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Averaging Effects on Irregularities in the Distribution of Primes in Arithmetic Progressions

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Abstract. Let t be an integer taking on values between 1 and x (x real), let $\pi_{b,c}(t)$ denote the number of positive primes $\leqslant t$ which are $\equiv c \pmod b$, and let $\operatorname{li} t$ denote the usual integral logarithm of t. Further, let the ratio of quadratic nonresidues of $b > 2$ to quadratic residues of b be $\gamma(b)$ to 1, and let

$$A_b(x) = (1/\gamma(b))\frac{1}{x}\left(\sum_{\substack{t=1\\ 1\leqslant c\leqslant b-1}}^{x} \pi_{b,c}(t) - \gamma(b)\sum_{\substack{t=1\\ 1\leqslant c'\leqslant b-1}}^{x} \pi_{b,c'}(t)\right)$$

where c runs over quadratic nonresidues and c' runs over quadratic residues of b.

Nearly periodic oscillations of $A_6(x) = (1/x)\sum_{t=1}^{x}(\pi_{6,5}(t) - \pi_{6,1}(t))$ about $h(x) = (1/x)\sum_{t=1}^{x} \operatorname{li}(t^{1/2})/2$ are depicted in Figures 2, 3, 4 over the range of integers less than 2.5×10^{11}. Over this range, $h(x)$ is a far better "axis of symmetry" for these oscillations than $s(x) = (1/x)\sum_{t=1}^{x} t^{1/2}/\log t$ (suggested by Shanks [29]).

On the other hand, recent work of W. J. Ellison [9], three letters from Andrzej Schinzel to the author, and my own considerations (see Section 4) lead to the following. In contradiction to a conjecture of Shanks [29],

$$\frac{1}{x}\left(\sum_{t=1}^{x}(\pi_{6,5}(t) - \pi_{6,1}(t))/(t^{1/2}/\log t)\right) \nrightarrow 1 \quad \text{as } x \to \infty.$$

Moreover, I prove in Theorem 4.1 that $A_6(x)/h(x) \nrightarrow 1$ as $x \to \infty$, and Schinzel has provided a heuristic argument that no amount of averaging of $A_6(x)$ will provide an asymptotic relationship of this sort. However, let $h^{(1)}(x) = h(x)$, $A_6^{(1)}(x) = A_6(x)$, and for $k > 1$ let

$$h^{(k+1)}(x) = \frac{1}{x}\sum_{t=1}^{x} h^{(k)}(t), \qquad A_6^{(k+1)}(x) = \frac{1}{x}\sum_{t=1}^{x} A_6^{(k)}(t).$$

Assuming the truth of the generalized Riemann hypothesis for $L(s,\chi)$, χ the nonprincipal character mod 6, we prove

$$\lim_{k\to\infty}\ \overline{\lim_{x\to\infty}}\ \frac{A_6^{(k)}(x)}{h^{(k)}(x)} = 1 = \lim_{k\to\infty}\ \underline{\lim_{x\to\infty}}\ \frac{A_6^{(k)}(x)}{h^{(k)}(x)}.$$

The behavior of $A_6(x)$ is a special case of a far more general phenomenon. In Section 3, reasons are given why $A_b(x)$ can be expected to oscillate more or less symmetrically about $h(x)$ for every modulus $b > 2$.

1. Introduction and Summary. Throughout b will denote a modulus > 2; c will always denote a quadratic nonresidue and c' a quadratic residue of b. Let t be an integer with $1 \leqslant t \leqslant x$ (x real) and let $\pi_{b,c}(t)$ and $\pi_{b,c'}(t)$ denote, respectively, the

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number of primes $\leqslant t$ which are $\equiv c \pmod b$ or to $c' \pmod b$. For each $b = 2^{\alpha_0} q_1^{\alpha_1} \cdots q_r^{\alpha_r}$, $q_1, \ldots, q_r$ distinct odd primes, it is well-known (see, e.g., [26]) that the ratio of quadratic nonresidues to residues is $2^{r+\beta-1} - 1$ to 1 where $\beta = 1$ if $\alpha_0 = 0$ or 1, $\beta = 2$ if $\alpha_0 = 2$, and $\beta = 3$ if $\alpha_0 \geqslant 3$. Consequently, one would expect that

$$A_b(x) = \frac{1}{(2^{r+\beta-1} - 1)x} \sum_{\substack{t=1 \\ 1 \leqslant c \leqslant b-1}}^{x} \pi_{b,c}(t) - \frac{1}{x} \sum_{\substack{t=1 \\ 1 \leqslant c' \leqslant b-1}}^{x} \pi_{b,c'}(t) \tag{1.1}$$

would oscillate more or less evenly about 0. Daniel Shanks [29] first observed that for $x < 3.10^6$, and certain values of b, this is not the case.

