

More Trigonometric Integrals

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Abstract. Integrals of the form

$$\int_0^{\pi/2} e^{ip\theta} \cos^q \theta \, d\theta, \quad \int_0^{\pi/2} e^{ip\theta} \sin^q \theta \, d\theta$$

(p real, $\operatorname{Re}(q) > -1$) are expressed in terms of Gamma and hypergeometric functions for integer and noninteger values of q and p . The results include those of [2] as special cases.

Introduction. The integrals considered here are

$$(1) \quad \int_0^{\pi/2} e^{ip\theta} \cos^q \theta \, d\theta,$$

$$(2) \quad \int_0^{\pi/2} e^{ip\theta} \sin^q \theta \, d\theta,$$

where p is real, and $\operatorname{Re}(q) > -1$. Values for some of the above integrals are recorded in [1, art. 3.631], but only for special (or integer) values of “ q ”, and not always in closed form. The integrals (1) and (2) are, of course, related since, with the change of variable $\theta \rightarrow \pi/2 - \theta$, (2) becomes

$$(3) \quad \int_0^{\pi/2} e^{ip\theta} \sin^q \theta \, d\theta = e^{ip\pi/2} \int_0^{\pi/2} e^{-ip\theta} \cos^q \theta \, d\theta$$

resulting in the following relations:

$$(4) \quad \int_0^{\pi/2} \sin^q \theta \cos p\theta \, d\theta = \sin \frac{p\pi}{2} \int_0^{\pi/2} \cos^q \theta \sin p\theta \, d\theta \\ + \cos \frac{p\pi}{2} \int_0^{\pi/2} \cos^q \theta \cos p\theta \, d\theta,$$

$$(5) \quad \int_0^{\pi/2} \sin^q \theta \sin p\theta \, d\theta = \sin \frac{p\pi}{2} \int_0^{\pi/2} \cos^q \theta \cos p\theta \, d\theta \\ - \cos \frac{p\pi}{2} \int_0^{\pi/2} \cos^q \theta \sin p\theta \, d\theta.$$

It is evident that, either with the aid of multiple angle formulae, or integration by parts, all of these integrals can be evaluated in finite form if either “ p ” or “ q ”, or both, are integers, see, e.g., [1, arts. 2.536-8]. For this reason, it will be assumed in the remainder of the paper that “ p ” and “ q ” are arbitrary, noninteger quantities, subject to the conditions stated above.

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1. Evaluation by Contour Integration. The integrals in question are evaluated by integrating the function

$$(6) \quad f(z) = (1 + z^2)^q (z^{p-q-1})$$

in the complex $z (= \rho e^{i\theta} \equiv x + iy)$ plane around the contour consisting of the portion of the real axis from $x = 0$ to $x = 1$, the quarter arc of the unit circle from $\theta = 0$ to $\theta = \pi/2$, and the portion of the imaginary axis from $y = 1$ to $y = 0$. The total line integral is zero, provided the contour is modified by small circular arcs about the branch points at $z = 0$ and $z = i$, since there is no contribution to the total value of the integral from these arcs when their radii approach zero, provided $\text{Re}(q) > -1$ and $\text{Re}(p - q) > 0$. The total line integral thus becomes a linear combination of the real integrals

$$(7) \quad \int_0^{\pi/2} \cos^q \theta \sin p\theta \, d\theta,$$

$$(8) \quad \int_0^{\pi/2} \cos^q \theta \cos p\theta \, d\theta,$$

$$(9) \quad \int_0^1 (1 + t)^q t^{(p-q-2)/2} \, dt,$$

$$(10) \quad \int_0^1 (1 - t)^q t^{(p-q-2)/2} \, dt.$$

The integral (10), as is well known, can be expressed by Gamma functions:

$$(11) \quad \int_0^1 (1 - t)^q t^{(p-q-2)/2} \, dt = \frac{\Gamma(1 + q) \Gamma\left(\frac{p-q}{2}\right)}{\Gamma\left(\frac{2+p+q}{2}\right)},$$

while (9) can be evaluated as a hypergeometric function:

$$(12) \quad \int_0^1 (1 + t)^q t^{(p-q-2)/2} \, dt = \frac{2}{p-q} {}_2F_1\left(-q, \frac{p-q}{2}; \frac{1+p-q}{2}; -1\right)$$

(see, e.g., [1, art 9.111], or [4, p. 12]).

