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Supplement to Semidiscrete and Single Step Fully Discrete Approximations for Second Order Hyperbolic Equations With Time-Dependent Coefficients

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8. Proofs of Theorem 5.1 and Theorem 5.2.

The following four lemmas will be used to prove Theorem 5.1. Throughout these proofs the general positive constant C will be independent of α.

Lemma 8.1. For any nonnegative integer m and any $\phi \in L^2$

$$(8.1) \qquad \|T^{1/2} L^{(m)} T^{1/2} \phi\| \le C\|\phi\|$$

and

$$(8.2) \qquad \|L^{1/2} T^{(m)} L^{1/2} \phi\| \le C\|\phi\| .$$

Proof. Assume ϕ is smooth so that $T^{1/2} L^{(m)} T^{1/2} \phi$ is in L^2. For any $\psi \in L^2$

$$(8.3) \qquad (T^{1/2} L^{(m)} T^{1/2} \phi,\psi) = (L^{(m)} T^{1/2} \phi , T^{1/2} \psi) = a^{(m)}(T^{1/2} \phi, T^{1/2} \psi) .$$

Therefore,

$$(8.4) \qquad (T^{1/2}L^{(m)}T^{1/2}\phi,\psi) \le C||T^{1/2}\phi||_1||T^{1/2}\psi||_1 .$$

Since $||T^{1/2}\phi||_1 \le C||\phi||$ and $||T^{1/2}\psi||_1 \le C||\psi||$, it follows that

$$(8.5) \qquad (T^{1/2}L^{(m)}T^{1/2}\phi,\psi) \le C||\phi|| \, ||\psi|| .$$

(8.5) proves that

$$||T^{1/2}L^{(m)}T^{1/2}\phi|| = \sup_{\psi \in L_2} \frac{(T^{1/2}L^{(m)}T^{1/2}\phi,\psi)}{||\psi||} \le C||\phi|| .$$

Since smooth functions are dense in L^2, (8.1) holds for all $\phi \in L^2$. (8.2) is proved using (8.1) and induction on m since

$$T^{(m)} = -\sum_{j=0}^{m-1} \binom{m}{j} T L^{(m-j)} T^{(j)}$$

and

$$L^{1/2}T^{(m)}L^{1/2} = -\sum_{j=0}^{m-1} \binom{m}{j} T^{1/2}L^{(m-j)}T^{1/2}L^{1/2}T^{(j)}L^{1/2} .$$

The next three lemmas contain bounds for terms in (5.5). These bounds will be given in the following special norm on $L^2 \times L^2$.

$$|||\Phi||| \equiv (||\phi_1||^2 + (T\phi_2,\phi_2))^{1/2}$$

for $\Phi = \binom{\phi_1}{\phi_2} \in L^2 \times L^2$.

<u>Lemma 8.2.</u> For any positive integer m

(8.6) $$|||\hat{\mathcal{L}}\hat{\mathcal{T}}^{(m)}\Phi||| \le (C_1 + C_2\alpha)|||\Phi||| ,$$

where C_1 and C_2 are constants which are independent of α.

<u>Proof</u>. Define $\hat{L} = L + \alpha^2 I$ and $\hat{T}^{(m)} = (\hat{L}^{-1})^{(m)}$. The following two estimates are from Sammon [20] and [21]. For integer $m \ge 0$ and $f \in L^2$

(8.7) $$||\hat{T}^{(m)}f|| \le \frac{C}{\alpha^2}||f|| ,$$

and

(8.8) $$||\hat{L}\hat{T}^{(m)}f|| \le C||f||$$

where the constants are independent of α. The following proofs of (8.7) and (8.8) are from Sammon [20] and [21]. Let $\{\phi_i\}_{i=1}^{\infty}$ and $\{\lambda_i\}_{i=1}^{\infty}$ be the eigenfunctions and eigenvalues of L. Since $\hat{T}f = \sum_{i=1}^{\infty}(\lambda_i + \alpha^2)^{-1}(f,\phi_i)\phi_i$,

(8.9) $$||\hat{T}f|| \le \frac{1}{\alpha^2}||f||$$

and

(8.10) $$||L\hat{T}f|| \le C||f||$$

where C is independent of α. Now since

$$||\hat{L}\hat{T}^{(m)}f|| = ||\sum_{\ell=0}^{m-1} \binom{m}{\ell} L^{(m-\ell)}\hat{T}^{(\ell)}f||$$

it follows by induction that

$$||\hat{L}\hat{T}^{(m)}f|| \leq C \sum_{\ell=0}^{m-1} ||L\hat{T}^{(\ell)}f|| \leq C||f||$$

which is (8.8). The estimate

$$||\hat{T}^{(m)}f|| = ||\hat{T}\hat{L}\hat{T}^{(m)}f|| \leq \frac{C}{\alpha^2}\,||f||$$

proves (8.7). In addition to (8.7) and (8.8) the following two estimates will be needed.

$$(8.11)\qquad ||T^{1/2}\hat{T}^{(m)}L^{1/2}f|| \leq \frac{C}{\alpha^2}\,||f|| \quad \text{and}$$

$$(8.12)\qquad ||T^{1/2}\hat{L}\hat{T}^{(m)}L^{1/2}f|| \leq C||f||$$

where the constant is independent of α. (8.12) is proved by induction on m. For $m = 1$,

$$T^{1/2}\hat{L}\hat{T}^{(1)}L^{1/2} = -\,T^{1/2}L^{(1)}\hat{T}L^{1/2} = -\,T^{1/2}L^{(1)}T^{1/2}L\hat{T}$$

Using Lemma 8.1 and (8.10), it follows that

$$||T^{1/2}\hat{L}\hat{T}^{(1)}L^{1/2}f|| \leq C||f|| \ .$$

Now assume (8.12) for $m \leq n - 1$. Since

$$\hat{L}\hat{T}^{(n)} = -\sum_{\ell=0}^{n-1}\binom{n}{\ell}\hat{L}^{(n-\ell)}\hat{T}^{(\ell)}$$

and

$$T^{1/2}\hat{L}\hat{T}^{(n)}L^{1/2} = -\sum_{\ell=0}^{n-1}\binom{n}{\ell}(T^{1/2}L^{(n-\ell)}T^{1/2}(L\hat{T})(T^{1/2}\hat{L}\hat{T}^{(\ell)}L^{1/2} ,$$

Lemma 8.1, (8.10) and the induction hypothesis imply (8.12).