Figures 2 and 3 at the conclusion of this paper depict oscillations of

$$A_6(x) = \frac{1}{x} \sum_{t=1}^{x} (\pi_{6,5}(t) - \pi_{6,1}(t))$$

about

$$h(x) = \frac{1}{x} \sum_{t=1}^{x} \frac{\operatorname{li} t^{1/2}}{2} \tag{1.2}$$

on standard scales and in Figure 4 they are depicted for $x < 250{,}000{,}000{,}000$ on a logarithmic scale. These figures show clearly that $h(x)$ is an excellent "axis of symmetry" for these nearly periodic oscillations. Indeed, over this considerable range, the approximation (1.2) is markedly superior to $s(x) = (1/x)\sum_{t=1}^{x} t^{1/2}/\log t$ although $h(x)$ and $s(x)$ are asymptotically equal (as we will see $h(x) - s(x) \sim (4/3)x^{1/2}/\log^2 x$).

In Section 4 we show that averaging in the ordinary sense (as in (1.1) and (1.2)) only postpones the swamping of the Chebyshev phenomenon by the giant fluctuations discovered by Hardy and Littlewood (see [16]).

In particular, with the help of A. Schinzel we prove

THEOREM 4.1. *It is false that*

$$\lim_{x \to \infty} \frac{A_6(x)}{h(x)} = 1.$$

Moreover, we give a heuristic argument that no amount of ordinary averaging will yield such a limit. This is interesting as it suggests that the Abelian averaging employed by, e.g., Knapowski and Turán [17]–[22] is in a sense stronger than ordinary averaging. Of course, the results of Knapowski and Turán are contingent on the truth of the generalized Riemann hypothesis as is our

THEOREM 4.2. *Let $A_6^{(k)}(x)$ and $h^{(k)}(x)$ denote the kth averaging of $A_6(x)$ and $h(x)$. Then*

$$\lim_{k \to \infty} \overline{\lim_{x \to \infty}} \frac{A_6^{(k)}(x)}{h^{(k)}(x)} = 1 = \lim_{k \to \infty} \underline{\lim_{x \to \infty}} \frac{A_6^{(k)}(x)}{h^{(k)}(x)}.$$

The nearly periodic behavior of the oscillations of $A_6(x)$ about $h(x)$ is interesting and should be compared with the important works of Pólya [27], Bloch and Pólya [5], and Grosswald [10].

2. Preliminaries–Computation of $s(x)$ and $h(x)$. Integration by parts and a straightforward treatment of the error term (see, e.g., [1]), yields for each fixed k_1,

$$
\begin{aligned}
s(x) = \frac{1}{x}\sum_{t=1}^{x} t^{1/2}/\log t &= \frac{2x^{1/2}}{3\log x} + \frac{4x^{1/2}}{9\log^2 x} + \frac{16x^{1/2}}{27\log^3 x} \\
&+ \frac{32x^{1/2}}{27\log^4 x} + \cdots + \frac{\left(\frac{2}{3}\right)^{k_1}(k_1-1)!x^{1/2}}{\log^{k_1} x} + \mathcal{O}\frac{x^{1/2}}{\log^{k_1+1} x}.
\end{aligned} \tag{2.1}
$$

Our computer program utilized a simple trapezoidal rule. Five terms of the above expansions are sufficient for a high level of accuracy.

