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Pisot Numbers in the Neighborhood of a Limit Point. II

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Abstract. Let S denote the set of real algebraic integers greater than one, all of whose other conjugates lie within the unit circle. In an earlier paper, we introduced the notion of "width" of a limit point α of S and showed that, if the width of α is smaller than $1.28\ldots$ then there is an algorithm for determining all members of S in a neighborhood of α. Recently, we introduced the "derived tree" in order to deal with limit points of greater width. Here, we apply these ideas to the study of the limit point α_3, the zero of $z^4 - 2z^3 + z - 1$ outside the unit circle. We determine the smallest neighborhood $\theta_1 < \alpha_3 < \theta_2$ of α_3 in which all elements of S other than α_3 satisfy one of the equations $z^n(z^4 - 2z^3 + z - 1) \pm A(z) = 0$, where $A(z)$ is one of $z^3 - z^2 + 1$, $z^3 - z + 1$ or $z^4 - z^3 + z - 1$. The endpoints θ_1 and θ_2 are elements of S of degrees 23 and 42, respectively.

1. Introduction. As usual, let S denote the set of Pisot (Pisot-Vijayaraghavan) numbers. In an earlier paper [1], we gave an algorithm for determining all elements of S in a given interval of the real line provided there are only a finite number of elements of S in this interval. We also showed how to determine all elements of S in the neighborhood of certain limit points.

We demonstrated the algorithm by finding all points of S in [1, 1.86675] and [1.868, 1.932], intervals containing, respectively, three and two limit points of S. The reason for the gap (1.86675, 1.868) is the presence of the limit point $\alpha_3 = 1.8667603992$, the zero of $x^4 - 2x^3 + x - 1$ in $|x| > 1$. In the terminology of [1], this limit point has width $1.7548\ldots$ and hence cannot be dealt with by the methods of [1].

In the first paper of this series [2], we showed how to extend the algorithm of [1] to deal with limit points such as α_3. The basic new idea is that of the derived tree. Briefly, each Pisot number θ is associated with a certain set of rational functions $f = A/Q = u_0 + u_1 z + \cdots$ with integer coefficients. The set $\mathscr{C}$ consists of all such f as θ varies over S. The sequences of coefficients $\{u_k\}$ are paths to infinity in a tree $\mathscr{T}$ defined by the inequalities of Schur's algorithm. An N-neighborhood of f in $\mathscr{C}$ consists of all g in $\mathscr{C}$ whose first $N + 1$ coefficients $u_0, \ldots, u_N$ agree with those of f. If f is a limit point of $\mathscr{C}$ in the topology defined by these neighborhoods then, at each level n of the corresponding path in $\mathscr{T}$, a subtree $\mathscr{T}_n(f)$ branches off. The derived tree $\mathscr{T}'(f)$ describes the asymptotic behavior of $\mathscr{T}_n(f)$ as $n \to \infty$.

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If $\mathscr{T}'(f)$ is essentially finite (see [2] for this and other undefined terminology), there is an effective constant N such that all g in the N-neighborhood of f can be completely determined. Under quite general conditions [2, Theorem 8.5], this N-neighborhood consists of $f(z) = A(z)/Q(z)$ and the functions

$$\frac{A(z) \pm z^{n+r}Q(z^{-1})}{Q(z) \pm z^{n+r}A(z^{-1})}$$

for $n \geqslant N$, where $r \leqslant \max(\deg A, \deg Q)$ is a certain integer depending on f.

Our purpose here is to demonstrate the practicality of the method of [2] by filling in the gap (1.86675, 1.868) around α_3. There are three limit points of $\mathscr{C}$ associated with α_3, namely

$$g_{1,3} = (1 - z + z^3)/(1 - 2z + z^3 - z^4),$$
$$g_{2,3} = (1 - z^2 + z^3)/(1 - 2z + z^3 - z^4),$$
$$g_{3,3} = (1 - z + z^3 - z^4)/(1 - 2z + z^3 - z^4).$$

The first two of these have width 1 and were already treated in [1]. Thus we can confine ourselves to $g_{3,3}$ which we discussed briefly in [2].

We will show that the only f in the 22-neighborhood of $g_{3,3}$ are

$$f_n = \frac{(1 - z + z^3 - z^4) \pm z^{n-3}(1 - z + 2z^3 - z^4)}{(1 - 2z + z^3 - z^4) \pm z^{n-3}(1 - z + z^3 - z^4)}, \tag{1.1}$$

for $n \geqslant 23$.

