ satisfying*

$$D_\Omega(\psi,\theta) = \langle\sigma,\theta\rangle \quad \text{for all } \theta \in H^1(\Omega). \tag{3.2}$$

Furthermore, the map $\sigma \to \nabla\psi$ is a homeomorphism of $H^{-1/2}(\Gamma)/\mathcal{N}_\Gamma$ onto K_H. A similar result holds with $\mathcal{N}_\Gamma$, $\mathring{H}^1(\Omega)$, $H^1(\Omega)$ and K_H replaced by $\mathcal{N}_{\Gamma_c}$, $\mathring{W}_0^1(\Omega_c)$, $W_0^1(\Omega_c)$ and $K_H(\Omega_c)$ respectively.

Remark. σ is the generalized outward normal derivative of ψ on Γ.

LEMMA 7. *Let w be in $\underline{L^2(\Omega)}$. Then $u = Tw$ is the unique function in $W_0^1(R^3)$ satisfying*

$$D_{R^3}(u,\phi) = (w,\nabla\phi)_\Omega \quad \text{for all } \phi \in W_0^1(R^3). \tag{3.3}$$

We postpone the proof of the last three lemmas until after the proof of the theorem and its corollary.

Proof of Theorem 1. Lemma 7 and the definition of A imply that $Aw = \nabla u$ where u is the solution of (3.3). Substituting $\phi = Tv$ in (3.3) gives

$$(3.4) \qquad (Aw, Av)_{R^3} = (w, Av)_\Omega$$

and implies that A is symmetric and bounded, hence selfadjoint.

If w is in K_0, then $w = \nabla v$ for some $v \in H_0^1(\Omega)$ and v extended by zero is in $W_0^1(R^3)$. Hence, v extended by zero is the solution of (3.3) and thus $Aw = \nabla v = w$ which proves (ii).

We next show that there exists $\lambda_0 > 0$ satisfying

$$(3.5) \qquad \lambda_0 \|w\|_{\underline{L^2(\Omega)}} \leq \|Aw\|_{\underline{L^2(\Omega)}} \quad \text{for all } w \in K_H.$$

Let w be in K_H then $w = \nabla\psi$ for some $\psi \in H^1(\Omega)$. Let σ_w be the distribution guaranteed by Lemma 6 satisfying

$$(3.6) \qquad D_\Omega(\psi, \phi) = \langle \sigma_w, \phi \rangle \quad \text{for all } \phi \in H^1(\Omega).$$

If u is the solution of (3.3), then ∇u is in $K_H(\Omega_c)$. Let σ_u be the corresponding distribution satisfying

$$(3.7) \qquad D_{\Omega_c}(u, \phi) = \langle \sigma_u, \phi \rangle \quad \text{for all } \phi \in W_0^1(\Omega_c).$$

Let ϕ be an arbitrary function in $H^{1/2}(\Gamma)$ and extend ϕ to R^3 by

$$\bar{\phi} \equiv \begin{cases} E_\Omega(\phi) & \text{on } \Omega, \\ E_{\Omega_c}(\phi) & \text{on } \Omega_c. \end{cases}$$

Then $\bar{\phi}$ is in $W_0^1(R^3)$ and using (3.3), (3.6), and (3.7)

$$\begin{aligned} |\langle \sigma_w, \phi \rangle| &= |(w, \nabla\bar{\phi})_\Omega| = |D_\Omega(u, \bar{\phi}) + \langle \sigma_u, \phi \rangle| \\ &\leq c\left\{\|\nabla u\|_{\underline{L^2(\Omega)}} + |\sigma_u|_{H^{-1/2}(\Gamma)}\right\}|\phi|_{H^{1/2}(\Gamma)} \\ &\leq c\|Aw\|_{\underline{L^2(R^3)}}|\phi|_{H^{1/2}(\Gamma)}. \end{aligned}$$

Thus by Lemma 6 and the definition of the norm in $H^{-1/2}(\Gamma)$ we have

$$\|w\|^2_{\underline{L^2(\Omega)}} \leq c|\sigma_w|^2_{H^{-1/2}(\Gamma)} \leq \frac{1}{\lambda_0}\|Aw\|^2_{\underline{L^2(R^3)}}.$$

Thus (3.5) follows from (3.4) and the Schwarz inequality.

By (3.3) and the definition of N, N is contained in Ker A and (i) follows from (ii) and (3.5). We also note that (3.4) with $v = w$ and the Schwarz inequality imply that

$$\|Aw\|_{\underline{L^2(\Omega)}} \leq \|w\|_{\underline{L^2(\Omega)}}$$

and thus the theorem will follow once (iii) is verified.

From the definition $Aw = \nabla Tw$, the range of A is contained in K. Let w be in K_H, u solve (3.3) and σ_u be defined as above. If $v = \nabla\psi$ is in K_0 then Lemma 7 implies that

$$0 = (w, \nabla\psi)_\Omega = D_\Omega(u, \psi) + \langle \sigma_u, \psi \rangle = (\nabla u, \nabla\psi)_\Omega.$$

Thus $Aw = \nabla u$ is in K_H and the proof of the theorem is complete.