Moreover,

$$
\begin{aligned}
\frac{\operatorname{li} x^{1/2}}{2} &= \frac{x^{1/2}}{\log x} + \frac{2x^{1/2}}{\log^2 x} + \frac{8x^{1/2}}{\log^3 x} + \frac{48x^{1/2}}{\log^4 x} \\
&+ \cdots + \frac{2^{k_2-1}(k_2-1)!x^{1/2}}{\log^{k_2} x} + \mathcal{O}\left(\frac{x^{1/2}}{\log^{k_2+1} x}\right).
\end{aligned} \tag{2.2}
$$

Consequently, we may integrate by parts to obtain for each fixed k_3,

$$
\begin{aligned}
h(x) &\sim \lim_{\eta\to 0^+} \frac{1}{x}\left(\left(\int_0^{1-\eta} + \int_{1+\eta}^{x}\right)\frac{\operatorname{li} t^{1/2}}{2}\,dt\right) \\
&= \frac{2x^{1/2}}{3\log x} + \frac{16x^{1/2}}{9\log^2 x} + \frac{208x^{1/2}}{27\log^3 x} + \frac{1280x^{1/2}}{27\log^4 x} + \frac{30976x^{1/2}}{81\log^5 x} \\
&+ \cdots + \frac{2^{k_3}(k_3-1)!(1+3+9+\cdots+3^{k_3-1})x^{1/2}}{3^{k_3}\log^{k_3} x} + \mathcal{O}\left(\frac{x^{1/2}}{\log^{k_3+1} x}\right) \\
&= \sum_{j=1}^{k_3} \frac{2^{j-1}(j-1)!(3^j-1)x^{1/2}}{3^j\log^j x} + \mathcal{O}\left(\frac{x^{1/2}}{\log^{k_3+1} x}\right).
\end{aligned} \tag{2.3}
$$

The reader may find it easier to verify (2.3) by differentiation than to derive it by integration. The implied constant in (2.3) is significantly larger than in (2.1). Consequently, the most serious error present in Figures 2, 3, 4 arises from the failure to take $k_3 > 5$ in using (2.3). However, even at 10^{11}, the use of seven or eight terms would raise the curve representing $h(x)$ by an almost imperceptible amount.

Let $h^{(2)}(x) = (1/x)\sum_{t=1}^{x} h(t)$. By means of a second integration by parts we have for each fixed k_4,

$$
\begin{aligned}
h^{(2)}(x) &\sim \frac{4x^{1/2}}{9\log x} + \frac{40x^{1/2}}{27\log^2 x} + \frac{576x^{1/2}}{81\log^3 x} + \frac{11136x^{1/2}}{243\log^4 x} + \cdots \\
&+ \frac{2^{k_4+1}(k_4-1)!(1+1+3+1+3+9+\cdots+1+3+9+\cdots+3^{k_4-1})x^{1/2}}{3^{k_4+1}\log^{k_4} x} \\
&+ \mathcal{O}\left(\frac{x^{1/2}}{\log^{k_4+1} x}\right) \\
&= \sum_{j=1}^{k_4} \frac{2^{j-1}(j-1)!(3^{j+1}-2j-3)x^{1/2}}{3^{j+1}\log^j x} + \mathcal{O}\left(\frac{x^{1/2}}{\log^{k_4+1} x}\right).
\end{aligned} \tag{2.4}
$$

Remark. The numerically inclined reader may be interested in plotting $(1/x)\sum_{t=1}^{x} A_6(t)$ versus $h^{(2)}(x)$ for his own curiosity. In doing so, one might choose to use (over a small range)

$$\text{(2.5)} \qquad \operatorname{li} x = \lim\left(\int_0^{1-\eta} + \int_{1-\eta}^{x}\right)\frac{du}{\log u} = \gamma + \log\log x + \sum_{k=1}^{\infty}\frac{(\log x)^k}{k!k}$$

in preference to the asymptotic expansion for li x;

$$\gamma = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n} - \log N \approx .577.$$

However, good convergence for this expansion at, say 10^{11}, requires nearly 80 terms.

3. The Isolation of the "Negative Part" of $A_b(x)$. An excellent account of the standard argument for the modulus 4 may be found in Ingham [16, pp. 106–107] (although the replacement of $\pi(x^{1/2})/2$ by $x^{1/2}/\log x$ on p. 107 is unnecessary).

Recently, the author has given a purely elementary argument [14] which suggests that $\pi_{4,3}(x) - \pi_{4,1}(x)$ oscillates about $\pi(x^{1/2})/2$ at least for small x. A generalization (see [14]) of this argument motivates what follows.

