

The Maximal Modulus of an Algebraic Integer

By David W. Boyd*

Abstract. The maximal modulus of an algebraic integer is the absolute value of its largest conjugate. We compute the minimum of the maximal modulus of all algebraic integers of degree d which are not roots of unity, for d at most 12. The computations suggest that the minimum is never attained for a reciprocal algebraic integer. The truth of this conjecture would show that the conjecture of Schinzel and Zassenhaus follows from a theorem of Smyth. We further test our conjecture by computing the minimum of the maximal modulus of all reciprocal algebraic integers of degree d which are not roots of unity, for d at most 16. Our computations strongly suggest that the best constant in the conjecture of Schinzel and Zassenhaus is $1.5 \log \theta_0$, where θ_0 is the smallest P.V. number. They also shed some light on a recent conjecture of Lind concerning the Perron numbers.

1. Introduction. Let α be an algebraic integer of degree d , with conjugates $\alpha_1, \dots, \alpha_d$. As usual, let $|\bar{\alpha}| = \max |\alpha_i|$ denote the maximal modulus of α . Clearly, $|\bar{\alpha}| \geq 1$, and a theorem of Kronecker [4] tells us that $|\bar{\alpha}| = 1$ if and only if α is a root of unity. Schinzel and Zassenhaus [9] have made the following conjecture:

CONJECTURE (SZ). There is a constant $c_1 > 0$ such that if α is not a root of unity, then $|\bar{\alpha}| \geq 1 + c_1/d$.

In this paper we describe the computation of the minimum of $|\bar{\alpha}|$ for α of degree d , with $d \leq 12$. The results suggest a conjecture which, when combined with a result of Smyth [10], implies (SZ). Our results also suggest that the best constant c_1 in (SZ) should be $\frac{3}{2} \log \theta_0$, where $\theta_0 = 1.3247\dots$ is the smallest Pisot number (the real zero of $x^3 - x - 1$).

The results also shed some light on a conjecture of Lind concerning the "Perron numbers" introduced in [6] and [7].

2. Conjectures Implying (SZ). The best results to date concerning (SZ) have been obtained as corollaries to results on a question of Lehmer. Let $M(\alpha) = \prod_{i=1}^d \max(|\alpha_i|, 1)$ denote the Mahler measure of α . Lehmer [5] asked:

(L) Does there exist a constant $c_0 > 1$ so that $M(\alpha) \geq c_0$ for all α not roots of unity?

A positive answer to (L) would prove (SZ), for, if ν is the number of α_i satisfying $|\alpha_i| > 1$, then clearly $M(\alpha) \leq |\bar{\alpha}|^\nu$. Thus,

$$|\bar{\alpha}| \geq M(\alpha)^{1/\nu} \geq M(\alpha)^{1/d} \geq c_0^{1/d} \geq 1 + c_1/d.$$

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TABLE 1

Extreme values of $|\bar{\alpha}|$ for fixed degree d . The minimum $m(d)$ is attained for α with minimal polynomial $P_d(x)$ having ν roots outside the unit circle.

d	ν	$m(d)$	$P_d(x)$
1	1	2	$x - 2$
2	2	$2^{1/2} = 1.4142135624$	$x^2 - 2, x^2 + x + 2$ or $x^2 + 2x + 2$
3	2	$\theta_0^{1/2} = 1.1509639253$	$x^3 + x - 1$
4	2	1.1837518186	$x^4 + x^3 + 1$ or $x^4 + x + 1$
5	4	1.1216451786	$x^5 - x^3 + x^2 + x - 1$
6	4	$\theta_0^{1/4} = 1.0728298678$	$P_3(x^2)$
7	4	1.0928455996	$x^7 + x^6 + x^3 - x - 1$
n8	6	1.0756204773	$x^8 + x^7 + x^4 - x^2 + 1$
9	6	$\theta_0^{1/6} = 1.0479821944$	$P_3(x^3)$
10	8	1.0590775130	$P_5(x^2)$
11	8	1.0571248570	$x^{11} + x^{10} + x^7 + x^6 - x^4 + x^2 - 1$
12	8	$\theta_0^{1/8} = 1.0357750083$	$P_3(x^4)$

However, it is conceivable that (SZ) could be true, and yet the answer to (L) could be negative.

Smyth [10] proved that if α is nonreciprocal (i.e., α_i^{-1} is not a conjugate of α for any i), then $M(\alpha) \geq \theta_0$. Hence, $|\bar{\alpha}| \geq 1 + (\log \theta_0)/d$ for nonreciprocal α . Smyth also pointed out that the α with minimal polynomial $x^{3k} + x^{2k} - 1$ (so $d = 3k$), has $|\bar{\alpha}| = \theta_0^{1/(2k)} = \theta_0^{3/(2d)}$, so one cannot improve this beyond $|\bar{\alpha}| \geq 1 + \frac{3}{2}(\log \theta_0)/d$.

On the other hand, it is known that, for reciprocal α , one can definitely have $1 < M(\alpha) < \theta_0$. Indeed, Lehmer [5] provided an example α_0 with $d = 10$ for which $M(\alpha_0) = 1.17628\dots < \theta_0$. It is widely felt that α_0 may be the best constant in (L). There are many other examples in [1]. For reciprocal α , Dobrowolski [3] has shown that

$$M(\alpha) \geq 1 + c_2 \left(\frac{\log \log d}{\log d} \right)^3,$$

from which a result slightly weaker than (SZ) follows.

