Analysis of Some Finite Elements for the Stokes Problem

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Abstract. We study some finite elements which are used in the approximation of the Stokes problem, so as to obtain error estimates of optimal order.

Résumé. Nous étudions deux éléments finis utilisés pour l'approximation du problème de Stokes et obtenons des estimations d'erreur d'ordre optimal.

I. Introduction. Let \( \Omega \) be a bounded polyhedral domain in \( \mathbb{R}^d \), \( d = 2 \) or \( 3 \). We consider the standard variational formulation of the stationary Stokes equations: for \( f \) given in \( H^{-1}(\Omega)^d \), find \( (u, p) \) in \( H^1_0(\Omega)^d \times L^2_0(\Omega) \) such that

\[
\begin{align*}
(\forall v \in H^1_0(\Omega)^d, \quad &v(\nabla u, \nabla v) - (p, \text{div} v) = (f, v), \\
(\forall q \in L^2_0(\Omega), \quad &q, \text{div} u) = 0,
\end{align*}
\]

where we denote by \((\cdot, \cdot)\) the inner product of \( L^2(\Omega) \) (or \( L^2(\Omega)^d \) or \( L^2(\Omega)^d \)). Hereafter \( L^2_0(\Omega) \) is the space \( \{ q \in L^2(\Omega); \int_\Omega q \, dx = 0 \} \). Now let \( h \) be a real positive parameter tending to zero. We introduce two finite-dimensional subspaces \( X_h \) and \( M_h \) of \( H^1_0(\Omega)^d \) and \( L^2_0(\Omega) \) respectively, satisfying the usual condition: for any \( q_h \) in \( M_h \), \( q_h \neq 0 \), there exists \( v_h \) in \( X_h \) such that \( (q_h, \text{div} v_h) \neq 0 \). We consider the discretized problem: find \( (u_h, p_h) \) in \( X_h \times M_h \) such that

\[
\begin{align*}
(\forall v_h \in X_h, \quad &v(\nabla u_h, \nabla v_h) - (p_h, \text{div} v_h) = (f, v_h), \\
(\forall q_h \in M_h, \quad &q, \text{div} u_h) = 0.
\end{align*}
\]

We recall that problem (1.1) (respectively problem (1.2)) has a unique solution \( (u, p) \) in \( H^1_0(\Omega)^d \times L^2_0(\Omega) \) (respectively \( (u_h, p_h) \) in \( X_h \times M_h \)). Moreover, when \( (u, p) \) belongs to the space \( H^{m+1}(\Omega)^d \times H^m(\Omega) \), it is well-known (see [7]) that the error estimate

\[
\|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq Ch^m(\|u\|_{m+1,\Omega} + \|p\|_{m,\Omega})
\]

holds whenever the following additional hypotheses are satisfied:

(H1) for any \( q \) in \( H^m(\Omega) \cap L^2_0(\Omega) \), one has

\[
\inf_{q_h \in M_h} \|q - q_h\|_{0,\Omega} \leq Ch^m\|q\|_{m,\Omega};
\]

(H2) there exists a linear operator \( \Pi_h \) from \( H^{m+1}(\Omega)^d \cap H^0_0(\Omega)^d \) into \( X_h \) such that

\[
\forall v \in H^{m+1}(\Omega)^d \cap H^1_0(\Omega)^d, \quad \begin{cases} 
(\forall q_h \in M_h, \quad (q_h, \text{div}(v - \Pi_h v)) = 0, \\
\|v - \Pi_h v\|_{1,\Omega} \leq Ch^m\|v\|_{m+1,\Omega};
\end{cases}
\]

Received May 24, 1983.
1980 Mathematics Subject Classification. Primary 65N30.
(H3) for each \( q_h \) in \( M_h \), there exists a function \( v_h \) in \( X_h \) such that

\[
\langle \text{div} \: v_h, q_h \rangle \geq \beta \| q_h \|_{0,\Omega} \| v_h \|_{1,\Omega},
\]

where \( \beta > 0 \) is a constant independent of \( h \).

