

Modular Multiplication Without Trial Division

By Peter L. Montgomery

Abstract. Let $N > 1$. We present a method for multiplying two integers (called N -residues) modulo N while avoiding division by N . N -residues are represented in a nonstandard way, so this method is useful only if several computations are done modulo one N . The addition and subtraction algorithms are unchanged.

1. Description. Some algorithms [1], [2], [4], [5] require extensive modular arithmetic. We propose a representation of residue classes so as to speed modular multiplication without affecting the modular addition and subtraction algorithms.

Other recent algorithms for modular arithmetic appear in [3], [6].

Fix $N > 1$. Define an N -residue to be a residue class modulo N . Select a radix R coprime to N (possibly the machine word size or a power thereof) such that $R > N$ and such that computations modulo R are inexpensive to process. Let R^{-1} and N' be integers satisfying $0 < R^{-1} < N$ and $0 < N' < R$ and $RR^{-1} - NN' = 1$.

For $0 \leq i < N$, let i represent the residue class containing $iR^{-1} \bmod N$. This is a complete residue system. The rationale behind this selection is our ability to quickly compute $TR^{-1} \bmod N$ from T if $0 \leq T < RN$, as shown in Algorithm REDC:

function REDC(T)

$m \leftarrow (T \bmod R)N' \bmod R$ [so $0 \leq m < R$]

$t \leftarrow (T + mN)/R$

if $t \geq N$ **then return** $t - N$ **else return** t ■

To validate REDC, observe $mN \equiv TN'N \equiv -T \bmod R$, so t is an integer. Also, $tR \equiv T \bmod N$ so $t \equiv TR^{-1} \bmod N$. Thirdly, $0 \leq T + mN < RN + RN$, so $0 \leq t < 2N$.

If R and N are large, then $T + mN$ may exceed the largest double-precision value. One can circumvent this by adjusting m so $-R < m \leq 0$.

Given two numbers x and y between 0 and $N - 1$ inclusive, let $z = \text{REDC}(xy)$. Then $z \equiv (xy)R^{-1} \bmod N$, so $(xR^{-1})(yR^{-1}) \equiv zR^{-1} \bmod N$. Also, $0 \leq z < N$, so z is the product of x and y in this representation.

Other algorithms for operating on N -residues in this representation can be derived from the algorithms normally used. The addition algorithm is unchanged, since $xR^{-1} + yR^{-1} \equiv zR^{-1} \bmod N$ if and only if $x + y \equiv z \bmod N$. Also unchanged are

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the algorithms for subtraction, negation, equality/inequality test, multiplication by an integer, and greatest common divisor with N .

To convert an integer x to an N -residue, compute $xR \bmod N$. Equivalently, compute $\text{REDC}((x \bmod N)(R^2 \bmod N))$. Constants and inputs should be converted once, at the start of an algorithm. To convert an N -residue to an integer, pad it with leading zeros and apply Algorithm REDC (thereby multiplying it by $R^{-1} \bmod N$).

To invert an N -residue, observe $(xR^{-1})^{-1} \equiv zR^{-1} \bmod N$ if and only if $z \equiv R^2x^{-1} \bmod N$. For modular division, observe $(xR^{-1})(yR^{-1})^{-1} \equiv zR^{-1} \bmod N$ if and only if $z \equiv x(\text{REDC}(y))^{-1} \bmod N$.

The Jacobi symbol algorithm needs an extra negation if $(R/N) = -1$, since $(xR^{-1}/N) = (x/N)(R/N)$.

Let $M|N$. A change of modulus from N (using $R = R(N)$) to M (using $R = R(M)$) proceeds normally if $R(M) = R(N)$. If $R(M) \neq R(N)$, multiply each N -residue by $(R(N)/R(M))^{-1} \bmod M$ during the conversion.

2. Multiprecision Case. If N and R are multiprecision, then the computations of m and mN within REDC involve multiprecision arithmetic. Let b be the base used for multiprecision arithmetic, and assume $R = b^n$, where $n > 0$. Let $T = (T_{2n-1}T_{2n-2} \cdots T_0)_b$ satisfy $0 \leq T < RN$. We can compute $TR^{-1} \bmod N$ with n single-precision multiplications modulo R , n multiplications of single-precision integers by N , and some additions:

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c ← 0
for i := 0 step 1 to n - 1 do
  (dTi+n-1 ··· Ti)b ← (0Ti+n-1 ··· Ti)b + N*(TiN' mod R)
  (cTi+n)b ← c + d + Ti+n
  [T is a multiple of bi+1]
  [T + cbi+n+1 is congruent mod N to the original T]
  [0 ≤ T < (R + bi)N]
end for
if (cT2n-1 ··· Tn)b ≥ N then
  return (cT2n-1 ··· Tn)b - N
else
  return (T2n-1 ··· Tn)b
end if

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Here variable c represents a delayed carry—it will always be 0 or 1.

3. Hardware Implementation. This algorithm is suitable for hardware or software. A hardware implementation can use a variation of these ideas to overlap the multiplication and reduction phases. Suppose $R = 2^n$ and N is odd. Let $x = (x_{n-1}x_{n-2} \cdots x_0)_2$, where each x_i is 0 or 1. Let $0 \leq y < N$. To compute $xyR^{-1} \bmod N$, set $S_0 = 0$ and S_{i+1} to $(S_i + x_i y)/2$ or $(S_i + x_i y + N)/2$, whichever is an integer, for $i = 0, 1, 2, \dots, n-1$. By induction, $2^i S_i \equiv (x_{i-1} \cdots x_0)y \bmod N$ and $0 \leq S_i < N + y < 2N$. Therefore $xyR^{-1} \bmod N$ is either S_n or $S_n - N$.

1. J. M. POLLARD, "Theorems on factorization and primality testing," *Proc. Cambridge Philos. Soc.*, v. 76, 1974, pp. 521–528.
2. J. M. POLLARD, "A Monte Carlo method for factorization," *BIT*, v. 15, 1975, p. 331–334.
3. GEORGE B. PURDY, "A carry-free algorithm for finding the greatest common divisor of two integers," *Comput. Math. Appl.* v. 9, 1983, pp. 311–316.
4. R. L. RIVEST, A. SHAMIR & L. ADLEMAN, "A method for obtaining digital signatures and public-key cryptosystems," *Comm. ACM*, v. 21, 1978, pp. 120–126; reprinted in *Comm. ACM*, v. 26, 1983, pp. 96–99.
5. J. T. SCHWARTZ, "Fast probabilistic algorithms for verification of polynomial identities," *J. Assoc. Comput. Mach.*, v. 27, 1980, pp. 701–717.
6. GUSTAVUS J. SIMMONS, "A redundant number system that speeds up modular arithmetic," Abstract 801-10-427, *Abstracts Amer. Math. Soc.*, v. 4, 1983, p. 27.