Let $b = 2^{\alpha_0}q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_r^{\alpha_r}$ where $q_1,\ldots,q_r$ are distinct odd primes. The ratio of quadratic nonresidues of b to quadratic residues of b is $2^{r+\beta-1}-1$ to 1 where $\beta = 1$ if $\alpha_0 = 0$ or 1, $\beta = 2$ if $\alpha_0 = 2$, and $\beta = 3$ if $\alpha_0 \geqslant 3$ (see, e.g., [26, p. 167]). With this in mind, we define

$$\text{(3.1)} \qquad \tau_b(s) = \begin{cases} 2^{r+\beta-1}-1 & \text{if } s \text{ is a quadratic residue of } b, \\ -1 & \text{if } s \text{ is a quadratic nonresidue of } b, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we define analogues of $\pi(x)$ and $\Pi(x)$ by

$$\text{(3.2)} \qquad \pi(x,\tau_b) = \sum_{p\leqslant x}\tau_b(p)$$

where p is prime, and

$$\text{(3.3)} \qquad \Pi(x,\tau_b) = \sum_{p^m\leqslant x}\frac{\tau_b(p^m)}{m}.$$

In the following theorem we let c' denote a quadratic residue of b and c a quadratic nonresidue of b.

THEOREM 3.1. *Let $b = 2^{\alpha_0}\cdot q_1^{\alpha_1}\cdot q_2^{\alpha_2}\cdot q_3^{\alpha_3}\cdots q_r^{\alpha_r}$, where $b > 2$ and $q_1,\ldots,q_r$ are distinct odd primes. Then,*

$$\text{(3.4)} \qquad \begin{aligned} &\sum_{1\leqslant c'\leqslant b-1}\pi_{b,c'}(x) - \frac{1}{2^{r+\beta-1}-1}\Bigg(\sum_{1\leqslant c\leqslant b-1}\pi_{b,c}(x)\Bigg) \\ &\qquad = \frac{\pi(x^{1/2})}{2}\left[\left(-1 + \frac{2}{2^{r+\beta-1}-1}\left(\frac{\Pi(x,\tau_b)}{\pi(x^{1/2})}\right)\right) + o(1)\right]. \end{aligned}$$

Proof. Note that

$$\pi(x, \tau_b) = (2^{r+\beta-1} - 1) \sum_{1 \leqslant c' \leqslant b-1} \pi_{b,c'}(x) - \sum_{1 \leqslant c \leqslant b-1} \pi_{b,c}(x). \tag{3.5}$$

Moreover, since $\tau_b(p^2) = +1$ for every prime p with $(b, p) = 1$, we have (see p. 107 of [16])

$$\Pi(x, \tau_b) - \pi(x, \tau_b) = \left(\frac{(2^{r+\beta-1} - 1)\pi(x^{1/2})}{2} \right) + \mathcal{O}(x^{1/3} \log x). \tag{3.6}$$

Now (3.4) follows easily from (3.5) and (3.6).

The extent to which (3.2) and (3.3) are natural analogues of $\pi(x)$ and $\Pi(x)$, respectively, is readily seen by noting that Theorem 3.1 reduces to the classical result in Ingham [16] if $b = 4$ (or $b = 6$). Specifically, if $b = 4$ then τ_b is the real nonprincipal Dirichlet character, χ_1, and $\Pi(x, \tau_b)$ is the function "naturally associated with $\log L(s, \chi_1)$". As such, $\Pi(x, \chi_1)$ can be expected to have values fairly evenly distributed about zero. Consequently, $-\pi(x^{1/2})/2$ is called by Ingham the "negative part" and $\Pi(x, \chi_1)$ the "oscillating part" of (3.4) when $b = 4$.

Let $b > 2$ be a modulus admitting a primitive root, i.e., $b = 4$, q^α, or $2q^\alpha$, where q is an odd prime. Then τ_b is the real nonprincipal Dirichlet character (or Kronecker symbol) and the above argument is seen to be a natural special case.