Our aim is to give some examples of finite-element spaces such that hypotheses (H1), (H2) and (H3) are satisfied. To this end, we introduce a family \( (\mathcal{T}_h)_h \) of triangulations of \( \Omega \), where \( \mathcal{T}_h \) is made of \( d \)-simplices with diameters bounded by \( h \).

For any integer \( k \), \( P_k(K) \) denotes the space of polynomials of degree \( \leq k \) on \( K \). We set

\[
M_h^{(m)} = \{ q_h \in L^2_0(\Omega) \mid \forall K \in \mathcal{T}_h, q_{h/K} \in P_{m-1}(K) \}.
\]

Then hypothesis (H1) is satisfied (see [2] for instance). Finally, we set

\[
X_h = \{ v \in H^0(\Omega)^d \cap H^1_0(\Omega)^d \mid \forall K \in \mathcal{T}_h, v_{h/K} \in P_K \};
\]

hereafter we study some examples of spaces \( P_K \) introduced by Fortin [6] such that hypotheses (H2) and (H3) are satisfied.

More precisely, we give in Section II an example of a simplicial element of order \( m = 1 \) and, in Section III, an example of a three-dimensional tetrahedral element of order \( m = 2 \).

From now on we denote by \( \| \cdot \|_{m,\Omega} \) and \( | \cdot |_{m,\Omega} \) the usual norm and seminorm on the Sobolev space \( H^m(\Omega) \).

II. A Simplicial Element of Order 1 (\( d = 2 \) or 3). Let us consider a \( d \)-simplex \( K \) with vertices \( a_1, \ldots, a_{d+1} \). For \( 1 \leq i \leq d + 1 \), we denote by \( \lambda_i \) the barycentric coordinate associated with \( a_i \), by \( F_i \) the face which does not contain \( a_i \), and by \( n_i \) the unit outward normal to \( F_i \), and we set

\[
p_i = n_i \prod_{j=1, j\neq i}^{d+1} \lambda_j.
\]

Then, we consider

\[
P_K = P_1(K)^d \oplus \text{Span}(p_i, 1 \leq i \leq d + 1).
\]

(Note that \( \dim P_K = (d + 1)^2 \).) As far as the degrees of freedom are concerned, we can choose the values at the vertices \( a_i \), \( 1 \leq i \leq d + 1 \), and the flux through the faces \( F_i \), \( 1 \leq i \leq d + 1 \).

**Lemma II.1.** For any \( v \) in \( C^0(K)^d \), there exists a unique \( \Pi_K v \) in \( P_K \) such that

\[
\begin{align*}
\Pi_K v(a_i) &= v(a_i), \\
\int_{F_i} (v - \Pi_K v) \cdot n_i \, d\sigma &= 0, \quad 1 \leq i \leq d + 1.
\end{align*}
\]

Moreover, \( \Pi_K v_{/F_i} \) depends only on \( v_{/F_i} \), \( 1 \leq i \leq d + 1 \).

**Proof.** Let us denote by \( \bar{\Pi}_K v \) the classical Lagrange interpolate of \( v \) in \( P_1(K)^d \), i.e.,

\[
\bar{\Pi}_K v = \sum_{i=1}^{d+1} v(a_i) \lambda_i.
\]
Then, as the \( p_i \)'s are equal to 0 at any vertex, one has

\[
\Pi_K v = \tilde{\Pi}_K v + \sum_{i=1}^{d+1} \alpha_i p_i, \\
\text{with } \alpha_j = \left( \int_{F_i} (v - \tilde{\Pi}_K v) \cdot n_j \, d\sigma \right) / \int_{F_i} \prod_{j=1, j \neq i}^{d+1} \lambda_j \, d\sigma.
\]

Moreover, on \( F_i \),

\[
\Pi_K v_{/F_i} = \sum_{j=1, j \neq i}^{d+1} v(a_j) \lambda_j + \alpha_i p_i,
\]

so that \( \Pi_K v_{/F_i} \) depends only on \( v(a_j), j \neq i \), and on \( \int_{F_i} v \cdot n_j \, d\sigma \).