(8.11) follows from (8.9) and (8.12) since

$$||T^{1/2}\hat{T}^{(m)}L^{1/2}f|| = ||\hat{T}T^{1/2}\hat{L}\hat{T}^{(m)}L^{1/2}f|| \leq \frac{C}{\alpha^2}\,||f|| \ .$$

The estimates (8.8) and (8.12) are used to prove the lemma. Since

$$\hat{\mathcal{L}} = \begin{pmatrix} -\alpha I & I \\ -L & -\alpha I \end{pmatrix} \quad \text{and}$$

$$\hat{\mathcal{T}}^{(m)} = \begin{pmatrix} -\alpha\hat{T}^{(m)} & -\hat{T}^{(m)} \\ -\alpha^2\hat{T}^{(m)} & -\alpha\hat{T}^{(m)} \end{pmatrix} ,$$

it follows that

$$\hat{\mathcal{L}}\hat{\mathcal{T}}^{(m)} = \begin{pmatrix} 0 & 0 \\ \alpha\hat{L}\hat{T}^{(m)} & \hat{L}\hat{T}^{(m)} \end{pmatrix} ,$$

so that for $\Phi \in L^2 \times L^2$

$$|||\hat{\mathcal{L}}\hat{\mathcal{T}}^{(m)}\Phi||| = ||T^{1/2}(\alpha\hat{L}\hat{T}^{(m)}\Phi_1 + \hat{L}\hat{T}^{(m)}\Phi_2)||$$

$$\leq \alpha||T^{1/2}\hat{L}\hat{T}^{(m)}\phi_1|| + ||T^{1/2}\hat{L}\hat{T}^{(m)}\phi_2|| .$$

Since $T^{1/2}$ is a bounded operator on L^2 and $T^{1/2}\hat{L}\hat{T}^{(m)} = T^{1/2}\hat{L}\hat{T}^{(m)}L^{1/2}T^{1/2}$, (8.8) and (8.12) imply that

$$|||\hat{L}\hat{\mathcal{T}}^{(m)}\phi||| \leq C\alpha||\phi_1|| + C||T^{1/2}\phi_2||$$

(8.6) follows from this estimate.

<u>Lemma 8.3</u>. For α sufficiently large $(I - m\,\hat{\mathcal{T}}^{(1)})$ is invertible on $L^2 \times L^2$ and

(8.13) $$|||(I-m\,\hat{\mathcal{T}}^{(1)})^{-1}\phi||| \leq C|||\phi||| ,$$

where C is independent of α.

<u>Proof</u>. Since $\hat{\mathcal{T}}^{(1)} = \begin{pmatrix} -\alpha\hat{T}^{(1)} & \hat{T}^{(1)} \\ -\alpha^2\hat{T}^{(1)} & -\alpha\hat{T}^{(1)} \end{pmatrix}$, if $(I + m\alpha\hat{T}^{(1)})$ is invertible, it follows that

$$(I - m\hat{\mathcal{T}}^{(1)})^{-1} = (I + 2m\alpha\hat{T}^{(1)})^{-1}\begin{pmatrix} I+m\alpha\hat{T}^{(1)} & -m\hat{T}^{(1)} \\ -m\alpha^2\hat{T}^{(1)} & I+m\alpha\hat{T}^{(1)} \end{pmatrix}$$

so that

(8.14) $$|||(I + m\hat{\mathcal{T}}^{(1)})^{-1}\phi|||^2$$

$$= ||(I + 2m\alpha\hat{T}^{(1)})^{-1}((I + m\alpha\hat{T}^{(1)})\phi_1 - m\hat{T}^{(1)}\phi_2)||^2$$

$$+ ||T^{1/2}(I + 2m\alpha\hat{T}^{(1)})^{-1}(-m\alpha^2\hat{T}^{(1)}\phi_1 + (I + m\alpha\hat{T}^{(1)})\phi_2)||^2 .$$

(8.7) states that $||\hat{T}^{(1)}f|| \leq \frac{C}{\alpha^2}||f||$ so that if α is large enough $||2m\alpha\hat{T}^{(1)}f|| \leq \gamma_1||f||$ where $\gamma_1 < 1$ and for α large can be chosen independent of α. Writing $(I+2m\alpha\hat{T}^{(1)})^{-1} = I-2m\alpha\hat{T}^{(1)} + (2m\alpha\hat{T}^{(1)})^2 - \ldots$ gives

(8.15) $$||(I+2m\alpha\hat{T}^{(1)})^{-1}f|| \leq (1+\gamma_1+\gamma_1^2+\ldots)||f|| \leq \frac{1}{1-\gamma_1}||f|| .$$

Also,

(8.16) $$||T^{1/2}(I+2m\alpha\hat{T}^{(1)})^{-1}L^{1/2}f||$$

$$= ||(I+2m\alpha T^{1/2}\hat{T}^{(1)}L^{1/2})^{-1}f|| \leq \frac{1}{1-\gamma_2}||f||$$

since $||2m\alpha T^{1/2}\hat{T}^{(1)}L^{1/2}f|| \leq \frac{C}{\alpha}||f||$ (from (8.11)), where $\gamma_2 < 1$ and for α large can be chosen independent of α.