Equating the real and imaginary parts of the resulting integrals gives

$$(13) \quad \begin{aligned} \text{(a)} \quad 2^{1+q} \int_0^{\pi/2} \cos^q \theta \cos p\theta \, d\theta &= \sin\left(\frac{p-q}{2}\pi\right) \frac{\Gamma(1+q) \Gamma\left(\frac{p-q}{2}\right)}{\Gamma\left(\frac{2+p+q}{2}\right)}, \\ \text{(b)} \quad 2^{1+q} \int_0^{\pi/2} \cos^q \theta \sin p\theta \, d\theta &= \frac{2}{(p-q)} {}_2F_1\left(-q, \frac{p-q}{2}; \frac{2+p-q}{2}; -1\right) \\ &\quad - \cos\left(\frac{p-q}{2}\pi\right) \frac{\Gamma(1+q) \Gamma\left(\frac{p-q}{2}\right)}{\Gamma\left(\frac{2+p+q}{2}\right)}, \end{aligned}$$

and using the relations (4) and (5),

$$\begin{aligned}
 & \text{(a)} \quad 2^{1+q} \int_0^{\pi/2} \sin^q \theta \cos p\theta \, d\theta \\
 & \qquad = \frac{2 \sin(p\pi/2)}{p-q} {}_2F_1\left(-q, \frac{p-q}{2}, \frac{2+p-q}{2}; -1\right) \\
 & \qquad \quad - \sin\left(\frac{q\pi}{2}\right) \cdot \frac{\Gamma(1+q)\Gamma\left(\frac{p-q}{2}\right)}{\Gamma\left(\frac{2+p+q}{2}\right)}, \\
 & \text{(14)} \quad \text{(b)} \quad 2^{1+q} \int_0^{\pi/2} \sin^q \theta \sin p\theta \, d\theta \\
 & \qquad = \frac{-2 \cos(p\pi/2)}{p-q} {}_2F_1\left(-q, \frac{p-q}{2}, \frac{2+p-q}{2}; -1\right) \\
 & \qquad \quad + \cos\left(\frac{q\pi}{2}\right) \cdot \frac{\Gamma(1+q)\Gamma\left(\frac{p-q}{2}\right)}{\Gamma\left(\frac{2+p+q}{2}\right)}.
 \end{aligned}$$

(It may be noted that, of the four relations in (13) and (14), only (13a) is given in [1] for arbitrary values of “ p ” and “ q ”, and also that, although these relations were established under the assumption that $\text{Re}(p - q) > 0$, the various functions on the right of (13) and (14) are defined for all values of “ p ” and “ q ”, provided $(p - q) \neq 0, -2, -4, \dots, -2n$, and therefore the results may be extended to the wider range $\text{Re}(q) > -1$ by analytic continuation.)

2. Expressions When $\frac{1}{2}(p - q)$ is a Negative Integer or Zero. From principles of continuity, it follows that all of the results given by Eqs. (13) and (14) must hold if “ p ” is replaced by “ $-p$ ”. In particular, from (13), we get, after some obvious transformations of the Gamma function:

$$\begin{aligned}
 & \text{(a)} \quad 2^{1+q} \int_0^{\pi/2} \cos^q \theta \cos p\theta \, d\theta = \pi \frac{\Gamma(1+q)}{\Gamma\left[1 + \frac{1}{2}(p+q)\right] \Gamma\left[1 - \frac{1}{2}(p-q)\right]}, \\
 & \text{(b)} \quad 2^{1+q} \int_0^{\pi/2} \cos^q \theta \sin p\theta \, d\theta \\
 & \text{(15)} \quad = \frac{2}{p+q} {}_2F_1\left(-q, -\frac{1}{2}(p+q); 1 - \frac{1}{2}(p+q); -1\right) \\
 & \qquad \quad - \pi \cot(p+q) \frac{\pi}{2} \frac{\Gamma(1+q)}{\Gamma\left[1 + \frac{1}{2}(p+q)\right] \Gamma\left[1 - \frac{1}{2}(p-q)\right]},
 \end{aligned}$$

and the above equations now yield determinate expressions when $\frac{1}{2}(p - q)$ is a negative integer or zero. Specifically if we set $p = q + 2k$, and make use of the relation

$${}_2F_1(a, b; c; z) = (1-z)^{-b} {}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right),$$