Now, for each \( h \), we consider a triangulation \( \mathcal{T}_h \) of \( \Omega \) made of \( d \)-simplices with diameters bounded by \( h \) and we assume that the family \( (\mathcal{T}_h)_h \) is regular, i.e., (see [2]) there exists a constant \( \sigma \) such that

\[
(11.4) \quad \forall h, \forall K \in \mathcal{T}_h, \quad h_K \leq \sigma \rho_K,
\]

where \( h_K \) is the diameter of \( K \), and \( \rho_K \) the diameter of the sphere inscribed in \( K \).

With each \( K \) in \( \mathcal{T}_h \), we associate the space \( P_K \) defined by (11.1); then Lemma II.1 allows us to define an operator \( \Pi_h \) from \( \mathcal{C}^0(\Omega)^d \cap H^1(\Omega)^d \) into \( X_h \) by

\[
\forall K \in \mathcal{T}_h, \quad \Pi_h v_{/K} = \Pi_K v.
\]

**Lemma II.2.** The operator \( \Pi_h \) satisfies (H2) for \( m = 1 \).

**Proof.** Clearly, one has

\[
\int_K \text{div}(v - \Pi_K v) \, dx = \sum_{i=1}^{d+1} \int_{F_i} (v - \Pi_K v) \cdot n_i \, d\sigma = 0,
\]

so that \( \forall q_h \in M_h^{(1)}, \langle q_h, \text{div}(v - \Pi_h v) \rangle = 0 \).

Moreover, we know that (see [2], for instance), for \( k = 0 \) and 1,

\[
|v - \tilde{\Pi}_K v|_{k,K} \leq C h^{2-k} |v|_{2,K}.
\]

Let us compute \( \Pi_K v - \tilde{\Pi}_K v = \Sigma_{i=1}^{d+1} \alpha_i p_i \). We consider an affine invertible mapping \( F_K: \hat{x} \mapsto x = B_K \hat{x} + b_K \) which maps the \( d \)-simplex \( \hat{K} = \{ \hat{x} \in \mathbb{R}^d; \forall i, 1 \leq i \leq d, \hat{x}_i \geq 0 \text{ and } \Sigma_{i=1}^d \hat{x}_i \leq 1 \} \) onto \( K \), and use the notations \( x = F_K(\hat{x}), v = \hat{v} \circ F_K^{-1} \).

Clearly, one has

\[
|p_i|_{k,K}^2 = \int_K \left\| D^k \left( \prod_{j=1, j \neq i}^{d+1} \lambda_j \right) \right\|^2 \, dx
\]

\[
\leq C \int_K \left\| D^k \left( \prod_{j=1, j \neq i}^{d+1} \lambda_j \right) \right\|^2 \left\| B_K^{-1} \right\|^{2k} |\det B_K| \, d\hat{x} \leq C |\det B_K| \left\| B_K \right\|^{-2k}
\]

so that, by the regularity of the family \( (\mathcal{T}_h)_h \),

\[
(11.6) \quad |p_i|_{k,K} \leq C h_i^{d/2-k}.
\]

But, since

\[
\int_{F_i} \prod_{j=1, j \neq i}^{d+1} \lambda_j \, d\sigma = |\det B_{K/F_i}| \int_{\hat{F}_i} \prod_{j=1, j \neq i}^{d+1} \lambda_j \, d\hat{\sigma},
\]