Incorporating the results from [1], we find an interval (θ_1, θ_2) of α_3 in which the only elements of S are the roots of $z^n(z^4 - 2z^3 + z - 1) \pm A(z)$, where $A(z)$ is one of $z^3 - z^2 + 1$, $z^3 - z + 1$ or $z^4 - z^3 + z - 1$. The endpoint $\theta_1 = 1.8667463463$ is an element of S of degree 23 associated with $g_{2,3}$, while $\theta_2 = 1.8667627119$ is an element of S of degree 42 associated with $g_{3,3}$. The minimal polynomials for θ_1 and θ_2 are (writing $a_0z^d + a_1z^{d-1} + \cdots + a_d \equiv a_0a_1 \cdots a_d$),

(1.2) $P_1 = 1\ {-1}\ {-1}\ {-2}\ 1\ 1\ 1\ {-1}\ {-1}\ 0\ 0\ 0\ {-1}\ 0\ 0\ 1\ {-1}\ {-1}\ {-1}\ 1\ 1\ 1\ 0\ {-1}\ {-1},$

(1.3) $P_2 = 1\ {-2}\ 1\ {-1}\ {-2}\ 3\ {-3}\ 3\ 1\ {-2}\ 4\ {-4}\ 1\ 0\ {-3}\ 3\ {-2}\ 1\ 1\ {-1}\ 1$
$\quad {-1}\ 1\ {-1}\ 1\ 1\ 0\ {-2}\ 2\ {-3}\ 1\ 1\ {-2}\ 4\ {-2}\ 1\ 1\ {-3}\ 2\ {-2}\ 0\ 1\ {-1}\ 1.$

Here $-P_1(z)/z^{23}P_1(z^{-1})$ is in $\mathscr{T}_{17}(g_{2,3})$ while $P_2(z)/z^{42}P_2(z^{-1})$ is in $\mathscr{T}_{22}(g_{3,3})$.

2. The Derived Tree for $g_{3,3}$. If f is a limit point of $\mathscr{C}$, the derived tree $\mathscr{T}'(f)$ consists of all sequences $(c_0, \ldots, c_k)$ of integers satisfying the inequalities $c_0 \neq 0$ and

$$W_k(c_0, \ldots, c_{k-1}) \leqslant c_k \leqslant W_k^*(c_0, \ldots, c_{k-1}), \tag{2.1}$$

where W_k and W_k^* are defined by the recurrence relations (4.16) to (4.20) of [2].

Because of the symmetry $W_k(-c_0, \ldots, -c_{k-1}) = -W_k^*(c_0, \ldots, c_{k-1})$, we may assume $c_0 > 0$. Table 1 gives the values of W_k and W_k^* for $\mathscr{T}' = \mathscr{T}'(g_{3,3})$ truncated to 6 decimal places. The integers in the columns headed M_k and M_k^* will be defined in Section 5. The tree $\mathscr{T}'$ is infinite but we have truncated it at nodes where $W_k = W_k^*$ to obtain a finite tree.

TABLE 1

k	c_{k-1}	W_k	W_k^*	M_k	M_k^*
1	1	1.675282	4.045357	10	11
2	2	3.898594	5.019511	18	10
3	4	8.051692	8.420621	50	26
3	5	11.991607	12.068292	15	16
4	12	27.970103	28	21	19
5	28	62	62	24	17
2	3	6.497327	8.834473	21	31
3	7	15.421767	17	34	1
4	16	34.989364	36.454884	37	35
5	35	76.001155	76.043389	98	43
5	36	77.788620	79.043389	41	43
6	78	165.900195	166.603279	51	55
7	166	347.051834	347.394384	82	60
6	79	168.824854	168.992408	50	90
4	17	39	39	1	1
3	8	17.897284	20.043389	27	27
4	18	40.426227	40.817428	45	42
4	19	42.458813	44.603279	42	39
5	43	94.073654	95.692095	66	44
6	95	204.179028	205.763565	66	61
7	205	433.131847	434.714303	74	70
8	434	904.807006	906.374505	65	74
9	905	1866.786707	1867.463636	77	77
10	1867	3816.608989	3817.193336	88	84
11	3817	7744.362025	7744.879502	95	109
9	906	1869.373630	1870.513744	82	87
10	1870	3823.956019	3825.085007	88	91
11	3824	7761.021063	7761.190135	127	93
11	3825	7763.546869	7763.861294	100	116
5	44	96.660124	98.394384	46	44
6	97	208.959271	210.052343	55	44
7	209	442.830908	442.987753	68	91
7	210	445.947206	446.146552	47	58
8	446	932.819400	932.974648	61	85
6	98	211.542501	212.761293	54	60
7	212	449.480521	450.623590	66	59
8	450	941.125009	942.258594	84	63
9	942	1947.904856	1948.703271	72	82
10	1948	3990.600850	3990.936074	87	98
4	20	45.822361	45.992408	34	74
2	4	9.822045	10	18	23
3	10	24	24	29	36