When b does not admit a primitive root, $2^{r+\beta-1} - 1 > 1$. The resultant awkwardness in the appearance of (3.1) is (regrettably) necessary if one wishes to extend the above argument to arbitrary modulus. The reason that I find most compelling for feeling that for every b the isolation of the term,

$$\frac{-\pi(x^{1/2})}{2}(1 + o(1)) \tag{3.7}$$

in (3.4) is meaningful (at least for small x), is the fact that the elementary theory in Section 6 of [14] (see (6.14) of [14]) suggests that

$$\frac{1}{2^{r+\beta-1} - 1} \left(\sum_{1 \leqslant c \leqslant b-1} \pi_{b,c}(t) \right) - \sum_{1 \leqslant c' \leqslant b-1} \pi_{b,c'}(t) \tag{3.8}$$

oscillates about $(\pi(t^{1/2})/2)(1 + o(1))$.

Of course, if $b = 4$ or 6, the result of Littlewood [25] shows that the "negative part" is overcome infinitely often. (The case for general moduli is far from solved; for an interesting recent result, see Stark [32].) Moreover, the results of Ellison [9] and Theorem 4.1 of this paper show that (for $b = 4$ or 6), even in the mean, the "oscillatory part" is not negligible relative to the "negative part."

4. Limitations of ordinary averaging. Recently W. J. Ellison [9] has disproven the conjecture of Shanks [29] that

$$\lim_{x \to \infty} \frac{1}{x} \sum_{t=1}^{x} \frac{\pi_{4,3}(t) - \pi_{4,1}(t)}{t^{1/2}/\log t} = 1. \tag{4.1}$$

A modification of this argument suggested to me by A. Schinzel (personal communication, May 30, 1977) yields the following theorem.

THEOREM 4.1. *It is false that*

$$\lim_{x\to\infty} \frac{A_6(x)}{h(x)} = 1. \tag{4.2}$$

Proof. Let $\chi(n) = (n/3)$ and

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \Pi(x,\chi) = \sum_{p^m \leqslant x} \frac{\pi(p^m)}{m} \tag{4.3}$$

so that

$$\Pi(t,\chi) - \pi(t,\chi) = \frac{\pi(t^{1/2})}{2} + \mathcal{O}(t^{1/3}\log t). \tag{4.4}$$

Assume that $A_6(x)/h(x) \to 1$ as $x \to \infty$.

If x is an integer, we have

$$\begin{aligned} \int_1^x \Pi(t,\chi)\,dt &= \int_1^x \pi(t,\chi)\,dt + \int_1^x \frac{\pi(t^{1/2})}{2}\,dt + \mathcal{O}\left(\int_1^x t^{1/3}\log t\,dt\right) \\ &= \sum_{t=1}^{x-1} \pi(t_1,\chi) + \sum_{t=1}^{x-1} \frac{\pi(t^{1/2})}{2} + \mathcal{O}(x^{4/3}\log x) \\ &= -(x-1)A_6(x-1) + (x-1)h(x-1) + \mathcal{O}(x^{4/3}\log x). \end{aligned} \tag{4.5}$$

But, because of our assumption,

$$-A_6(x-1) + h(x-1) = o(h(x-1)) = o(x^{1/2}/\log x) \tag{4.6}$$

so that

$$-(x-1)A_6(x-1) + (x-1)h(x-1) = o(x^{3/2}/\log x). \tag{4.7}$$

If x is not an integer,

$$\int_1^x \Pi(t,\chi)\,dt = \int_1^{[x]} \Pi(t,\chi)\,dt + \mathcal{O}(\Pi(x,\chi)), \tag{4.8}$$

and this is

$$= o(x^{3/2}/\log x) + \mathcal{O}(x/\log x) = o(x^{3/2}/\log x). \tag{4.9}$$

Thus, in any case,

$$\Pi_1(x,\chi) = \int_1^x \Pi(t,\chi)\,dt = o(x^{3/2}/\log x). \tag{4.10}$$

On the other hand, for $\operatorname{Re} s > 1$,

$$\begin{aligned} \log L(s,\chi) &= s\int_1^\infty \frac{\Pi(x,\chi)}{x^{s+1}}\,dx \\ &= s \lim_{y\to\infty} \frac{\Pi_1(y,\chi)}{y^{s+1}} + (s+1)\int_1^y \frac{\Pi_1(x,\chi)}{x^{s+2}}\,dx \\ &= s(s+1)\int_1^\infty \frac{\Pi_1(x,\chi)}{x^{s+2}}\,dx; \end{aligned} \tag{4.11}$$

see (6.23) of [9].