Using (8.15) and (8.16) in (8.14) gives

(8.17) $$|||(I+m\hat{\mathcal{T}}^{(1)})^{-1}\phi|||^2 \leq C(||(I+m\alpha\hat{T}^{(1)})\phi_1 - m\hat{T}^{(1)}\phi_2||^2$$

$$+ ||T^{1/2}(-m\alpha^2\hat{T}^{(1)}\phi_1 + (I+m\alpha\hat{T}^{(1)})\phi_2)||^2)$$

where the constant C is independent of α. Since

$$\hat{T}^{(1)}L^{1/2} = T^{1/2}(L\hat{T})(T^{1/2}\hat{L}\hat{T}^{(1)}L^{1/2}) ,$$

(8.10) and (8.12) imply that

(8.18) $$||\hat{T}^{(1)}L^{1/2}g|| \leq C||g|| .$$

where the constant C is independent of α. Now using (8.7), (8.18), the fact that $T^{1/2}$ is a bounded operator on L^2, and (8.11) in (8.17) gives

$$|||(I+m\hat{\mathcal{F}}^{(1)})^{-1}\Phi|||^2 \leq C(||\phi_1||^2 + ||T^{1/2}\phi_2||^2)$$

for α large, where C can be chosen indpendent of α. This completes the proof of the lemma.

<u>Lemma 8.4</u> For any $\Phi \in L^2 \times L^2$

$$(8.19) \qquad |||\hat{\mathcal{F}}\Phi||| \leq \frac{1}{\alpha}|||\Phi||| .$$

<u>Proof</u>. Let $\{\phi_j\}_{j=1}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty}$ be the orthonormal eigenfunctions and eigenvalues of L. $\hat{\mathcal{L}}$ has a complete orthonormal set of eigenfunctions given by $\Phi_{\pm j} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_j \\ \pm i\sqrt{\lambda_j}\,\phi_j\end{pmatrix}$ with eigenvalues $-\alpha \pm i\sqrt{\lambda_j}$, for $j = 1, \ldots, \infty$. Let $\Phi = \sum_{j=-\infty}^{\infty} C_j\Phi_j$. Then

$$\hat{\mathcal{F}}\Phi = \sum_{j=-\infty}^{\infty} \frac{C_j}{(\text{sgn } j)\, i\sqrt{\lambda_{|j|}} - \alpha}\,\Phi_j$$

where $(\text{sgn } j)$ is the sign of j. Since $\frac{1}{|(\text{sgn } j)\, i\sqrt{\lambda_{|j|}} - \alpha|} \leq \frac{1}{\alpha}$, it follows that

$$|||\hat{\mathcal{F}}\Phi|||^2 = \sum_{j=-\infty}^{\infty} \frac{|C_j|^2}{|(\text{sgn } j) i\sqrt{\lambda_{|j|}} - \alpha|^2} \leq \frac{1}{\alpha^2}|||\Phi|||^2 ,$$

This gives (8.19).

Lemmas 8.1, 8.2, 8.3, and 8.4 are now used to prove Theorem 5.1.

<u>Proof of Theorem 5.1.</u> The theorem is proved by induction on m. Assume that it is true for $\hat{E}_i$, $i = 1, \ldots, m$, $(m \geq 2)$. Also, as part of the induction hypothesis, assume that each of the four components of $\hat{E}_i^{-1}$, $i = 1, \ldots, m$, is a bounded operator on H^ℓ for $\ell \geq 0$ (i.e. writing $\hat{E}_i^{-1} = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$, the operators A_{11}, A_{12}, A_{21} and A_{22} are assumed to be bounded where the bound can depend on α and ℓ) and that $|||\hat{E}_i^{-1}\Phi||| \leq \frac{C}{\alpha}|||\Phi|||$, for $i = 1, \ldots, m$ and for any $\Phi \in H^1 \times L^2$ where the constant C is independent of α. (It is straightforward using Lemmas 8.3 and 8.4 and by writing out formulas for $\hat{E}_1^{-1}$ and $\hat{E}_2^{-1}$ to see that the induction hypothesis is true for $\hat{E}_1$ and $\hat{E}_2$.)

From the induction hypothesis it is easy to see that (1) in Theorem 5.1 is satisfied for $\hat{E}_{m+1}$. Let $\Phi \in (H^2 \cap H_0^1) \times H^1$. In order to prove (2) in Theorem 5.1 we first show that

$$(8.20) \qquad \frac{\alpha}{C}|||\Phi||| \leq |||\hat{E}_{m+1}\Phi|||$$

where the constant C is independent of α when α is sufficiently large. From Lemma 8.2 it follows that

$$\sum_{\ell=0}^{m-2} \binom{m}{\ell} |||\hat{\mathcal{L}}\hat{\mathcal{F}}^{(m-1)}\hat{E}_{\ell+2}^{-1}, \ldots, \hat{E}_m^{-1}\Phi|||$$

$$\leq \sum_{\ell=0}^{m-2} C\alpha|||\hat{E}_{\ell+2}^{-1} \cdots \hat{E}_m^{-1}\Phi||| .$$