(15b) becomes

$$(16) \quad 2^{1+q} \int_0^{\pi/2} \cos^q \theta \sin(q+2k)\theta d\theta = \frac{2^{k+q}}{k+q} {}_2F_1(1-k, -k-q; 1-k-q; \frac{1}{2}) - \pi \cot q\pi \frac{\Gamma(1+q)}{\Gamma(1+k+q)\Gamma(1-k)},$$

and if “ k ” is a positive integer, the second term vanishes, while the hypergeometric series terminates. The result then becomes equivalent to formula 3.632(3) of [1]. On the other hand, if “ k ” is a negative integer or zero, the second term remains, and the series is no longer finite.

As a special result in the second case, we obtain the following closed form expression for the integral

$$(17) \quad \int_0^{\pi/2} \cos^q \theta \sin q\theta d\theta = \frac{1}{2q} {}_2F_1(1, -q; 1-q; \frac{1}{2}) - \frac{\pi}{2^{q+1}} \cot q\pi,$$

which has not been given previously, except for integer “ q ”, in which case the above expression is indeterminate [see Eq. (20) below].

3. Alternative Integral Representations When $p - q$ is an Even Integer. A different, and much simpler representation for the integrals in (16) and (17) can be obtained by re-expressing (13b) in terms of (11) and (12):

$$(18) \quad \begin{aligned} & 2^{1+q} \int_0^{\pi/2} \cos^q \theta \sin p\theta d\theta \\ &= \int_0^1 [(1+t)^q - 1] t^{(p-q)/2-1} dt \\ &+ 2 \frac{1 - \cos \frac{\pi}{2}(p-q)}{p-q} + \cos \frac{\pi}{2}(p-q) \int_0^1 [1 - (1-t)^q] t^{(p-q)/2-1} dt. \end{aligned}$$

In the limit, as $p \rightarrow q$, this gives, for arbitrary “ q ” > -1 ,

$$(19) \quad \begin{aligned} 2^{1+q} \int_0^{\pi/2} \cos^q \theta \sin q\theta d\theta &= \int_0^1 \frac{(1+t)^q - 1}{t} dt + \int_0^1 \frac{1 - (1-t)^q}{t} dt \\ &= \int_0^2 \frac{u^q - 1}{u - 1} du \end{aligned}$$

and if “ q ” is a positive integer, “ n ”, the following result quoted in [1, formula 3.631(16)]:

$$(20) \quad 2^{1+n} \int_0^{\pi/2} \cos^n \theta \sin n\theta d\theta = \int_0^2 \frac{u^n - 1}{u - 1} du = 2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^n}{n}.$$

The result of Eq. (19) is significant in that it shows that the integral on the left can be evaluated in terms of elementary functions whenever “ q ” is a rational number, (n/m), since then, with the change of variable

$$u = v^n$$

on the right-hand side, the integrand becomes the quotient of two polynomials. For example,

$$\int_0^{\pi/2} \cos^{1/2} \theta \sin \frac{1}{2} \theta d\theta = 2^{-1/2} \int_0^{2^{1/2}} \frac{v dv}{1+v} = 1 - 2^{-1/2} \ln(2^{1/2} + 1);$$

$$(21) \quad \int_0^{\pi/2} \cos^{1/4} \theta \sin \frac{1}{4} \theta d\theta = 2^{+3/4} \int_0^{2^{1/4}} \frac{v^3 dv}{(1+v)(1+v^2)}$$

$$= 2 - 2^{-1/4} [\ln(2^{1/4} + 1) = \frac{1}{2} \ln(2^{1/2} + 1) + \tan^{-1}(2^{1/4})].$$

Application of the same transformation to the integral in (16) gives, for $k > 0$,

$$(22) \quad 2^{1+q} \int_0^{\pi/2} \cos^q \theta \sin(q + 2k)\theta d\theta = \int_0^2 u^q (u - 1)^{k-1} du,$$

and after expanding the term $(u - 1)^{k-1}$ by the binomial theorem and integrating term-by-term, we obtain the expansion of formula 3.632(3) of [1]. On the other hand, successive integrations by parts applied to the right-hand side of (22) leads to a different representation:

$$(23) \quad \int_0^{\pi/2} \cos^q \theta \sin(q + 2k)\theta d\theta$$

$$= \frac{1}{q+1} \left[1 - \frac{k-1}{q+2} \cdot 2 + \frac{(k-1)(k-2)}{(q+2)(q+3)} \cdot 2^2 + \dots \right].$$