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we obtain by (II.3)
\[ |\alpha| \leq C|\det B_{K/\hat{K}}|^{-1} \int_{F_i} |v - \hat{\Pi}_K v| d\sigma \leq C \int_{\hat{F}_i} |\hat{v} - \hat{\Pi}_K \hat{v}| d\hat{\sigma}; \]
therefore, as \( P_1(\hat{K})^d \) is invariant under \( \hat{\Pi}_K \),
\[ |\alpha| \leq C|\hat{\Pi}_{2,K}^{-1/2} B_K^{1/2} v|_{2,K} \leq C h_K^{2-d/2} |v|_{2,K}. \]
The previous inequalities yield, for \( k = 0 \) and \( 1 \),
\[ |v - \Pi_K v|_{k,K} \leq C h_K^{2-k} |v|_{2,K}, \]
so that
\[ \|v - \Pi_K v\|_{1,\Omega} \leq C h |v|_{2,\Omega}. \]

We recall the proof of the following inequality only for the reader’s convenience.

**Lemma II.3.** For any \( v \) in \( H^1(K) \), we have
\[ \|v\|_{0,F_i} \leq C|\text{mes } F_i|^{1/2} h_K^{-d/2} \left\{ \|v\|_{0,K} + h_K |v|_{1,K} \right\}. \]

*Proof.* As the trace mapping is continuous from \( H^1(K) \) into \( L^2(\hat{F}_i) \),
\[ \|v\|^2_{0,F_i} = |\det B_{K/\hat{K}}| \int_{\hat{F}_i} \hat{\beta}^2 d\hat{\sigma} \leq C|\det B_{K/\hat{K}}| \left\{ \|\hat{v}\|^2_{0,K} + |\hat{\beta}|^2_{1,K} \right\} \]
\[ \leq C|\text{mes } F_i| h_K^{-d} \left\{ \|v\|^2_{0,K} + h_K^2 |v|^2_{1,K} \right\}. \]

Let us now study the hypothesis (H3). We know (see [7, Chapter I, Lemma 3.2]) that, for each \( q_h \) in \( M_h^{(1)} \), there exists \( v \) in \( H^1_0(\Omega)^d \) such that
\[ \text{div } v = q_h \quad \text{and} \quad \|v\|_{1,\Omega} \leq C\|q_h\|_{0,\Omega}. \]
Hence, the hypothesis (H3) is an immediate consequence of the following

**Lemma II.4.** For any \( v \) in \( H^1_0(\Omega)^d \), there exists \( v_h \) in \( X_h \) such that
\[ \forall q_h \in M_h^{(1)}, \quad \left( \langle q_h, \text{div}(v - v_h) \rangle = 0 \right) \quad \text{and} \quad \|v_h\|_{1,\Omega} \leq C\|v\|_{1,\Omega}. \]

*Proof.* Let us denote by \( w_h \) the interpolate of \( v \) in the space
\[ \left\{ u_h \in C^0(\bar{\Omega}) \cap H^1_0(\Omega); \forall K \in T_h, u_{h/K} \in P_1(K) \right\}^d, \]
defined by local regularization as in [4] (see [1] for an explicit generalization to the case \( d = 3 \)). By the regularity of the family \( (T_h)_h \), we know that the following local interpolation error holds
\[ \|v - w_h\|_{0,K} + h_K |w_h|_{1,K} \leq C h_K \|v\|_{1,\Delta_K}, \]
where \( \Delta_K \) is the union of all \( K' \) in \( T_h \) such that \( K \cap K' \neq \emptyset \); moreover, each element of \( T_h \) is contained in at most \( M \) subsets \( \Delta_K \), where \( M \) is an integer independent of \( h \).

Then, we consider the element \( v_h \) in \( V_h \) defined by
\[ \begin{aligned}
\langle v_h(a_i) = w_h(a_i), \\
\int_{F_i} (v - v_h) \cdot n_i d\sigma = 0, \quad 1 \leq i \leq d + 1,
\end{aligned} \]
or, in other words, equal on $K$ to

$$
\begin{align*}
\mathbf{v}_h/K &= \mathbf{w}_h + \sum_{i=1}^{d+1} \alpha_i \mathbf{p}_i, \\
\text{with } \alpha_i &= \left( \int_{F_i} (\mathbf{v} - \mathbf{w}_h) \cdot \mathbf{n}_i \, d\sigma \right) / \int_{F_i} \prod_{j=1, j \neq i}^{d+1} \lambda_j \, d\sigma.
\end{align*}
$$

