Remark on a Lemma by R. Wong and J. P. McClure

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Abstract. A short proof is presented for a formula arising in the above-mentioned paper.

In [1] the authors prove the following lemma:

Let \( f \) and \( g \) be \( C^\infty \)-functions, and let \( n \) be a nonnegative integer. Then

\[
\begin{align*}
\left[ f(x)g^{n+1}(x) \right]^{(n+1)} &= - \sum_{p=0}^{n} \binom{n+1}{p+1} [f(x)g^{n-p}(x)]^{(n+1)}(-g(x))^{p+1} \\
&\quad + (n+1)!f(x)(g'(x))^{n+1}.
\end{align*}
\]

This formula is important in deriving a Taylor series expansion for the Dirac \( \delta \)-function. The proof of the lemma is rather complicated and covers two pages in print. Therefore the following simple proof may be of interest.

**Proof.** Obviously

\[
g(t) - g(x) = (t-x)(g'(x) + \epsilon(t,x)),
\]

where for fixed \( x \) the function \( \epsilon(t,x) \) belongs to \( C^\infty \) with respect to \( t \), and

\[
\lim_{t \to x} \epsilon(t,x) = 0.
\]

Then

\[
\begin{align*}
\left[ f(x)g^{n+1}(x) \right]^{(n+1)} &+ \sum_{p=0}^{n} \binom{n+1}{p+1} [f(x)g^{n-p}(x)]^{(n+1)}(-g(x))^{p+1} \\
&= \left( \frac{d^{n+1}}{dt^{n+1}} [f(t)(g(t) - g(x))^{n+1}] \right)_{t=x} \\
&= \left( \frac{d^{n+1}}{dt^{n+1}} [(t-x)^{n+1}f(t)(g'(x) + \epsilon(t,x))^{n+1}] \right)_{t=x} \\
&= (n+1)!f(x)(g'(x))^{n+1},
\end{align*}
\]

where the first equality follows from the binomial theorem, and the last equality is obtained by using the Leibniz rule.
The author is grateful for the referee's comments, which helped to improve the style of the presentation.

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