Proof of Corollary 1. Assume that Ω_c has no bounded components. Let w be in K_H and let u and σ_u be as in the proof of Theorem 1. Then by (3.3), Lemmas 1 and 4

$$(w, Aw)_\Omega = \|Aw\|^2_{\underline{L^2(\Omega)}} + \|\nabla u\|^2_{\underline{L^2(\Omega_c)}} \geq \|Aw\|^2_{\underline{L^2(\Omega)}} + c|u|^2_{H^{1/2}(\Gamma)}.$$

Now since $u = h_u^\Omega$, Lemma 5 implies

$$\|Aw\|_{\underline{L^2(\Omega)}} = \|\nabla u\|_{\underline{L^2(\Omega)}} \leqslant c|u|_{H^{1/2}(\Gamma)},$$

thus

$$(w, Aw)_\Omega \geqslant (1 + c)\|Aw\|^2_{\underline{L^2(\Omega)}}$$

with $c > 0$. This proves the first part of the corollary.

Suppose Ω_c has a bounded component which we will denote Ω_c^b. Define ϕ on Γ by

$$\phi(x) \equiv \begin{cases} 1 & \text{if } x \in \partial\Omega_c^b, \\ 0 & \text{if } x \in \Gamma - \partial\Omega_c^b. \end{cases}$$

Define $w = \nabla h_\phi^\Omega$; then by Lemma 5, w is in K_H. Furthermore, it is easily seen that

$$u = \begin{cases} h_\phi^\Omega & \text{on } \Omega, \\ h_\phi^{\Omega_c} & \text{on } \Omega_c \end{cases}$$

is constant on the components of Ω_c and hence is the solution of (3.3). Thus $Aw = w$ and the corollary follows.

We shall only sketch the proofs of Lemmas 5 and 6 since the lemmas correspond to results which are well known in the case of smooth domains.

Proof of Lemma 5. We prove Lemma 5 for the domain Ω. Given ϕ in $H^{1/2}(\Gamma)$ define $\psi = E_\Omega(\phi)$ and let v be the unique function in $H_0^1(\Omega)$ satisfying

$$D_\Omega(v + \psi, \theta) = 0 \quad \text{for all } \theta \in H_0^1(\Omega).$$

Then $h_\phi^\Omega = v + \psi$ has the desired properties. The uniqueness of the extension h_ϕ^Ω is a consequence of the fact that

$$H_0^1(\Omega) = \{\phi \in H^1(\Omega) \,|\, T_\Omega\phi = 0\}.$$

Proof of Lemma 6. We prove Lemma 6 for Ω, the proof for Ω_c is similar. Let σ be in $H^{-1/2}(\Gamma)/\mathcal{N}_\Gamma$; then the map $\theta \to \langle\sigma, \theta\rangle$ is a bounded linear functional on $\mathring{H}^1(\Omega)$. Since $\|\nabla\phi\|_{\underline{L^2(\Omega)}}$ is a norm on $\mathring{H}^1(\Omega)$, the Riesz Representation Theorem guarantees that there exists ψ in $\mathring{H}^1(\Omega)$ satisfying (3.2) for all θ in $\mathring{H}^1(\Omega)$. From the definition of $H^{-1/2}(\Gamma)/\mathcal{N}_\Gamma$, this also implies that ψ satisfies (3.2) for all functions θ in $H^1(\Omega)$. The map $S(\sigma) \equiv \nabla\psi$ is clearly bounded from $H^{-1/2}(\Gamma)/\mathcal{N}_\Gamma$ into K_H.

Given $w = \nabla\phi$ in K_H we have by Lemma 5

$$\left|D_\Omega\left(\psi, h_\psi^\Omega\right)\right| \leqslant C\|\nabla\psi\|_{\underline{L^2(\Omega)}}|\psi|_{H^{1/2}(\Gamma)} \quad \text{for all } \psi \in H^{1/2}(\Gamma).$$

Thus there exists a unique distribution σ in $H^{-1/2}(\Gamma)$ satisfying

$$\langle\sigma, \phi\rangle = D_\Omega\left(\psi, h_\phi^\Omega\right) \quad \text{for all } \phi \in H^{1/2}(\Gamma).$$

One then argues that the map $w \to \sigma$ is a bounded map of K_H into $H^{-1/2}(\Gamma)/\mathcal{N}_\Gamma$ and $S(\sigma) = w$. Hence S is a homeomorphism of $H^{-1/2}(\Gamma)/\mathcal{N}_\Gamma$ onto K_H.

Proof of Lemma 7. Let w be in $\underline{L^2(\Omega)}$. Choose $\{w_i\}_{i=1}^\infty$ in $\underline{\mathcal{D}(\Omega)}$ with w_i converging to w in $\underline{L^2(\Omega)}$. Clearly the functional $\phi \to (w, \nabla\phi)_\Omega$ is bounded on $W_0^1(R^3)$. Thus by Lemma 4 there exists a unique function $u_i \in W_0^1(R^3)$ satisfying

$$D_{R^3}(u_i, \phi) = (w_i, \nabla\phi)_\Omega \quad \text{for all } \phi \in W_0^1(R^3).$$

Clearly u_i converges to the solution u of (3.3) in $W_0^1(R^3)$. It is also clear from classical potential theory that

$$u_i = \frac{1}{4\pi}\int_{R^3} w_i \cdot \nabla_y\left(\frac{1}{r}\right) dy = \frac{1}{4\pi}\int_{\Omega} w_i \cdot \nabla_y\left(\frac{1}{r}\right) dy.$$

Thus, $Tw_i = u_i$ and the lemma follows by density.

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