Clearly, one has $\forall q_h \in M_h, \ (q_h, \text{div}(\mathbf{v} - \mathbf{v}_h)) = 0$. Moreover, by (II.6),

$$
\|\mathbf{v}_h\|_{1,K} \leq \|\mathbf{w}_h\|_{1,K} + \sum_{i=1}^{d+1} |\alpha_i| \|\mathbf{p}_i\|_{1,K} \leq \|\mathbf{w}_h\|_{1,K} + C h_K^{d/2-1} \sum_{i=1}^{d+1} |\alpha_i|.
$$

But, we also have

$$
|\alpha_i| \leq C |\det B_{K/F_i}|^{-1} \int_{F_i} (\mathbf{v} - \mathbf{w}_h) \cdot \mathbf{n}_i \, d\sigma \leq C |\text{mes } F_i|^{-1/2} \|\mathbf{v} - \mathbf{w}_h\|_{0,F_i}.
$$

Lemma II.3 implies

$$
(\text{II.11}) \quad |\alpha_i| \leq C h_K^{-d/2} \left\{ \|\mathbf{v} - \mathbf{w}_h\|_{0,K} + h_K |\mathbf{v} - \mathbf{w}_h|_{1,K} \right\}.
$$

Finally, we obtain

$$
\|\mathbf{v}_h\|_{1,K} \leq \|\mathbf{w}_h\|_{1,K} + h_K^{-1} \left\{ \|\mathbf{v} - \mathbf{w}_h\|_{0,K} + h_K |\mathbf{v} - \mathbf{w}_h|_{1,K} \right\},
$$

which, together with (II.10), yields $\|\mathbf{v}_h\|_{1,\Omega} \leq C \|\mathbf{v}\|_{1,\Omega}$.

As assumptions (H1) to (H3) are satisfied with $m = 1$, this element can be used to solve the Stokes problem with an $O(h)$-error estimate.

Remark II.1. In the two-dimensional case, we can also consider a triangulation $\mathcal{T}_h$ of $\Omega$ made of triangles and convex quadrilaterals. Then, if $K$ is a triangle, the space $P_K$ is defined by (II.1). If $K$ is a convex quadrilateral with vertices $a_1, \ldots$ and $a_4$, there exists an invertible mapping $F_K$ in $\hat{Q}_1^2$ which maps the unit square $\hat{K} = [0,1]^2$ onto $K$ ($\hat{Q}_1$ is the space of polynomials spanned by $x_1, x_2, x_3 = 1 - x_1$ and $x_4 = 1 - x_2$); for $1 \leq i \leq 4$, we denote by $F_i$ the edge with vertices $a_{i-1}$ and $a_i$ (of course, $a_0 = a_4$) and by $n_i$ the unit outward normal to $F_i$, and we set

$$
\mathbf{p}_i = n_i (\hat{q}_i \circ F_K^{-1}), \quad \hat{q}_i = \prod_{j=1, j \neq i}^4 \hat{x}_j.
$$

Then, we consider

$$
(\text{II.12}) \quad P_K = Q_1(K)^2 \oplus \text{Span} \{ \mathbf{p}_i, 1 \leq i \leq 4 \},
$$

where $Q_1(K) = \{ \hat{p} \circ F_K^{-1}, \hat{p} \in \hat{Q}_1 \}$. (Note that dim $P_K = 12$.) The degrees of freedom can be chosen as previously. If the family $(\mathcal{T}_h)_h$ is regular (see [3] for instance), the previous results are still valid.