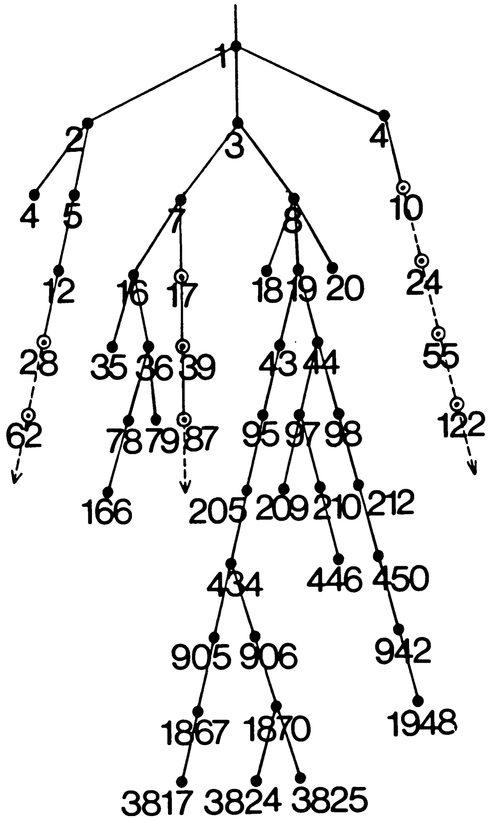

FIGURE 1

The derived tree for $g_{3,3}$

The tree is traversed in left preorder so that listing c_{k-1} suffices to identify a given node $(c_0,\ldots,c_{k-1})$. A comparison with Figure 1 reproduced from [2], may be helpful. For example, the row of the table giving

$$(2.2) \qquad c_4 = 35, \quad W_5 = 76.001155, \quad W_5^* = 76.043389$$

refers to the node $(1,3,7,16,35)$ of $\mathscr{T}'$. Since there is no integer c_5 satisfying (2.1) in this case, $(1,3,7,16,35)$ is a terminal node of $\mathscr{T}'$.

The values of W_k^* listed without fractional part are exactly integers. The branches of $\mathscr{T}'$ corresponding to $c_k = W_k^*$ are simple paths to infinity in $\mathscr{T}'(f)$ discussed more fully in [2, Section 9].

The trees $\mathscr{T}_n$ consist of all $(c_0,\ldots,c_k)$ for which $(u_0,\ldots,u_{n-1}, u_n + c_0,\ldots,u_{n+k} + c_k)$ are in $\mathscr{T}$. (Here $g_{3,3} = u_0 + u_1 z + \cdots$.) These trees are characterized by inequalities similar to (2.1):

$$(2.3) \qquad W_{n,k}(c_0,\ldots,c_{k-1}) \leqslant c_k \leqslant W_{n,k}^*(c_0,\ldots,c_{k-1}).$$

We will use $\mathscr{T}_n^+$ ($\mathscr{T}_n^-$) to refer to the sequences satisfying (2.3) and $c_0 > 0$ ($c_0 < 0$) respectively.

By Theorem 7.3 of [2], there are effective constants N_k such that, for $n \geqslant N_k$, each path $(c_0,\ldots,c_k)$ in $\mathscr{T}_n(f)$ is also a path in $\mathscr{T}'(f)$. This depends on the convergence $W_{n,k} \to W_k$ and $W_{n,k}^* \to W_k^*$ as $n \to \infty$.

As Figure 1 shows, the effective height of $\mathscr{T}'(g_{3,3})$ is 10 and hence we need only estimate N_{11}. For this, it suffices to find an N such that, for $n \geqslant N$, and $(c_0, \ldots, c_{k-1})$ a node in $\mathscr{T}'$ with $k - 1 \leqslant 10$, we have

$$[W_{n,k}, W_{n,k}^*] \cap \mathbf{Z} \subseteq [W_k, W_k^*] \cap \mathbf{Z}. \tag{2.4}$$

Inspection of Table 1 suggests that the most stringent requirement here is that $76 < W_{n,5} < 77$ for the node $(1, 3, 7, 16, 35)$ of (2.2). This is indeed the case and we will show that $N_5 = N_6 = \cdots N_{11} = 63$. Thus $\mathscr{T}_n^{\pm}(f)$, $n \geqslant 63$ has only one path to infinity, the "regular" path corresponding to f_n of (1.1). In fact this is true for $n \geqslant 23$ even though $\mathscr{T}_n$ in this case usually contains extraneous nodes which are not in the tree $\mathscr{T}'$. The tree $\mathscr{T}_{22}^+$ has two paths to infinity, the regular path and one corresponding to (1.3). The other $\mathscr{T}_n$ for small n are discussed more fully in Section 6.