The induction hypothesis implies that

$$|||\hat{E}_{\ell+2}^{-1} \cdots \hat{E}_m^{-1}\Phi||| \leq \frac{C}{\alpha}|||\Phi||| ,$$

so that

$$(8.21)\qquad \Big|\Big|\Big|\sum_{\ell=0}^{m-2} \binom{m}{\ell}\hat{\mathcal{L}}\hat{\mathcal{J}}^{(m-1)}\hat{E}_{\ell+2}^{-1}\cdots \hat{E}_m^{-1}\phi\Big|\Big|\Big| \le C_1|||\phi|||$$

where C_1 is independent of α. Lemma 5.3 and Lemma 5.4 imply that

$$C_2\alpha|||\phi||| \le \alpha|||(I-m\hat{\mathcal{J}}^{(1)})\phi||| \le |||\hat{\mathcal{L}}(I-m\hat{\mathcal{J}}^{(1)})\phi|||\ .$$

This estimate used with (8.21) and (5.5) proves that

$$(C_2\alpha - C_1)|||\phi||| \le |||\hat{E}_{m+1}\phi|||\ .$$

Choosing $\alpha \ge \frac{2C_1}{C_2}$, gives $C_2\alpha - C_1 \ge C_2\alpha - \frac{C_2}{2}\alpha \ge \frac{C_2}{2}\alpha$.

So defining a new constant to be $\frac{C_2}{2}$ completes the proof of (8.20).

(8.20) shows that if $F \in H^1 \times L^2$ is given, then the equation $\hat{E}_{m+1}\phi = F$ can have at most one solution in $(H^2 \cap H_0^1) \times H^1$. We now prove the existence of a solution of $\hat{E}_{m+1}\phi = F$ for $F \in H^1 \times L^2$. Since

$$\hat{\mathcal{L}}\hat{\mathcal{J}}^{(j)} = \begin{pmatrix} 0 & 0 \\ \alpha\hat{L}\hat{T}^{(j)} & \hat{L}\hat{T}^{(j)} \end{pmatrix}$$

for any positive integer j,

$$\hat{E}_{m+1} = \begin{pmatrix} -\alpha I & I \\ -L & -\alpha I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix}$$

where (using the induction hypothesis) B_1 and B_2 are bounded operators on H^ℓ (the bound depending on α), for any integer $\ell \ge 0$. Solving $\hat{E}_{m+1}\phi = F$ is equivalent to solving

$$\begin{pmatrix} -\alpha I & I \\ -L+B_1 & -\alpha I+B_2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

which is equivalent to solving

$$(8.22)\qquad \begin{pmatrix} -\alpha I & I \\ -L+B_1-\alpha^2 I+\alpha B_2 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2+(\alpha I-B_2)f_1 \end{pmatrix}.$$

(8.22) can be solved if $-L+B_1-\alpha^2I+\alpha B_2$ is invertible. This is equivalent to the invertibility of $I-T(B_1-\alpha^2I+\alpha B_2)$. Since T is a compact operator and $B_1-\alpha^2I+\alpha B_2$ is a bounded operator on L^2 (where the bound can depend on α), either the operator $I-T(B_1-\alpha^2I+\alpha B_2)$ is invertible or there exists a nonzero solution ψ_1 of $(I-T(B_1-\alpha^2I+\alpha B_2))\psi_1 = 0$. If ψ_1 exists and is nonzero, then a nonzero solution of $E_{m+1}\psi = 0$ in $(H^2 \cap H_0^1) \times H^1$ can be constructed using (8.22) (with $f_1 = 0$ and $f_2 = 0$ in (8.22)). However, this contradicts (8.20) (for α large). Therefore the operator $-L+B_1-\alpha^2I+\alpha B_2$ in (8.22) must be invertible.

If (8.22) is solved for $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, then

$$(8.23)\qquad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} (-L+B_1-\alpha^2I+\alpha B_2)^{-1}(f_2+(\alpha I-B_2)f_1) \\ f_1+\alpha(-L+B_1-\alpha^2I+\alpha B_2)^{-1}(f_2 + (\alpha I-B_2)f_1) \end{pmatrix}$$

$$= \hat{E}_{m+1}^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where

$$\hat{E}_{m+1}^{-1} \equiv (-L+B_1-\alpha^2 I+\alpha B_2)^{-1}\begin{pmatrix} \alpha I - B_2 & I \\ -L+B_1 & \alpha I \end{pmatrix} .$$

From the regularity properties of elliptic operators it follows that the components of $\hat{E}_{m+1}^{-1}$ are bounded operators on H^ℓ for any fixed integer $\ell \geq 0$ (this bound can depend on α) . From (8.20) it follows that

$$|||\hat{E}_{m+1}^{-1}F||| \leq \frac{C}{\alpha} |||F||| .$$

Also, the regularity properties of the operator $(-L+B_1-\alpha^2 I+\alpha B_2)^{-1}$ and (8.23) imply that

$$||\phi_1||_{\ell+2} \leq C(\alpha)||f_2+\alpha f_1-B_2 f_1||_\ell \leq C(\alpha)(||f_1||_{\ell+1} + ||f_2||_\ell)$$

and

$$||\phi_2||_{\ell+1} \leq ||f_1||_{\ell+1} + \alpha||\phi_1||_{\ell+1} \leq C(\alpha)(||f_1||_{\ell+1} + ||f_2||_\ell) .$$

These inequalities imply that (2) in Theorem 5.1 is satisfied for $\hat{E}_{m+1}^{-1}$.