III. A Tetrahedral Element of Order 2 ($d = 3$). Let us consider a tetrahedron $K$ with vertices $a_1, \ldots$ and $a_4$. We use the same notations as in Section II, in particular, we set

$$
\mathbf{p}_i = n_i \prod_{j=1, j \neq i}^4 \lambda_j, \quad 1 \leq i \leq 4;
$$

we also introduce the points $a_{ij} = \frac{1}{2} (a_i + a_j), 1 \leq i < j \leq 4$. Then, we consider

$$
(\text{III.1}) \quad P_K = P_2(K)^3 \oplus \text{Span} \{ \mathbf{p}_i, 1 \leq i \leq 4 \} \oplus (\text{Span} \{ \lambda_1 \lambda_2 \lambda_3 \lambda_4 \})^3.
$$
Let us remark that this space generalizes in the three-dimensional case the space studied in [5] for $d = 2$. As far as the degrees of freedom are concerned, we choose the values at the vertices $a_i$, $1 \leq i \leq 4$, and at the midpoints $a_{ij}$, $1 \leq i < j \leq 4$, the flux through the faces $F_i$, $1 \leq i \leq 4$, and the moments $\int_K x_l \text{div}(\cdot) \, dx$, $1 \leq l \leq 3$.

**Lemma III.1.** For any $v$ in $\mathcal{C}^0(K) \cap H^1(K)$, there exists a unique $\Pi_K v$ in $P_K$ such that

$$
\begin{align*}
\begin{cases}
\Pi_K v(a_i) = v(a_i), & 1 \leq i \leq 4, \\
\Pi_K v(a_{ij}) = v(a_{ij}), & 1 \leq i < j \leq 4, \\
\int_{F_i} (v - \Pi_K v) \cdot n_i \, d\sigma = 0, & 1 \leq i \leq 4, \\
\int_K x_l \text{div}(v - \Pi_K v) \, dx = 0, & 1 \leq l \leq 3.
\end{cases}
\end{align*}
$$

Moreover, $\Pi_K v_{/F_i}$ depends only on $v_{/F_i}, 1 \leq i \leq 4$.

**Proof.** Let us denote by $\Pi_K v$ the classical Lagrange interpolate of $v$ in $P_2(K)$, i.e.,

$$
\Pi_K v = \sum_{i=1}^4 v(a_i) \lambda_i (2\lambda_i - 1) + \sum_{1 \leq i < j \leq 4} v(a_{ij}) 4\lambda_i \lambda_j.
$$

Then, as the $p_i$'s and $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ are equal to 0 on any edge, $\Pi_K v$ can be written

$$
\Pi_K v = \Pi_K v + \sum_{i=1}^4 \alpha_i p_i + \beta \lambda_1 \lambda_2 \lambda_3 \lambda_4.
$$

Since $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ is equal to 0 on $\partial K$, we have

$$
\alpha_i = \left( \int_{F_i} (v - \Pi_K v) \cdot n_i \, d\sigma \right) / \int_{F_i} \prod_{j=1, j \neq i}^4 \lambda_j \, d\sigma, \quad 1 \leq i \leq 4.
$$

Then, setting

$$
\Pi_K v = \Pi_K v + \sum_{i=1}^4 \alpha_i p_i,
$$

and using the Green's formula, we obtain

$$
\beta = -\left( \int_K x_l \text{div}(v - \Pi_K v) \, dx \right) / \int_K \lambda_1 \lambda_2 \lambda_3 \lambda_4 \, dx, \quad 1 \leq l \leq 3.
$$

Moreover, on $F_i$, one has

$$
\Pi_K v_{/F_i} = \Pi_K v_{/F_i} + \alpha_i p_i,
$$

so that $\Pi_K v_{/F_i}$ depends only on $v_{/F_i}$.

Now, for each $h$, we consider a triangulation $\mathcal{T}_h$ of $\Omega$ made of tetrahedra with diameters bounded by $h$ and we assume that the family $(\mathcal{T}_h)_h$ is regular.

With each $K$ in $\mathcal{T}_h$, we associate the space $P_K$ defined by (III.1); then Lemma III.1 allows us to define an operator $\Pi_h$ from $\mathcal{C}^0(\Omega) \cap H^1(\Omega)$ into $X_h$ by (II.5).