3. Formulae for Estimating $W_{n,k} - W_k$**.** To estimate N_k, we will need estimates for $W_{n,k} - W_k$ and $W_{n,k}^* - W_k^*$. We develop estimates in detail only for $W_{n,k} - W_k$ since the treatment of $W_{n,k}^* - W_k^*$ is almost identical.

According to [2, (7.5) and (7.6)] we have

$$\frac{\Omega}{Q^2} \frac{P_{n,k}}{Q_{n,k}} = c_0 + \cdots + c_{k-1} z^{k-1} + W_{n,k} z^k + \cdots \qquad (n > k + 1) \tag{3.1}$$

and

$$\frac{\Omega}{Q^2} \frac{Gd_k}{\tilde{G}e_k} = c_0 + \cdots + c_{k-1} z^{k-1} + W_k z^k + \cdots. \tag{3.2}$$

Here $\Omega = (1 - z + z^3)(1 - z^2 + z^3)$, $Q = 1 - 2z + z^3 - z^4$, $Q_{n,k}$ is of degree $k + 3$ with $Q_{n,k}(0) = 1$ and $P_{n,k}(z) = -z^{k+3} Q_{n,k}(z^{-1})$. Also $\Omega(z) = G(z)\tilde{G}(z)/G(0)$, where G is monic, of degree 3, and has all its zeros in $|z| > 1$, while $\tilde{G}(z) = z^3 G(z^{-1})$. The polynomials d_k, e_k satisfy $e_k(0) = 1$, $\deg e_k = k$ and $d_k = -z^k e_k(z^{-1})$.

If $c_{m-1} \neq W_{m-1}$ or W_{m-1}^* for $m \leqslant k$ then these conditions and (3.2) characterize d_k and e_k. If $c_{m-1} = W_{m-1}$ or W_{m-1}^* for some $m \leqslant k$ then we define $e_k(z) = (1 + z)^{k-m} e_m(z)$. In this case $c_0, \ldots, c_{m-1}$ uniquely determine $c_m, c_{m+1}, \ldots$. Similar remarks apply to $P_{n,k}$ and $Q_{n,k}$ if $n > k + 1$.

LEMMA 3.1. *With the above notation, if* $n > k + 1$, *then*

$$P_{n,k}\tilde{G}e_k - Q_{n,k}Gd_k = z^k H_{n,k}, \tag{3.3}$$

where $H_{n,k}$ *is a polynomial of degree at most* $2r$ *which satisfies* $z^{2r} H_{n,k}(z^{-1}) = -H_{n,k}(z)$, *and*

$$W_{n,k} - W_k = H_{n,k}(0). \tag{3.4}$$

Proof. The left member of (3.3) is a polynomial R of degree at most $2k + 2r$ which satisfies

$$z^{2k+2r} R(z^{-1}) = -R(z). \tag{3.5}$$

By (3.1) and (3.2), it follows that

$$\Omega R / \left(Q^2 Q_{n,k} \tilde{G} e_k\right) = (W_{n,k} - W_k) z^k + \cdots, \tag{3.6}$$

so R has a zero of order k at $z = 0$. Thus $R(z) = z^k H_{n,k}(z)$ where $\deg H_{n,k} \leqslant k + 2r$. But by (3.5) the leading k coefficients of r also vanish so, in fact, $\deg H_{n,k} \leqslant 2r$. The remaining properties of $H_{n,k}$ follow from (3.5) and (3.6).

For our example, $r = 3$ so that

$$H_{n,k}(z) = a + bz + cz^2 - cz^4 - bz^5 - az^6, \tag{3.7}$$

where $a = W_{n,k} - W_k$. Let α_i, $(i = 1, 2, 3)$, be the zeros of G so α_1 is the real root of $1 - z + z^3 = 0$ and α_2, α_3 the complex roots of $1 - z^2 + z^3$. Numerically,

$$\alpha_1 = -1.3247179572, \qquad \alpha_2 = .8774388331 + i(.7448617666).$$

From (3.3), we have

$$H_{n,k}(\alpha_i) = \alpha_i^{-k} P_{n,k}(\alpha_i) \tilde{G}(\alpha_i) e_k(\alpha_i) \qquad (i = 1, 2, 3). \tag{3.8}$$

From (3.4) we have, for $n > k + 1$,

$$W_{n,k} - W_k = \sum_{i=1}^{3} h_i H_{n,k}(\alpha_i), \tag{3.9}$$

where h_1, h_2 and h_3 are obtained by solving the following 3×3 linear system for a:

$$a(1 - \alpha_i^6) + b(\alpha_i - \alpha_i^5) + c(\alpha_i^2 - \alpha_i^4) = H_{n,k}(\alpha_i). \tag{3.10}$$

Numerically,

$$h_1 = .057477943, \qquad |h_2| = |h_3| = .374123681. \tag{3.11}$$

The quantities $\alpha_i^k \tilde{G}(\alpha_i) e_k(\alpha_i)$ can be computed for each node of the derived tree. The quantities $P_{n,k}(\alpha_i)$ tend to zero geometrically. So Formulae (3.8) and (3.9) provide a means of estimating $W_{n,k} - W_k$. It is useful, though, to further manipulate (3.8).