Proof of (3) in Theorem 5.1 will complete the proof of the theorem. The proof is formally the same as in Sammon [20] and [21]. Assume $\hat{A}_\ell = \hat{E}_\ell \hat{A}_{\ell-1}$ for $0 \leq \ell \leq m$. Then

$$\hat{E}_{m+1}\hat{A}_m = \hat{\mathcal{L}}(I-m\hat{\mathcal{F}}^{(1)} - \sum_{\ell=0}^{m-2}\binom{m}{\ell}\hat{\mathcal{F}}^{(m-\ell)}\hat{E}_{\ell+2}^{-1} \cdots \hat{E}_m^{-1})\hat{A}_m$$

$$= \hat{\mathcal{L}}(\hat{A}_m - m\hat{\mathcal{F}}^{(1)}\hat{A}_m - \sum_{\ell=0}^{m-2}\binom{m}{\ell}\hat{\mathcal{F}}^{(m-\ell)}\hat{A}^{(\ell+1)})$$

$$= \hat{\mathcal{L}}(\hat{A}_m - \sum_{\ell=0}^{m-1}\binom{m}{\ell}\hat{\mathcal{F}}^{(m-\ell)}\hat{A}_{\ell+1}) .$$

Using (5.4)

$$\hat{E}_{m+1}\hat{A}_m = \hat{\mathcal{L}}(\hat{A}_m - \sum_{\ell=0}^{m-1}\binom{m}{\ell}\hat{\mathcal{F}}^{(m-\ell)}\sum_{j=0}^{\ell}\binom{\ell}{j}\hat{\mathcal{L}}^{(\ell-j)}\hat{A}_j)$$

$$= \hat{\mathcal{L}}(\hat{A}_m - \sum_{j=0}^{m-1}\sum_{\ell=j}^{m-1}\binom{m}{\ell}\binom{\ell}{j}\hat{\mathcal{F}}^{(m-\ell)}\hat{\mathcal{L}}^{(\ell-j)}\hat{A}_j)$$

$$= \hat{\mathcal{L}}(\hat{A}_m - \sum_{j=0}^{m-1}\sum_{k=0}^{m-1-j}\binom{m}{k+j}\binom{k+j}{j}\hat{\mathcal{F}}^{(m-k-j)}\hat{\mathcal{L}}^{(k)}\hat{A}_j)$$

$$= \hat{\mathcal{L}}(\hat{A}_m - \sum_{j=0}^{m-1}\binom{m}{j}(\sum_{k=0}^{m-1-j}\binom{m-j}{k}\hat{\mathcal{F}}^{(m-k-j)}\hat{\mathcal{L}}^{(k)})\hat{A}_j)$$

$$= \hat{\mathcal{L}}(\hat{A}_m - \sum_{j=0}^{m-1}\binom{m}{j}((\hat{\mathcal{F}}\hat{\mathcal{L}})^{(m-j)} - \hat{\mathcal{F}}\hat{\mathcal{L}}^{(m-j)})\hat{A}_j)$$

$$= \hat{\mathcal{L}}\hat{A}_m + \sum_{j=0}^{m-1}\binom{m}{j}\hat{\mathcal{L}}^{(m-j)}\hat{A}_j .$$

Using (5.4) again ($j = m$ in (5.4)) gives $\hat{E}_{m+1}\hat{A}_m = \hat{A}_{m+1}$. This completes the proof of Theorem 5.1.

The proof of the invertibility of the operators $\hat{E}_{1,h},\ldots,\hat{E}_{m,h}$ is similar to the proof of the invertibility of $\hat{E}_1,\ldots,\hat{E}_m$. The analogue of Lemma 8.1 is Lemma 3.1. The following lemma is the counterpart of Lemma 8.2.

Lemma 8.5. For any positive integer m and all $\Phi \in S_h \times S_h$

(8.24) $$|||\hat{\mathcal{L}}_h\hat{\mathcal{F}}_h^{(m)}\Phi|||_0 \leq (C_1 + C_2\alpha)\ |||\Phi|||_0 ,$$

where C_1 and C_2 are constants which are independent of α .

Proof. Define $\hat{L}_h = L_h + \alpha^2 I$ and $\hat{T}_h^{(m)} = (\hat{L}_m^{-1})^{(m)}$. Let $\{\phi_i\}_{i=1}^M$ and $\{\lambda_i\}_{i=1}^M$ be the eigenfunctions and eigenvalues of L_h. Since $\hat{T}_h f = \sum_{i=1}^M (\lambda_i + \alpha^2)^{-1}(f,\phi_i)\phi_i$, it follows that

$$(8.25)\qquad \|\hat{T}_h f\| \le \frac{1}{\alpha^2}\|f\|$$

and

$$(8.26)\qquad \|L_h \hat{T}_h f\| \le C\|f\|$$

where C is independent of α.

Also $\hat{L}_h \hat{T}_h^{(m)} = -\sum_{j=0}^{m-1} \binom{m}{j} \hat{L}_h^{(m-j)} \hat{T}_h^{(j)}$, so that

$$T_h^{1/2} \hat{L}_h \hat{T}_h^{(m)} L_h^{1/2} = -\sum_{j=0}^{m-1} \binom{m}{j} (T_h^{1/2} L_h^{(m-j)} T_h^{1/2})(L_h \hat{T}_h)(T_h^{1/2} \hat{L}_h \hat{T}_h^{(j)} L_h^{1/2}) .$$

By induction (using Lemma 3.1 and (8.26)),

$$(8.27)\qquad \|T_h^{1/2} \hat{L}_h \hat{T}_h^{(m)} L_h^{1/2} f\| \le C\|f\|$$

where the constant C is independent of α. Since

$$\hat{\mathcal{L}}_h \hat{\mathcal{T}}_h^{(m)} = \begin{pmatrix} 0 & 0 \\ \alpha \hat{L}_h \hat{T}_h^{(m)} & \hat{L}_h \hat{T}_h^{(m)} \end{pmatrix} ,$$

$$|||\hat{\mathcal{L}}_h \hat{\mathcal{T}}_h^{(m)} \Phi|||_0 = \|T_h^{1/2}(\alpha \hat{L}_h \hat{T}_h^{(m)} \phi_1 + \hat{L}_h \hat{T}_h^{(m)} \phi_2)\|$$

and

$$|||\hat{\mathcal{L}}_h \hat{\mathcal{T}}_h^{(m)} \Phi|||_0 \le \alpha \|T_h^{1/2} \hat{L}_h \hat{T}_h^{(m)} \phi_1\| + \|T_h^{1/2} \hat{L}_h \hat{T}_h^{(m)} \phi_2\| .$$

Using (8.27) and the fact that $T_h^{1/2}$ is a bounded operator gives

$$|||\hat{\mathcal{L}}_h \hat{\mathcal{T}}_h^{(m)} \Phi|||_0 \le C\alpha\|\phi_1\| + C\|T_h^{1/2}\phi_2\|$$

where C is independent of α. (8.24) follows from this estimate.