**Lemma III.2.** The operator $\Pi_h$ satisfies (H2) for $m = 2$. 

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Proof. Clearly, one has
\[ \int_K \text{div}(v - \Pi_K v) \, dx = \int_K x_i \text{div}(v - \Pi_K v) \, dx = 0, \quad 1 \leq l \leq 3, \]
so that \( \forall q_h \in M_h^{(2)}, (q_h, \text{div}(v - \Pi_K v)) = 0. \)
Moreover, we know that (see [2]), for \( k = 0 \) and 1,
\[ |v - \Pi_K v|_{k, K} \leq Ch_{1-k}^k |v|_{3, K}. \]
Let us compute \( \Pi_K v - \Pi_K v = \sum_{i=1}^4 \alpha_i p_i. \) As in Section II,
\[ |\alpha_i| \leq C \int_{F_i} |\hat{v} - \hat{\Pi}_K \hat{v}| \, d\hat{\sigma}; \]
therefore, as \( P_2(\hat{K})^3 \) is invariant under \( \hat{\Pi}_K, \)
\[ |\alpha_i| \leq C |\hat{v}|_{3, \hat{K}} \leq Ch_{1/2}^2 |v|_{3, K}. \]
The previous inequalities, together with (II.6), yield
\[ |v - \Pi_K v|_{k, K} \leq Ch_{1-k}^k |v|_{3, K}. \]
Finally, we compute \( \Pi_K v - \Pi_K v = \beta_1 \lambda_2 \lambda_3 \lambda_4. \) Clearly, one has
\[ (\text{III.7}) \quad |\lambda_1 \lambda_2 \lambda_3 \lambda_4|_{k, K} \leq C \left( \int_K \|D^k(\lambda_1 \lambda_2 \lambda_3 \lambda_4)\|^2 \|B_K\|_{2, K}^{2k} |\text{det} B_K| \, d\hat{x} \right)^{1/2}
\leq Ch_{1/2}^{2-k}, \]
and, by (III.6),
\[ |\beta_1| \leq C |\text{det} B_K|^{-1} \left| \int_K x_i \text{div}(v - \Pi_K v) \, dx \right|. \]
We use Green's formula
\[ |\beta_1| \leq C |\text{det} B_K|^{-1} \left\{ \left| \int_K (v - \Pi_K v) \, dx \right| + \left| \int_{\partial K} x_i (v - \Pi_K v) \cdot n \, d\sigma \right| \right\}
\leq C \left( |\text{det} B_K|^{-1/2} \|v - \Pi_K v\|_{0, K} + |\text{det} B_K|^{-1} \left| \int_{\partial K} x_i (v - \Pi_K v) \cdot n \, d\sigma \right| \right). \]
But we remark that, since \( x = B_K \hat{x} + b_K, \)
\[ \int_{\partial K} x_i (v - \Pi_K v) \cdot n \, d\sigma = \int_{\partial K} (B_K \hat{x})_i \left( v - \Pi_K v \right) \cdot \hat{n} |\text{det} B_{K/\hat{K}}| \, d\hat{\sigma}
+ b_K \int_{\partial K} \left( v - \Pi_K v \right) \cdot \hat{n} |\text{det} B_{K/\hat{K}}| \, d\hat{\sigma}. \]
Therefore,
\[ \left| \int_{\partial K} x_i (v - \Pi_K v) \cdot n \, d\sigma \right| \leq \|B_K\| \int_{\partial K} |v - \Pi_K v| \, d\sigma
+ |b_K| \int_{\partial K} (v - \Pi_K v) \cdot n \, d\sigma. \]
Since the last term is equal to 0, we obtain

\[ |\beta| \leq C \left( |\det B_K|^{1/2} \|v - \Pi_K v\|_{0,K} + |\det B_K|^{-1} \|B_K\| \sum_{i=1}^{4} |\mes F_i|^{1/2} \|v - \Pi_K v\|_{0,F_i} \right) \]

so that, by Lemma II.3,

\[ |\beta| \leq \left\{ h_{3/2}^{-2} h_{3/2}^3 + h_{3}^{-3} h_{3} h_{3/2} \right\} |v|_{3,K} \leq Ch_{3/2}^{3/2} |v|_{3,K}. \]