Now (4.10) implies the convergence of the integral on the right-hand side for $\operatorname{Re} s > \frac{1}{2}$, hence the nonvanishing of $L(s, \chi)$ to the right of $\sigma = \frac{1}{2}$. Consider the function

$$\begin{aligned} f(s) &= -\frac{d}{ds}\left(\frac{\log L(s,\chi)}{s(s+1)}\right) \\ &= \frac{-1}{s(s+1)}\frac{L'(s,\chi)}{L(s,\chi)} + \frac{2s+1}{s^2(s+1)^2}\log L(s,\chi). \end{aligned} \tag{4.12}$$

By (4.11), we have

$$f(s) = \frac{-d}{ds}\int_1^\infty \frac{\Pi_1(x,\chi)}{x^{s+2}}dx = \int_1^\infty \frac{\Pi_1(x,\chi)\log x}{x^{s+2}}dx. \tag{4.13}$$

If now, $s = \sigma + it$, $\sigma \to (1/2)^+$, and (4.10) holds, we have

$$|f(x)| = o\left(\int_1^\infty \frac{x^{3/2}}{|x^{s+2}|}dx\right) = o\left(\frac{1}{\sigma - \frac{1}{2}}\right), \tag{4.14}$$

a contradiction, since then $L(s, \chi)$ has no zero on the line $\sigma = \frac{1}{2}$!

Using the Riemann ζ function it is easy to modify the above to show the falsity of Shanks' conjecture [29] that

$$\lim_{x\to\infty} \frac{1}{x}\sum_{t=1}^{x} \frac{\operatorname{li} t - \pi(t)}{t^{1/2}/\log t} = 1. \tag{4.15}$$

The well-known result of Hardy and Littlewood [11] and Landau [23], [24], rephrased as on p. 276 of [29], states that there exists a positive constant k with

$$A_6(x) > kx^{1/2}/\log x \tag{4.16}$$

if and only if the generalized Riemann hypothesis is true for $L(s, \chi)$, χ the nonprincipal character mod 6. If (4.2) were true, one would have the very much stronger result,

$$A_6(x) \sim 2x^{1/2}/3\log x \quad (\text{as } x \to \infty). \tag{4.17}$$

Seen in this light the falsity of (4.1) is not entirely surprising in spite of the relatively good behavior of $A_6(x)$ over the first 250 billion integers.

Indeed, Schinzel has communicated the following heuristic argument that no amount of averaging will yield an asymptotic relationship of this sort. In particular, letting $h^{(1)}(x) = h(x)$, $A_6^{(1)}(x) = A_6(x)$, and for $k > 1$, letting

$$h^{(k+1)}(x) = \frac{1}{x}\sum_{t=1}^{x} h^{(k)}(t), \qquad A_6^{(k+1)}(x) = \frac{1}{x}\sum_{t=1}^{x} A_6^{(k)}(t), \tag{4.18}$$

it appears implausible that for any fixed k, we have

$$A_6^{(k)}(x) \sim h^{(k)}(x). \tag{4.19}$$

For note that (see, e.g., [23, Section 138])

$$\pi_1(x,\chi) = -\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)\log x} + \mathcal{O}\left(\frac{x^\theta}{(\log x)^2}\right), \tag{4.20}$$

where ρ runs through all zeros of $L(s, \chi)$ in the critical strip and θ is the upper bound of their real parts.

If (4.19) holds and $\theta = \frac{1}{2}$ we should have

$$\sum_{\rho} \frac{x^{\rho}}{\rho(\rho+1)^{k}} = o(x^{1/2}), \tag{4.21}$$

but every term on the left-hand side of (4.21) has order $x^{1/2}$.

Prior to receiving Schinzel's letter, dated September 4, 1977, I had overlooked the above argument although I had noted (as did Schinzel independently) that from

$$\lim_{k \to \infty} \sum_{\rho} \frac{1}{|\rho|\,|\rho+1|^{k}} = 0, \tag{4.22}$$

the following theorem follows at once, assuming the truth of the generalized Riemann hypothesis for $L(s, \chi)$.