The next lemma is the discrete counterpart of Lemma 8.3.

Lemma 8.6. For α sufficiently large $(I - m\hat{\mathcal{T}}_h^{(1)})$ is invertible on $S_h \times S_h$ and

$$(8.28)\qquad |||(I - m\hat{\mathcal{T}}_h^{(1)})^{-1}\Phi|||_0 \le C|||\Phi|||_0 ,$$

where the constant C is independent of α.

Proof. If $(I + 2m\alpha \hat{T}_h^{(1)})$ is invertible, then

$$(I - m\hat{\mathcal{T}}_h^{(1)})^{-1} = (I + 2m\alpha \hat{T}_h^{(1)})^{-1} \begin{pmatrix} I + m\alpha \hat{T}_h^{(1)} & -m\hat{T}_h^{(1)} \\ -m\alpha^2 \hat{T}_h^{(1)} & I + m\alpha \hat{T}_h^{(1)} \end{pmatrix}$$

so that

$$(8.29) \qquad |||(I+m\hat{\mathcal{T}}_h^{(1)})^{-1}\Phi|||_0^2$$

$$= ||(I+2m\alpha\hat{T}_h^{(1)})^{-1}((I+m\alpha\hat{T}_h^{(1)})\phi_1 - m\hat{T}_h^{(1)}\phi_2)||^2$$

$$+ ||T_h^{1/2}(I+2m\alpha\hat{T}_h^{(1)})^{-1}(-m\alpha^2\hat{T}_h^{(1)}\phi_1 + (I+m\alpha\hat{T}_h^{(1)})\phi_2)||^2 \ .$$

Since $\hat{T}_h f = \sum_{i=1}^{M} (\lambda_i + \alpha^2)^{-1}(f,\phi_i)\phi_i$, it follows that

$$L_h^{1/2}\hat{T}_h f = \sum_{i=1}^{M} \lambda_i^{1/2}(\lambda_i + \alpha^2)^{-1}(f,\phi_i)\phi_i \ .$$

This and the equality $\frac{\sqrt{\lambda_i}}{\lambda_i+\alpha^2} = \frac{\sqrt{\lambda_i}}{(\lambda_i+\alpha^2)^{1/2}(\lambda_i+\alpha^2)^{1/2}}$ imply that

$$(8.30) \qquad ||L_h^{1/2}\hat{T}_h f|| \le \frac{1}{\alpha} ||f|| \ .$$

Since

$$\hat{T}_h^{(1)} = -\hat{T}_h L_h^{(1)}\hat{T}_h = -\hat{T}_h L_h^{1/2}(T_h^{1/2}L_h^{(1)}T_h^{1/2})L_h^{1/2}\hat{T}_h$$

and $T_h^{1/2}\hat{T}_h^{(1)}L_h^{1/2} = -\hat{T}_h(T_h^{1/2}L_h^{(1)}T_h^{1/2})(L_h\hat{T}_h)$, Lemma 3.1, (8.30) , (8.25) and (8.26) prove that

$$(8.31) \qquad ||\hat{T}_h^{(1)}f|| \le \frac{C}{\alpha^2} ||f||$$

and

$$(8.32) \qquad ||T_h^{1/2}\hat{T}_h^{(1)}L_h^{1/2}f|| \le \frac{C}{\alpha^2} ||f||$$

where the constant C is independent of α . As in the proof of Lemma 8.3, (8.29), (8.31) and (8.32) show that for α large

$$(8.33) \qquad |||(I+m\hat{\mathcal{T}}_h^{(1)})^{-1}\Phi|||_0^2 \le C(||(I+m\alpha\hat{T}_h^{(1)})\phi_1 - \hat{T}_h^{(1)}\phi_2||^2$$

$$+ ||T_h^{1/2}(-m\alpha^2\hat{T}_h^{(1)}\phi_1 + (I+m\alpha\hat{T}_h^{(1)}))\phi_2||^2) \ .$$

Since $\hat{T}_h^{(1)}L_h^{1/2} = T_h^{1/2}(L_h\hat{T}_h)(T_h^{1/2}\hat{L}_h\hat{T}_h^{(1)}L_h^{1/2})$,

(8.26) and (8.27) show that

$$(8.34) \qquad ||\hat{T}_h^{(1)}L_h^{1/2} g|| \le C||g|| \ .$$

Using (8.31), (8.34) and (8.32) in (8.33) gives

$$|||(I+m\hat{\mathcal{T}}_h^{(1)})^{-1}\Phi|||_0^2 \le C(||\phi_1||^2 + ||T_h^{1/2}\phi_2||^2)$$

for α large, where C can be chosen independent of α . This completes the proof of the lemma.

The next lemma is the discrete counterpart of Lemma 8.4.

