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A Note on the Diophantine Equation $x^3 + y^3 + z^3 = 3$

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Abstract. Any integral solution of the title equation has $x \equiv y \equiv z$ (9).

The report of Scarowsky and Boyarsky [3] that an extensive computer search has failed to turn up any further integral solutions of the title equation prompts me to give the proof of a result which I noted many years ago and which might be of use in further work (cf. footnote on p. 505 of [2]).

THEOREM. *Any integral solution of*

$$x^3 + y^3 + z^3 = 3 \tag{1}$$

has

$$x \equiv y \equiv z\ (9). \tag{2}$$

Proof. Trivially,

$$x \equiv y \equiv z \equiv 1\ (3). \tag{3}$$

We work in the ring $\mathbf{Z}[\rho]$ of Eisenstein integers, where ρ is a cube root of unity. If $\alpha \in \mathbf{Z}[\rho]$ is prime to 3, then there is precisely one unit $\varepsilon = \pm\rho^j$ ($j = 0, 1, 2$) such that $\varepsilon\alpha \equiv 1$ (3). The supplement [1] to the law of cubic reciprocity states that if $\pi \in \mathbf{Z}[\rho]$ is prime, $\pi \equiv 1$ (3), then 3 is a cubic residue of π in $\mathbf{Z}[\rho]$ precisely when $\pi \equiv a$ (9) for some $a \in \mathbf{Z}$. It follows that if $\alpha \in \mathbf{Z}[\rho]$, $\alpha \equiv 1$ (3) and if 3 is congruent to a cube modulo α, then $\alpha \equiv b$ (9) for some $b \in \mathbf{Z}$.

Put

$$\alpha = -\rho^2 x - \rho y,$$

so

$$\alpha = x + (x - y)\rho \equiv 1\ (3)$$

by (3). By (1) we have $z^3 \equiv 3$ (α), so the preceding remarks apply. Hence $x - y \equiv 0$ (9). Finally, (2) follows by symmetry.

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1. G. Eisenstein, "Nachtrag zum cubischen Reciprocitätssatze...." *J. Reine Angew. Math.*, v. 28, 1844, pp. 28–35.

2. L. J. Mordell, "Integer solutions of $x^2 + y^2 + z^2 + 2xyz = n$," *J. London Math. Soc.*, v. 28, 1953, pp. 500–510.

3. M. Scarowsky & A. Boyarsky, "A note on the Diophantine equation $x^n + y^n + z^n = 3$," *Math. Comp.*, v. 42, 1984, pp. 235–236.