LEMMA 3.2. *With the above notation,*

$$P_{n,k} Q_{n,k+1} - Q_{n,k} P_{n,k+1} = z^k (1 - z)(W_{n,k} - c_k)\Omega. \tag{3.12}$$

Proof. Combine the definitions [2, (5.6) and (5.10)] of $P_{n,k}$ and $Q_{n,k}$ with the following formula from [3, p. 82],

$$D_{n+1} E_n - D_n E_{n+1} = (u_n - w_n) z^n (1 - z).$$

COROLLARY 3.3. *Let $f = g_{3,3}$. Then, for all n, k and $i = 1, 2, 3$,*

$$P_{n,k}(\alpha_i) = \alpha_i^{-1-n} Q_{n,k}(\alpha_i). \tag{3.13}$$

Proof. By (3.12), the ratio $P_{n,k}(\alpha_i)/Q_{n,k}(\alpha_i)$ is independent of k. When $k = 0$, $P_{n,k} = P_n$ and [2, (5.9)] gives

$$z^{n+r} P_n(z) A(z^{-1}) - z^r P_n(z^{-1}) Q(z) = E_n(z) \Omega(z). \tag{3.14}$$

Thus

$$P_n(\alpha_i)/P_n(\alpha_i^{-1}) = \alpha_i^{-n} Q(\alpha_i)/A(\alpha_i^{-1}). \tag{3.15}$$

For $f = g_{3,3}$ we readily verify that

$$Q(z) + z^2 A(z^{-1}) = -z^{-2}\Omega(z), \tag{3.16}$$

and hence $Q(\alpha_i)/A(\alpha_i^{-1}) = -\alpha_i^2$. Since $Q_{n,0}(z) = -z^3 P_n(z^{-1})$, we can now derive (3.13) for $k = 0$ and hence for all k.

The formula (3.8) can thus be written in the more useful form:

$$H_{n,k}(\alpha_i) = \alpha_i^{-n-k-1} \tilde{G}(\alpha_i) e_k(\alpha_i) Q_{n,k}(\alpha_i). \tag{3.17}$$

It only remains to estimate $Q_{n,k}(\alpha_i)$, to which we now turn.

4. Estimates for the Auxiliary Polynomials. Throughout this section, $P_{n,k}$ etc. refer to the auxiliary polynomials for $f = g_{3,3}$. The techniques apply to any f for which Ω has no multiple zeros. We regard polynomials of degree d with real coefficients as vectors in $\mathbf{R}^{d+1}$ and measure distance between two such polynomials by the l^∞-norm, i.e., $\|a_0x^d + \cdots + a_d\| = \max\{|a_i|: 1 \leqslant i \leqslant d\}$. Recall that $P_{n,0} = P_n$ and $Q_{n,0} = Q_n$.

LEMMA 4.1. *For $n \geqslant 50$,*

$$\|P_n - G\| = \|Q_n - \tilde{G}\| \leqslant C\delta^n \leqslant .007,$$

where $C \leqslant 7.871$ and $\delta = |\alpha_2|^{-1} \leqslant .868837$.

Proof. Temporarily let

$$G(z) = (z - \alpha_1)(z - \alpha_2)(z - \alpha_3) = a + bz + cz^2 + z^3 \tag{4.1}$$

and

$$P_n(z) = a_n + b_n z + c_n z^2 + z^3. \tag{4.2}$$

From (3.13) we have

$$P_n(\alpha_i) = -\alpha_i^{2-n} P_n(\alpha_i^{-1}). \tag{4.3}$$

Thus

$$a + b\alpha_i + c\alpha_i^2 = -\alpha_i^3 \tag{4.4}$$

and

$$a_n(1 + \alpha_i^{2-n}) + b_n(\alpha_i + \alpha_i^{1-n}) + c_n(\alpha_i^2 + \alpha_i^{-n}) = -\alpha_i^3 - \alpha_i^{-1-n}. \tag{4.5}$$