<u>Lemma 8.7</u>. For all $\Phi \in S_h \times S_h$

$$(8.35) \qquad ||\hat{\mathcal{T}}_h\Phi|||_0 \le \frac{C}{\alpha} |||\Phi|||_0$$

where the constant C is independent of α :

Proof. $\hat{\mathcal{L}}_h$ has a complete orthonormal set of eigenfunctions given by

$$\Phi_{\pm j} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_j \\ \pm i\sqrt{\lambda_j}\, \phi_j \end{pmatrix}$$

with eigenvalues $-\alpha \pm i\sqrt{\lambda_j}$, for $j = 1, \ldots, M$. Let

$$\Phi = \sum_{j=-M}^{M} C_j \Phi_j \ .$$

Then

$$\hat{\mathcal{J}}_h \Phi = \sum_{j=-M}^{M} \frac{C_j}{(\operatorname{sgn} j) i\sqrt{\lambda_{|j|}} - \alpha} \Phi_j$$

where $\operatorname{sgn} j$ is the sign of j . Since

$$\frac{1}{|(\operatorname{sgn} j) i\sqrt{\lambda_{|j|}} - \alpha|} \le \frac{1}{\alpha}$$

(8.35) follows (as in the proof of Lemma 8.4).

Proof of Theorem 5.2. Using Lemmas 8.5, 8.6, and 8.7 and the definition of $\hat{E}_{m,h}$ it follows that

$$(8.36) \qquad (\alpha C_1 - C_2)\ |||\Phi|||_0 \le |||\hat{E}_{m,h}\Phi|||_0$$

for all $\Phi \in S_h \times S_h$. (8.36) implies that $\hat{E}_{m,h}$ is invertible on $S_h \times S_h$ for sufficiently large α . (5.8) follows from exactly the same computation given at the end of the proof of Theorem 5.1.

The proof of (5.9) will complete the proof of Theorem 5.2. Assume

$$(8.37) \qquad ||||(\hat{E}_i^{-1} - \hat{E}_{i,h}^{-1})F|||_0 \le Ch^2(||f_1||_{s-2} + ||f_2||_{s-2})$$

for $i = 1, \ldots, m$. Let $W = \hat{E}_{m+1}^{-1} F$, $W_h = \hat{E}_{m+1,h}^{-1} F$,

$$\hat{R} = \sum_{\ell=0}^{m-2} \binom{m}{\ell} \hat{\mathcal{L}} \hat{\mathcal{J}}^{(m-\ell)} \hat{E}_{\ell+2}^{-1} \cdots \hat{E}_m^{-1} \quad \text{and}$$

$$\hat{R}_h = \sum_{\ell=0}^{m-2} \binom{m}{\ell} \hat{\mathcal{L}}_h \hat{\mathcal{J}}_h^{(m-\ell)} \hat{E}_{\ell+2,h}^{-1} \cdots \hat{E}_{m,h}^{-1} \ . \quad \text{Since}$$

$(\hat{\mathcal{L}}(I - m\hat{\mathcal{J}}^{(1)}) - \hat{R})W = F$ and $(\hat{\mathcal{L}}_h(I - m\hat{\mathcal{J}}_h^{(1)}) - \hat{R}_h)W_h = F$,
$W = (I - m\hat{\mathcal{J}}^{(1)})^{-1}\hat{\mathcal{J}}(F + \hat{R}W)$ and $W_h = (I - m\hat{\mathcal{J}}_h^{(1)})^{-1}\hat{\mathcal{J}}_h(F + \hat{R}_h W_h)$.

These equations imply that

$$(8.38) \qquad W - W_h = [(I - m\hat{\mathcal{J}}^{(1)})^{-1}\hat{\mathcal{J}} - (I - m\hat{\mathcal{J}}_h^{(1)})^{-1}\hat{\mathcal{J}}_h]F$$
$$+ [(I - m\hat{\mathcal{J}}^{(1)})^{-1}\hat{\mathcal{J}}\hat{R} - (I - m\hat{\mathcal{J}}_h^{(1)})^{-1}\hat{\mathcal{J}}_h\hat{R}_h]W$$
$$+ [(I - m\hat{\mathcal{J}}_h^{(1)})^{-1}\hat{\mathcal{J}}_h\hat{R}_h](W - W_h) \ .$$

Using Lemmas 8.5, 8.6 and 8.7 and (8.36), it follows that

$$(8.39) \qquad |||(I - m\hat{\mathcal{J}}_h^{(1)})^{-1}\hat{\mathcal{J}}_h\hat{R}_h(W - W_h)|||_0 \le \frac{C}{\alpha}\ |||W - W_h|||_0$$
$$\le \frac{1}{2}\ |||W - W_h|||_0$$

for $\alpha \ge 2C$.

$$(8.40)\qquad (I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}} = (I+2m\alpha\hat{T}^{(1)})^{-1}\begin{pmatrix} -\alpha\hat{T}-m\hat{T}^{(1)} & -\hat{T} \\ I+\alpha m\hat{T}^{(1)} - \alpha^2\hat{T} & -\alpha\hat{T} \end{pmatrix}$$

and

$$(8.41)\qquad (I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}^{(m-\ell)} = (I+2m\alpha\hat{T}^{(1)})^{-1}\begin{pmatrix} -\alpha\hat{T}^{(m-\ell)} & -\hat{T}^{(m-\ell)} \\ -\alpha^2\hat{T}^{(m-\ell)} & -\alpha\hat{T}^{(m-\ell)} \end{pmatrix}$$