When “ k ” is negative, similar, but more complex, expressions can be obtained for the integral in (16), by first subtracting from the integrands on the right side of (18) the first $(|k| + 1)$ terms of the Taylor series for $(1 + t)^q$ and $(1 - t)^q$. For example, if $k = -1$, we get

$$(24) \quad 2^{1+q} \int_0^{\pi/2} \cos^q \theta \sin(q - 2)\theta d\theta$$

$$= \lim_{p \rightarrow -2} \left\{ \int_0^1 [(1+t)^q - 1 - qt] t^{(p-q)/2-1} dt \right.$$

$$+ \cos \frac{\pi}{2} (p - q) \int_0^1 [1 - qt - (1-t)^q] t^{(p-q)/2-1} dt$$

$$\left. + 2 \frac{1 - \cos \frac{\pi}{2} (p - q)}{p - q} + 2q \frac{1 + \cos \frac{\pi}{2} (p - q)}{p - q + 2} \right\}$$

$$= \int_0^1 \frac{(1+t)^q - 1 - qt}{t^2} dt - \int_0^1 \frac{(1-qt - (1-t)^q)}{t^2} dt - 2$$

$$= \int_0^2 \frac{u^q - 1 - q(u-1)}{(u-1)^2} du - 2.$$

For integer values of $q = n$, this again becomes a terminating series, since

$$\frac{u^n - 1 - n(u-1)}{(u-1)^2} = 0; \quad n = 0, 1,$$

$$= u^{n-2} + 2u^{n-3} + \dots + (n-1); \quad n > 1,$$

and hence

$$\begin{aligned}
 & 2^{1+n} \int_0^{\pi/2} \cos^n \theta \sin(n-2)\theta \, d\theta \\
 (25) \quad & = \left[\frac{2^{n-1}}{n-1} + 2 \cdot \frac{2^{n-2}}{n-2} + \cdots + (n-1) \cdot 2 \right] - 2; \quad n > 1, \\
 & = -2; \quad n = 0, 1.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & 2^{1+q} \int_0^{\pi/2} \cos^q \theta \sin(q-4)\theta \, d\theta \\
 (26) \quad & = \int_0^2 \frac{u^q - 1 - q(u-1) - \frac{1}{2}q(q-1)(u-1)^2}{(u-1)^3} du - 2q
 \end{aligned}$$

and

$$\begin{aligned}
 & 2^{1+n} \int_0^{\pi/2} \cos^n \theta \sin(n-4)\theta \, d\theta \\
 (27) \quad & = \left[\frac{2^{n-2}}{n-2} + 3 \frac{2^{n-3}}{n-3} + 6 \frac{2^{n-4}}{n-4} + \cdots + \frac{(n-1)(n-2)}{2} \cdot 2 \right] \\
 & \quad - 2n; \quad n > 2, \\
 & = -2n; \quad n = 0, 1, 2.
 \end{aligned}$$

Alternatively, integration by parts leads to

$$(28) \quad \int_0^{\pi/2} \cos^q \theta \sin p\theta \, d\theta = \frac{1}{q-p} \left\{ 1 - q \int_0^{\pi/2} \cos^{q-1} \theta \sin(p+1)\theta \, d\theta \right\},$$

which, when applied successively k times gives, for $k > 0$, $q > k - 1$,

$$\begin{aligned}
 & \int_0^{\pi/2} \cos^q \theta \sin(q-2k)\theta \, d\theta \\
 & = \frac{q(q-1) \cdots (q-k+1)}{2^k k!} \\
 (29) \quad & \times \left\{ \int_0^{\pi/2} \cos^{q-k} \theta \sin(q-k)\theta \, d\theta \right. \\
 & \quad - \left[\frac{1}{(q-k+1)} + \frac{2 \cdot 1!}{(q-k+1)(q-k+2)} \right. \\
 & \quad \quad \left. \left. + \cdots + \frac{(2)^{k-1} (k-1)!}{(q-k+1)(q-k+2) \cdots q} \right] \right\},
 \end{aligned}$$

with the second integral in (29) given by (19). If $q = k$, the above result reduces to that of (20).