Define the 3×3 matrices V and U_n to have rows $(1, \alpha_i, \alpha_i^2)$ and $(\alpha_i^{2-n}, \alpha_i^{1-n}, \alpha_i^{-n})$ respectively, and define the column vectors p_n, g and h_n by $p_n = (a_n, b_n, c_n)^{\mathrm{tr}}$, $g = (a, b, c)^{\mathrm{tr}}$ and $h_n = (-\alpha_1^{-1-n}, -\alpha_2^{-1-n}, -\alpha_3^{-1-n})^{\mathrm{tr}}$. Then (4.4) and (4.5) combine into

$$(V - U_n)p_n = Vg + h_n, \tag{4.6}$$

or

$$p_n - g = (I - V^{-1}U_n)^{-1} V^{-1}(U_n g + h_n). \tag{4.7}$$

Using the l^∞-norm, we calculate

$$\begin{aligned} \|V^{-1}\| &= 1.125642247, \\ \|U_n\| &= (3.475681885)\delta^n, \qquad n \geqslant 2, \\ \|h_n\| &= \delta^{1+n}, \\ \|g\| &= \alpha_1^2 = 1.754877666, \end{aligned}$$

so that $(1 - \|V^{-1}U_n\|)^{-1} \leqslant 1.00347425$ for $n \geqslant 50$. Hence (4.7) gives

$$\|p_n - g\| \leqslant (1 - \|V^{-1}U_n\|)^{-1} \|V^{-1}\|(\|U_n\|\,\|g\| + \|h_n\|), \tag{4.8}$$

which proves the lemma.

LEMMA 4.2. *For $n \geqslant 51$,*

$$|Q_{n,k}(\alpha_i)| \leqslant C_i \rho_i^k, \tag{4.9}$$

where

$$C_1 = 4.302463189, \qquad C_2 = C_3 = 1.204779188,$$

and where ρ_i *is the larger root of* $x^2 = |1 + \alpha_i|x + 4|\alpha_i|$, *so that*

$$\rho_1 = 2.470005456, \qquad \rho_2 = \rho_3 = 3.381346122.$$

Proof. From [2, (5.19)], we deduce that

$$Q_{n,k+1} = (1+z)Q_{n,k} - \gamma_{n,k} z Q_{n,k-1}, \tag{4.10}$$

where $\gamma_{n,k} = (c_k - W_{n,k})/(c_{k-1} - W_{n,k-1})$. From [2, (5.21)], we have

$$W^*_{n,k} - W_{n,k} = 4(W^*_{n,k-1} - c_{k-1})(c_{k-1} - W_{n,k-1})/(W^*_{n,k-1} - W_{n,k-1}). \tag{4.11}$$

Combining (2.3) and (4.11) yields $0 \leqslant \gamma_{n,k} \leqslant 4$, provided $c_{k-1} \neq W_{n,k-1}$. (If $c_{k-1} = W_{n,k-1}$ then $Q_{n,m}$ for $m \geqslant k+1$ is not uniquely defined, but if we simply let $Q_{n,m} = (1+z)^{m-k} Q_{n,k}$ in this case, then (4.10) holds with $0 \leqslant \gamma_{n,k} \leqslant 4$.)

The initial conditions in (4.10) are

$$Q_{n,0} = Q_n \quad \text{and} \quad Q_{n,-1} = Q_{n-1}. \tag{4.12}$$

If we demonstrate (4.9) for $k = 0, 1$ it will follow by induction for $k \geqslant 2$ from (4.10) and $|\gamma_{n,k}| \leqslant 4$. By Lemma 4.1, $\|Q_n - \tilde{G}\| \leqslant .007$ for $n \geqslant 50$, so

$$|Q_n(\alpha_i)| \leqslant |\tilde{G}(\alpha_i)| + (.007)\big(|\alpha_i| + |\alpha_i|^2 + |\alpha_i|^3\big).$$

Since

$$\tilde{G}(\alpha_1) = -4.264632994 \quad \text{and} \quad |\tilde{G}(\alpha_2)| = 1.176776497,$$

we do have (4.9) for $k = 0$ and $i = 1, 2, 3$.

To estimate $Q_{n,1}(\alpha_i)$, we use (4.10) and (4.12) where

$$\gamma_{n,0} = (c_0 + u_n - w_n)/(u_{n-1} - w_{n-1})$$

and $c_0 = \pm 1$. From Lemma 4.1

$$|(u_n - w_n) - w| \leqslant \|P_n - G\| \leqslant .007 \quad \text{if } n \geqslant 50.$$

Hence $|\gamma_{n,0} - \gamma_0| \leqslant .010291843$, for $n \geqslant 51$, where $\gamma_0 = (c_0 + w)/w = 1.569840291$ (if $c_0 = 1$) or $\gamma_0 = .430159709$ (if $c_0 = -1$).