In order to estimate the terms in (8.38) we show that for $j \geq 0$

$$(8.42)\qquad ||((I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}\hat{T}_h^{(j)})\phi|| \leq C(\alpha)h^2||\phi||_{s-2}$$

and

$$(8.43)\qquad ||T_h^{1/2}((I+2m\alpha\hat{T}^{(1)})^{-1} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}P)\phi|| \leq C(\alpha)h^2||\phi||_{s-2}$$

where P is the L^2 orthogonal projection onto S_h. To show (8.42) write

$$(I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}\hat{T}_h^{(j)}$$
$$= ((I+2m\alpha\hat{T}^{(1)})^{-1} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}P)\hat{T}^{(j)}$$
$$+ (I+2m\alpha\hat{T}_h^{(1)})^{-1}P(\hat{T}^{(j)} - \hat{T}_h^{(j)})$$
$$= (I+2m\alpha\hat{T}_h^{(1)})^{-1}P(P+2m\alpha\hat{T}_h^{(1)} - I-2m\alpha\hat{T}^{(1)})(I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)}$$
$$+ (I-P)(I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)}$$
$$+ (I+2m\alpha\hat{T}_h^{(1)})^{-1}P(\hat{T}^{(j)} - \hat{T}_h^{(j)}) .$$

This identity and $||(I-P)\psi|| \leq ||(TL-T_hL)\psi|| \leq Ch^s||L\psi||_{s-2}$ imply that

$$||((I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}\hat{T}^{(j)})\phi||$$
$$\leq C(\alpha)h^s(||L(I+2m\alpha\hat{T}^{(1)})^{-1}\hat{T}^{(j)}\phi||_{s-2} + ||\phi||_{s-2})$$
$$\leq C(\alpha)h^s(||L(\hat{L}+2m\alpha\hat{L}\hat{T}^{(1)})^{-1}\hat{L}\hat{T}^{(j)}\phi||_{s-2} + ||\phi||_{s-2})$$
$$\leq C(\alpha)h^s||\phi||_{s-2} .$$

This is (8.42). The proof of (8.43) is similar.

$$T_h^{1/2}((I+2m\alpha\hat{T}^{(1)})^{-1} - (I+2m\alpha\hat{T}_h^{(1)})^{-1}P)$$
$$= T_h^{1/2}(I+2m\alpha\hat{T}_h^{(1)})^{-1}P(P+2m\alpha\hat{T}_h^{(1)} - I-2ma\alpha\hat{T}^{(1)})(I+2m\alpha\hat{T}^{(1)})^{-1}$$

Therefore,

$$||T_h^{1/2}((I+2m\alpha\hat{T}^{(1)})^{-1} - (I+2m\alpha\hat{T}_h^{(1)})P)\phi||_{s-2}$$
$$\leq C(\alpha)h^s||(I+2m\alpha\hat{T}^{(1)})^{-1}\phi||_{s-2}$$

$$\leq C(\alpha)h^s||(\hat{L}+2m\alpha\hat{L}\hat{T}^{(1)})^{-1}\hat{L}\phi||_{s-2}$$

$$\leq C(\alpha)h^s||\phi||_{s-2}\ .$$

This is (8.43). Using (8.42) and (8.43) to estimate the difference between (8.40) and the corresponding formula for $(I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h$ gives

$$|||((I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}} - (I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h)F||| \tag{8.44}$$

$$\leq C(\alpha)h^s(||f_1||_{s-2} + ||f_2||_{s-2})\ .$$

Estimating (8.38) with (8.39) and (8.44) gives

$$|||W-W_h|||_0 \leq C(\alpha)h^s(||f_1||_{s-2} + ||f_2||_{s-2}) \tag{8.45}$$

$$+\ 2|||((I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}\hat{R} - (I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h\hat{R}_h)W|||_0\ .$$

Since

$$(I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}\hat{R} = \sum_{\ell=0}^{m-2}\binom{m}{\ell}(I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}^{(m-\ell)}\hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1}$$

and

$$(I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h\hat{R}_h = \sum_{\ell=0}^{m-2}\binom{m}{\ell}(I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h^{(m-\ell)}\hat{E}_{\ell+2,h}^{-1}\cdots\hat{E}_{m,h}^{-1}\ ,$$ it

follows that

$$|||((I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}\hat{R} - (I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h\hat{R}_h)W|||_0$$

$$\leq \sum_{\ell=0}^{m-2}\binom{m}{\ell}|||((I-m\hat{\mathcal{T}}^{(1)})^{-1}\hat{\mathcal{T}}^{(m-\ell)} - (I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h^{(m-\ell)})$$

$$\circ\ \hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1}W|||_0$$

$$+\sum_{\ell=0}^{m-2}\binom{m}{\ell}|||((I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h^{(m-\ell)})(\hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1} - \hat{E}_{\ell+2,h}^{-1}$$

$$\cdots\hat{E}_{m,h}^{-1})W|||_0\ .$$

Using (8.42) and (8.43) to estimate the difference between (8.41) and the corresponding formula for $(I-m\hat{\mathcal{T}}_h^{(1)})^{-1}\hat{\mathcal{T}}_h^{(m-\ell)}$ gives

$$|||W-W_h|||_0 \leq C(\alpha)(h^s(||f_1||_{s-2} + ||f_2||_{s-2})$$

$$+\sum_{\ell=0}^{m-2}|||(\hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1} - \hat{E}_{\ell+2,h}^{-1}\cdots\hat{E}_{m,h}^{-1})W|||_0)\ .$$

Since $\hat{E}_{\ell+2}^{-1}\cdots\hat{E}_m^{-1} - \hat{E}_{\ell+2,h}^{-1}\cdots\hat{E}_{m,h}^{-1}$

$$= \sum_{j=\ell+2}^{m}\hat{E}_{\ell+2,h}^{-1}\cdots\hat{E}_{j-1,h}^{-1}(\hat{E}_j^{-1} - \hat{E}_{j,h}^{-1})\hat{E}_{j+1}^{-1}\cdots\hat{E}_m^{-1}\ ,$$

(8.36), (8.37) and (2) in Theorem 5.1 imply that

$$|||W-W_h|||_0 \leq C(\alpha)h^s(||f_1||_{s-2} + ||f_2||_{s-2})\ .$$

This completes the proof of Theorem 5.2.