4. The Special Case: $q = 1$. In the case of Eq. (14b), the condition

$$\operatorname{Re}(q) > -1$$

may be modified to

$$\operatorname{Re}(q) \geq -1.$$

For $q = -1$, the first term on the right becomes

$$(30) \quad \begin{aligned} & -2 \frac{\cos \frac{1}{2} p \pi}{p+1} {}_2F_1\left(1, \frac{1}{2}(1+p); \frac{1}{2}(3+p); -1\right) \\ & = -\cos \frac{1}{2} p \pi \int_0^1 (1+t)^{-1} t^{(p-1)/2} dt = -\cos \frac{1}{2} p \pi \beta\left(\frac{1+p}{2}\right), \end{aligned}$$

where

$$\beta(x) = \frac{1}{2} \left[\Psi\left(\frac{1+x}{2}\right) - \Psi\left(\frac{x}{2}\right) \right]$$

(see [1, formula 8.371(1)], or [3, formula 15.1.23]), while the second term reduces to $\pi/2$. Hence we obtain

$$(31) \quad \int_0^{\pi/2} \frac{\sin p\theta}{\sin \theta} d\theta = \frac{\pi}{2} - \frac{1}{2} \cos \frac{1}{2} p \pi \left[\Psi\left(\frac{3+p}{4}\right) - \Psi\left(\frac{1+p}{4}\right) \right],$$

a result given previously by the author [2, Eq. (11)]. In a similar manner, by subtracting the result of Eq. (14a) from the corresponding one when $p = 0$, and taking the limit as $q \rightarrow -1$, we get the expression for

$$(32) \quad \begin{aligned} \int_0^{\pi/2} \frac{1 - \cos p\theta}{\sin \theta} d\theta & = \Psi\left(\frac{1+p}{2}\right) - \Psi\left(\frac{1}{2}\right) \\ & \quad - \frac{1}{2} \sin \frac{1}{2} p \pi \left[\Psi\left(\frac{3+p}{4}\right) - \Psi\left(\frac{1+p}{4}\right) \right], \end{aligned}$$

as given by Eq. (11) of the above reference.

5. Watson's Integral. In [5, p. 313], the following value is given for the integral

$$(33) \quad \int_0^\pi \cos^m \theta \cos p\theta d\theta = \frac{(-)^m \sin p\pi}{2^m (p+m)} {}_2F_1\left(-m, -\frac{p+m}{2}; 1 - \frac{p+m}{2}; -1\right),$$

where the sign of the second parameter has been corrected (cf. *Math. Comp.*, v. 14, 1960, p. 221). The above result has been reproduced with the incorrect sign in [4, p. 16], as well as in [1, art. 3.631(18)], with reference to the wrong page of [5], but is given correctly in art. 9.114 of [1].

The relation (33) is easily derived from (13a) and (14b) by bisecting the range of integration, and it can also be shown that the restriction $p \neq 0, \pm n$ is not necessary, since from elementary considerations, the integral is, in this case, zero except when $m \geq n$ and $m - n$ is even, in which instance it is equal to

$$(34) \quad \frac{\pi}{2^m} \binom{m}{\frac{m-n}{2}}.$$

On the other hand, after replacing p by $-p$ in the right side of (33), the latter expression becomes

$$(35) \quad \frac{(-)^m \sin p\pi}{2^m (p-m)} {}_2F_1\left(-m, \frac{p-m}{2}; 1 + \frac{p-m}{2}; -1\right),$$

while the hypergeometric series may be written in the form

$$(36) \quad \left(\frac{1}{p-m}\right)_2 {}_2F_1\left(-m, \frac{p-m}{2}; 1 + \frac{p-m}{2}; -1\right) = \sum_{k=0}^m \frac{\binom{m}{k}}{p-m+2k},$$

which will be finite unless $(p-m)$ is a negative even integer, or zero. It follows that, since

$$\lim_{p \rightarrow m-2k} \left(\frac{\sin p\pi}{p-m+2k}\right) = \pi(-1)^m,$$

the expression (35) will attain the value given by (34) when p is an integer: $n \leq m$, and $m-n$ is even, but will vanish otherwise.

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