Now write

$$Q_{n,1} = (1+z)\tilde{G} - \gamma_0 z\tilde{G} + (1+z)(Q_n - \tilde{G}) - \gamma_{n,0} z(Q_{n-1} - \tilde{G}) \\ - (\gamma_{n,0} - \gamma_0) z\tilde{G},$$

and use the estimates for $|\gamma_{n,0} - \gamma_0|$ and $\|Q_n - \tilde{G}\|$ given above to verify (4.9) for $k = 1$, $n \geqslant 51$. This completes the proof of the lemma.

Remark. The estimate (4.9) is rather unrealistic for large k since we know [2, Lemma 7.1] that $Q_{n,k} \to \tilde{G}e_k$ as $n \to \infty$ at interior nodes of the derived tree (nodes where $W_{k-1} < c_{k-1} < W^*_{k-1}$). The estimate can be improved by recursively estimating the differences $\gamma_{n,k} - \gamma_k$ and using the known values of γ_k, rather than resorting to the estimate $0 \leqslant \gamma_{n,k} \leqslant 4$. This turns out to be unnecessary for our purposes.

5. Estimates of N_k. Combining (3.9), (3.11), (3.17) and (4.9), at each node of the derived tree we have

$$|W_{n,k} - W_k| \leqslant \sum_{i=1}^{3} A_{k,i} |e_k(\alpha_i)|\, |\alpha_i|^{-n}, \tag{5.1}$$

where $A_{k,i} = |h_i| C_i |\alpha_i|^{-1-k} \rho_i^k$, so that

$$A_{k,1} = (.796116505)(1.864551955)^k, \tag{5.2}$$

$$A_{k,2} = A_{k,3} = (.460845058)(2.937838492)^k. \tag{5.3}$$

Tracing through the analysis, we find that Lemmas 4.1 and 4.2 are valid for $P^*_{n,k}$ and hence that

$$|W^*_{n,k} - W^*_k| \leqslant \sum_{i=1}^{3} A_{k,i}|e^*_k(\alpha_i)|\,|\alpha_i|^{-n}. \tag{5.4}$$

Let U_k be the largest integer strictly smaller than W_k and V^*_k be the smallest integer strictly larger than W^*_k, so

$$W_k = U_k + \delta_k, \qquad 0 < \delta_k \leqslant 1, \tag{5.5}$$

$$W^*_k = V^*_k - \delta^*_k, \qquad 0 < \delta^*_k \leqslant 1. \tag{5.6}$$

For each node in the derived tree, let $m_k = m_k(c_0,\ldots,c_{k-1})$ and m^*_k be such that $m \geqslant m_k$ implies $|W_{n,k} - W_k| < \delta_k$ and $m > m^*_k$ implies $|W^*_{n,k} - W^*_k| < \delta^*_k$. If M_k (M^*_k) are such that the right members of (5.1) and (5.4) are less than δ_k (δ^*_k) for $m \geqslant M_k$ (M^*_k), then $m_k \leqslant \max(M_k, 51)$ and $m^*_k \leqslant \max(M^*_k, 51)$. The quantities M_k and M^*_k are given in Table 1.

Suppose $(c_0,\ldots,c_{k-1})$ is a path in $\mathscr{T}'$ and in $\mathscr{T}_n$. Then, for $n \geqslant \max(m_k, m^*_k)$, (2.4) holds. As shown in [2, Theorem 8.4(b)], no path in $\mathscr{T}_n$ can follow any of the paths to infinity in $\mathscr{T}'$ to a height > 4 unless it coincides with one of $\pm(1,3,7,17,39,\ldots)$. Thus if $\mathscr{T}'$ is truncated at height 4 on these paths and $N = \max(m_k, m^*_k)$ over all nodes in the truncated $\mathscr{T}'$ then $n \geqslant N$ implies that $\mathscr{T}_n$ is a subtree of $\mathscr{T}'$.

The bounds M_k, M^*_k given in Table 1 together with values for the nodes where $c_k = W^*_k$ and $k \leqslant 4$ give $N \leqslant 127$. By computing $\mathscr{T}_n$ for $n \leqslant 126$ as described in the next section, we obtain exact values for m_k and m^*_k. For example,

$$m_5(1,3,7,16,35) = 63 \quad\text{and}\quad m_{11}(1,3,\ldots,1870,3824) = 47.$$

If we use $Q_{n,k}(\alpha_i) \approx \tilde{G}(\alpha_i)e_k(\alpha_i)$ rather than (4.9) we obtain $m_5(1,\ldots,35) \approx 66$ and $m_{11}(1,\ldots,3824) \approx 48$, so we see that it is (4.9) that leads to the more pessimistic values 98 and 127 given in Table 1 for these two quantities.