The previous inequalities yield, for \( k = 0 \) and \( 1 \),

\[ |v - \Pi_K v|_{k,K} \leq Ch_{3-k}^{3/2} |v|_{3,K}. \]

By (II.8), the hypothesis (H3) is an immediate consequence of

**Lemma III.3.** For any \( v \) in \( H_0^1(\Omega)^3 \), there exists \( v_h \) in \( X_h \) such that

(III.8)

\[ \forall q_h \in M_h^{(2)}, \left\{ \begin{array}{l} (q_h, \text{div}(v - v_h)) = 0 \\ \text{and } \|v_h\|_{1,\Omega} \leq C\|v\|_{1,\Omega}. \end{array} \right. \]

**Proof.** Let us denote by \( w_h \) the interpolate of \( v \) in the space

\[ \left\{ u_h \in \mathcal{C}^0(\overline{\Omega}) \cap H_0^1(\Omega); \forall K \in \mathcal{T}_h, u_{h,K} \in P_2(K) \right\}^3, \]

defined by local regularization as in [1], so that (II.10) is still satisfied.

Then, we consider the element \( v_h \) in \( V_h \) equal on \( K \) to

\[ v_h = w_h + \sum_{i=1}^{4} \alpha_i p_i + \beta \lambda_1 \lambda_2 \lambda_3 \lambda_4 \]

with

\[ \alpha_i = \left( \int_{F_i} (v - w_h) \cdot n_i \, d\sigma \right) / \int_{F_i} \prod_{j=1, j \neq i}^{4} \lambda_j \, d\sigma, \]

\[ \beta_i = -\int_K x_i \text{div} \left( v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right) \, dx / \int_K \lambda_1 \lambda_2 \lambda_3 \lambda_4 \, dx. \]

Clearly, one has \( \forall q_h \in M_h^{(2)}, (q_h, \text{div}(v - v_h)) = 0. \) Moreover, by (II.6) and (III.7),

\[ \|v_h\|_{1,K} \leq \|w_h\|_{1,K} + Ch_{3/2}^{1/2} \left( \sum_{i=1}^{4} |\alpha_i| + |\beta| \right). \]

The \( \alpha_i \)'s still satisfy (II.11). We also have

\[ |\beta| \leq C|\det B_K|^{-1} \left( \left| \int_K \left( v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right) \, dx \right| \right. \]

\[ \left. + \left| \int_{\partial K} x_i \left( v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right) \cdot n \, d\sigma \right| \right). \]
By the same way as in the proof of Lemma III.2,

$$|eta| \leq C \left[ \left| \det B_K \right|^{-1/2} \left\| v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right\|_{0,K} + \left| \det B_K \right|^{-1} \left\| B_K \right\| \times \sum_{i=1}^{4} \text{mes } E_i \right] \left\| \left( v - w_h - \sum_{i=1}^{4} \alpha_i p_i \right) + h_K \right\|_{1,K}$$

$$\leq C \left( h_K^{-3/2} \left\| v - w_h \right\|_{0,K} + h_K^{-1/2} \left\| v - w_h \right\|_{1,K} + \sum_{i=1}^{4} |\alpha_i| \right).$$

Finally, we obtain

$$\left\| v_h \right\|_{1,K} \leq \left\| w_h \right\|_{1,K} + C h_K^{-1} \left( \left\| v - w_h \right\|_{0,K} + h_K \left\| v - w_h \right\|_{1,K} \right),$$

which, together with (11.10), yields $\|v_h\|_{1,\Omega} \leq C \|v\|_{1,\Omega}$.

Consequently, this element can be used to solve the Stokes problem in the three-dimensional case with an $O(h^2)$-error estimate.

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