THEOREM 4.2. *Let $A^{(k)}(x)$ and $h^{(k)}(x)$ be defined as above. Then*

$$\lim_{k \to \infty} \overline{\lim_{x \to \infty}} \frac{A_6^{(k)}(x)}{h^{(k)}(x)} = 1 = \lim_{k \to \infty} \underline{\lim_{x \to \infty}} \frac{A_6^{(k)}(x)}{h^{(k)}(x)}. \tag{4.23}$$

Remark. Modifications of the above arguments for $b = 4$ or for $b = 2$ (li x versus $\pi(x)$) are easily obtained. It is, consequently, improbable that for any fixed $k \geqslant 1$, the kth average of li $x - \pi(x)$ is asymptotic to the kth average of li $x^{1/2}/2$. It is possible that Theorem 4.1 could be generalized to rigorously prove this but I have not been able to obtain this.

5. Data on $\pi_{6,5}(t) - \pi_{6,1}(t)$ and Pictorial Description of the Oscillations of $A_6(x)$ About $h(x)$. Our computations were carried out to 2.5×10^{11} largely in the hope of finding the smallest integer t with $\pi_{6,5}(t) < \pi_{6,1}(t)$; for the eventual success of this venture, see [4].

Figure 1 gives a value of t with $\pi_{6,5}(t) - \pi_{6,1}(t) > \pi(t^{1/2})$—a rare "inverted axis crossing"—and some "near axis crossings" for $2{,}000{,}000 < t < 250{,}000{,}000{,}000$. A "near axis crossing" is a value of t with $\pi_{6,5}(t) - \pi_{6,1}(t) < 1500$. The values listed in Figure 1 are pictured in Figure 4. Interestingly, they mostly occur in the vicinity of an inflection point of the graph of $A_6(x)$.

Point (see Figure 4)	t	$\pi_{6,5}(t) - \pi_{6,1}(t)$	$\pi(t^{1/2})$
a	344,558,471	2310	2125
b	2,471,075,683	627	5106
c	4,450,687,051	608	6654
d	5,278,986,859	573	7190
e	7,068,537,481	543	8197
f	13,121,332,621	1117	10834
g	24,411,171,451	1193	14379
h	27,182,104,207	730	15088
i	121,269,714,529	1220	29834

FIGURE 1

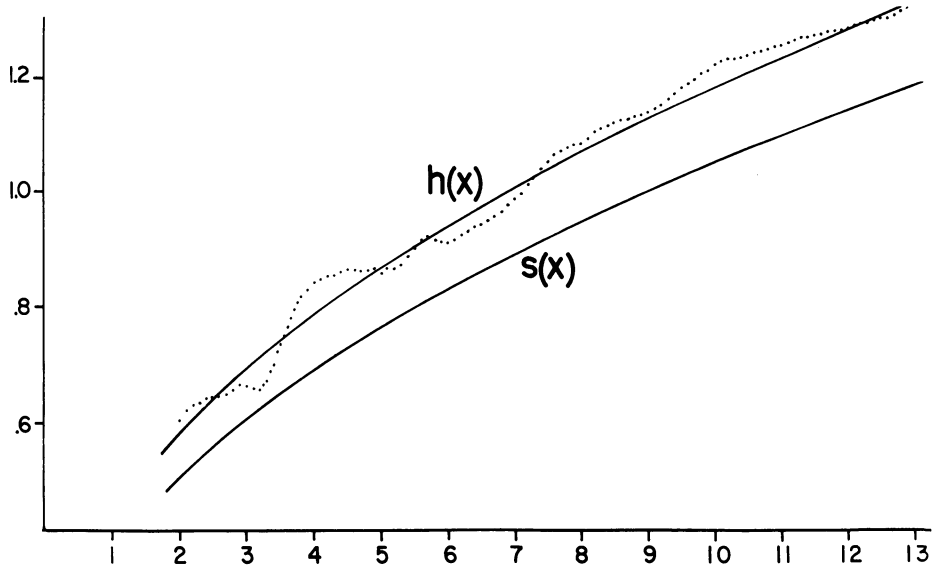

FIGURE 2

In this figure, and in Figure 3, *values for* $h(x)$, $s(x)$, *and* $A_6(x)$ *have been plotted on standard (increasing) scales; the dotted line represents* $A_6(x)$. *Values for* x *are on the horizontal axis and extend from* $2 \cdot 10^8$ *to* $13 \cdot 10^8$.

