A Note on the Diophantine Equation

\[ x^3 + y^3 + z^3 = 3 \]

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Abstract. Any integral solution of the title equation has \( x = y = z \) (9).

The report of Scarowsky and Boyarsky [3] that an extensive computer search has failed to turn up any further integral solutions of the title equation prompts me to give the proof of a result which I noted many years ago and which might be of use in further work (cf. footnote on p. 505 of [2]).

**Theorem.** Any integral solution of

(1) \[ x^3 + y^3 + z^3 = 3 \]

has

(2) \[ x \equiv y \equiv z \pmod{9} \].

Proof. Trivially,

(3) \[ x \equiv y \equiv z \equiv 1 \pmod{3} \].

We work in the ring \( \mathbb{Z}[\rho] \) of Eisenstein integers, where \( \rho \) is a cube root of unity. If \( \alpha \in \mathbb{Z}[\rho] \) is prime to 3, then there is precisely one unit \( \epsilon = \pm \rho^j \) (\( j = 0, 1, 2 \)) such that \( \epsilon \alpha \equiv 1 \pmod{3} \). The supplement [1] to the law of cubic reciprocity states that if \( \pi \in \mathbb{Z}[\rho] \) is prime, \( \pi \equiv 1 \pmod{3} \), then 3 is a cubic residue of \( \pi \) in \( \mathbb{Z}[\rho] \) precisely when \( \pi \equiv a \pmod{9} \) for some \( a \in \mathbb{Z} \). It follows that if \( \alpha \in \mathbb{Z}[\rho] \), \( \alpha \equiv 1 \pmod{3} \) and if 3 is congruent to a cube modulo \( \alpha \), then \( \alpha \equiv b \pmod{9} \) for some \( b \in \mathbb{Z} \).

Put

\[ \alpha = -\rho^2 x - \rho y, \]

so

\[ \alpha = x + (x - y)\rho \equiv 1 \pmod{3} \]

by (3). By (1) we have \( z^3 \equiv 3 \pmod{9} \), so the preceding remarks apply. Hence \( x - y \equiv 0 \pmod{9} \). Finally, (2) follows by symmetry.

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