For the sake of interest,

$$W^*_{62,5}(-1,-3,-7,-16,-35) = 75.999171,$$

and

$$W_{46,11}(1,3,\ldots,1870,3824) = 7760.998730,$$

correct to 6 decimal places.

6. The Trees $\mathscr{T}_n(g_{3,3})$. By the calculations of the previous section, if $n \geqslant 127$, each tree $\mathscr{T}_n^{\pm}(g_{3,3})$ contains only the single path to infinity $\pm(1,3,7,17,\ldots)$ corresponding to f_n of (1.1). It thus remains to examine the trees $\mathscr{T}_n^{\pm}$ for $n < 127$ to determine the finite number of exceptional elements of $\mathscr{C}$ in the neighborhood of $g_{3,3}$.

In the following discussion of the sizes of $\mathscr{T}_n^{\pm}$, we have truncated these trees at the nodes where $c_k = W_{n,k}$ or $W^*_{n,k}$. Since the expansion of $g_{3,3}$ begins

$$g_{3,3} = 1,1,2,4,7,13,24,45,\ldots$$

it is clear that $|\mathscr{T}_4^+| = \infty$ since it contains a path corresponding to

$$(1-z)/(1-2z) = 1,1,2,4,8,\ldots$$

which is in the second derived set of $\mathscr{C}$.

Similarly, $|\mathscr{T}_5^+| = \infty$ since the limit point $\beta_3 = 1.927\ldots$ has an expansion

$$(1 - z^4)/(1 - z - z^2 - z^3 - z^4) = 1, 1, 2, 4, 7, 14, \ldots.$$

All other $\mathscr{T}_n^{\pm}$ with $n \geqslant 4$ are finite. Although both of $\mathscr{T}_6^{\pm}$ are small, $\mathscr{T}_7^-$ is quite large, containing 1671 nodes and 102 paths corresponding to elements of $\mathscr{C}$. These expansions begin

$$1, 1, 2, 4, 7, 13, 24, 44, \ldots$$

and the corresponding Pisot numbers lie in a rather small neighborhood of $\beta_2 = 1.839\ldots$ rather than of $\alpha_3 = 1.866\ldots$ and are among those discussed at the end of Section 3 of [1].

For $n \geqslant 23$, there are no paths to infinity other than those corresponding to (1.1). For $n = 22$, we obtain P_2 of (1.2) corresponding to a path $(1, 3, 8, \ldots, 3682615)$ in $\mathscr{T}_{22}^+$.

In spite of the restriction $n > k + 1$, in (3.1), the trees $\mathscr{T}_n^{\pm}$ can be computed correctly for all n by using (3.1). That is, $P_{n,k}$ and $Q_{n,k}$ are computed from (4.10) and $W_{n,k}$ from (3.1), and similarly with $P_{n,k}^*$, $Q_{n,k}^*$. If $k + 1 \geqslant n$ then the values for c_k, $W_{n,k}$ and $W_{n,k}^*$ are incorrect, but the quantities $c_k - W_{n,k}$ and $W_{n,k}^* - c_k$ are correct, so the recurrence relation (4.10) gives the correct $P_{n,k}$ and $Q_{n,k}$.

The starting values P_n and P_n^* are computed using the recurrence relations [2, (5.18)]. This was done in integer arithmetic and, for comparison, using double precision floating-point arithmetic. The floating-point calculation proves to be accurate to over 13 decimal places indicating the exceptional stability of the recurrence relation [2, (5.18)].

For $n \leqslant 65$, the trees $\mathscr{T}_n^{\pm}$ were computed using integer arithmetic as in [1]. For $n > 65$, the computations were done in the double precision floating-point arithmetic. Since the trees in this case are of height at most 11, it is easy to see that the rounding error gives values of $W_{n,k}$ and $W_{n,k}^*$ correct to at least 5 decimal places even under the pessimistic (and false) assumption that $\gamma_{n,k} = 4$ for all n, k.

The integer calculations for $n \leqslant 65$ make the detection of $P_{n,k}$ and $P_{n,k}^*$ with integer coefficients easy. These correspond to D_{n+k} and D_{n+k}^* whose roots are Pisot numbers according to the formulae [2, (5.8) and (5.13)]. A comparison with the results of [1] revealed that the only Pisot number in the range [1, 1.932] actually missed in [1] was θ_2, the root of P_2 of (1.3).

The computations of this section were performed on an AMDAHL 470/V8. The numbers in Table 1 were computed on an Apple II+ and verified by a computation on the AMDAHL 470/V8. In fact, all of the computations described here are well within the capabilities of a microcomputer except for the computation of some of the larger $\mathscr{T}_n$ such as $\mathscr{T}_7^-$.

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