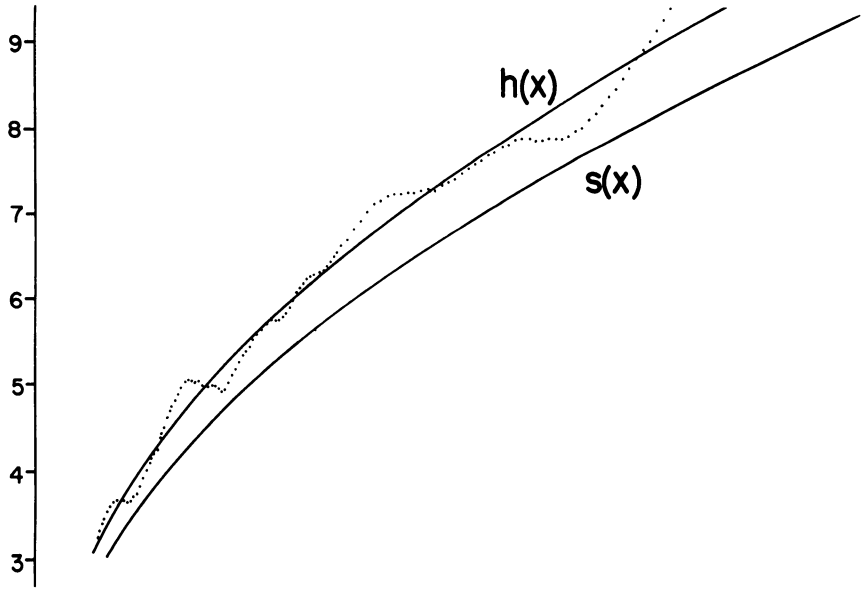

FIGURE 3

In this figure, x *ranges from* $1 \cdot 10^9$ *to* $13 \cdot 10^9$.

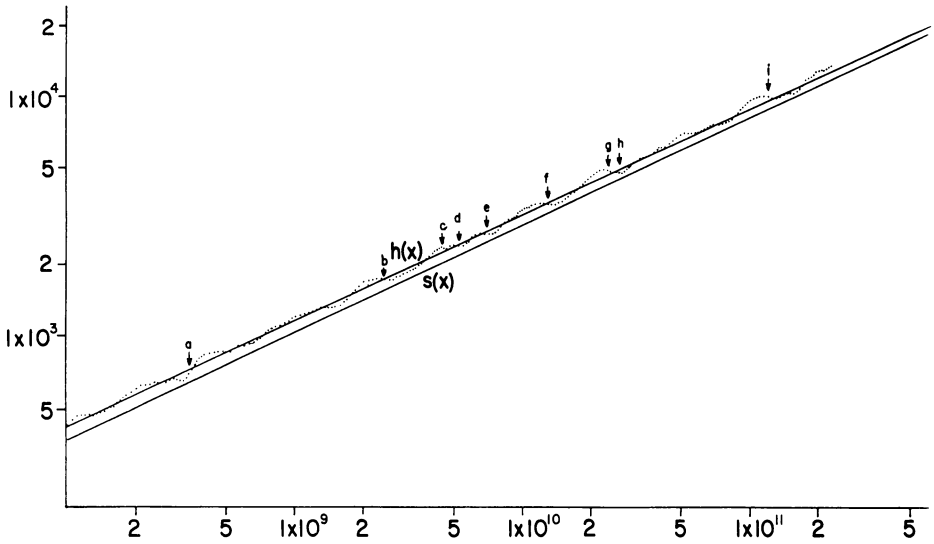

FIGURE 4

Values of $h(x)$, $s(x)$, and $A_6(x)$ have been plotted on a log log *scale. Values for x extend from 10^8 to $2.5 \cdot 10^{11}$. The locations of points listed in Figure* 1 *are indicated by arrows.*

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