It should be pointed out that the known reciprocal α with small measure (as listed in [1], for example) do not have $|\bar{\alpha}|$ particularly small, since ν is too small. For example, Lehmer's 10th degree α_0 has $\nu = 1$ and, hence, $|\bar{\alpha}_0| = M(\alpha_0) = 1.17628\dots$. Even the naive guess $\alpha = \sqrt[10]{2}$ has $|\bar{\alpha}| = 1.07177\dots$, while the minimum of $|\bar{\alpha}|$ for degree 10 is $1.05907\dots$, which is considerably smaller (see Table 1).

Let $m(d)$ denote the minimum of $|\bar{\alpha}|$ over α of degree d which are not roots of unity. It is easy to see that this is an attained minimum. Let an α attaining $m(d)$ be called extremal. Then our computations, as summarized in Tables 1 and 2 suggest the following:

CONJECTURE (A). Extremal α are always nonreciprocal.

CONJECTURE (B). If $d = 3k$, then the extremal α has minimal polynomial $x^{3k} + x^{2k} - 1$ (or $x^{3k} - x^{2k} - 1$).

CONJECTURE (C). The extremal α of degree d have $\nu \sim \frac{2}{3}d$ as $d \rightarrow \infty$.

TABLE 2

Extreme values of $|\bar{\alpha}|$ for reciprocal α of even degree d . The minimum $m_R(d)$ is attained for an α with minimal polynomial $R_d(x)$ having ν roots outside the unit circle.

d	ν	$m_R(d)$	$R_d(x)$
2	1	2.6180339887	1 - 3 1
4	2n	1.5392223384	1 1 3 1 1
6	2	1.3216631562	1 2 2 1 2 2 1
8	2	1.1692830298	1 0 0 1 1 1 0 0 1
10	2	1.1257148215	1 0 1 1 0 1 0 1 1 0 1
12	2	1.1080548536	1 1 1 0 -1 -1 -1 -1 -1 0 1 1 1
14	4	1.0939016857	1 0 0 0 1 1 0 1 0 1 1 0 0 0 1
16	4	1.0813339123	$R_8(x^2)$

Perhaps (C) seems far-fetched on the basis of Table 1. However, the evidence for (B) is clear, and it does appear that $\nu(d)$ is monotone. These would imply (C).

Note that (A) implies (SZ) with $c_1 = \log \theta_0$, while (C) implies that the best constant is $c_1 = \frac{3}{2} \log \theta_0$.

Since the computation for $d = 12$ was rather lengthy, it is not feasible to extend it to $d \geq 13$. However, we were able to test (A) up to $d = 16$ by computing $m_R(d)$, the minimum of $|\bar{\alpha}|$ over reciprocal α of degree d which are not roots of unity. Since $m_R(2k) > m(k)^{1/2} \geq m(2k)$ for $k \leq 8$, we thus have verified (A) for $d \leq 16$.

3. Perron Numbers. Lind [6] has defined a Perron number to be a real algebraic integer $\alpha = \alpha_1$ such that $\alpha_1 > |\alpha_i|$ for $i \geq 2$.

By the Perron-Frobenius theorem, if A is a matrix with nonnegative integer entries and such that A^k has positive entries for some k , then the dominant eigenvalue α of A is a Perron number. Lind has proved the converse [6], [7]. (Note that the dimension of A may have to be larger than $\deg(\alpha)$, e.g., if α has negative trace.)

In private correspondence, Lind conjectured that the smallest Perron number of degree $d \geq 2$ should have minimal polynomial $x^d - x - 1$. This turns out to be true if $d = 2, 3, 4, 6, 7, 8, 10, 12$, but false if $d > 3$ and $d \equiv 3$ or $5 \pmod{6}$. A slight modification of the conjecture is true up to degree 12.

The reason for the modification is the following: It is known [8] that if $(n, m) = 1$, then $x^n - x^m - 1$ is either irreducible or the product of $x^2 - x + 1$ and an irreducible polynomial. (One can now derive this in a few lines from Smyth's theorem [10] and the fact that $M(x^n - x^m - 1) \leq \sqrt{3} < \theta_0^2$.) For $(n, m) = 1$, $x^n - x^m - 1$ can have the factor $x^2 - x + 1$ only if $n \equiv 1$ or $5 \pmod{6}$ and $m + n \equiv 3 \pmod{6}$. Let us now compare the size of α , the positive root of $x^d - x - 1$, with β , the positive root of $x^{d+2} - x^m - 1$. If $d > 3$ and $m \leq 4$, then

$$\begin{aligned} \alpha^{d+2} - \alpha^m - 1 &\geq \alpha^{d+2} - \alpha^4 - 1 = \alpha^2(\alpha + 1) - \alpha^4 - 1 \\ &= -(\alpha - 1)(\alpha^3 - \alpha - 1) > 0, \end{aligned}$$

since $\alpha^3 - \alpha - 1 < \alpha^d - \alpha - 1 = 0$. Thus $\beta < \alpha$.

On the other hand, if $m \geq 5$, then

$$\begin{aligned} \alpha^{d+2} - \alpha^m - 1 &\leq \alpha^{d+2} - \alpha^5 - 1 = \alpha^2(\alpha + 1) - \alpha^5 - 1 \\ &= -(\alpha^2 - 1)(\alpha^3 - 1) < 0; \end{aligned}$$

so $\beta > \alpha$.

Thus, if $m \leq 4$ and $x^{d+2} - x^m - 1$ is divisible by $x^2 - x + 1$, then β is of degree d , and $|\beta| = \beta < |\alpha| = \alpha$. This occurs exactly when $d \equiv 5 \pmod{6}$, and $m = 2$ or $d \equiv 3 \pmod{6}$, and $m = 4$.

This suggests the following modification of Lind’s conjecture. It has been verified for $d \leq 12$:

CONJECTURE (D). The smallest Perron number of degree $d \geq 2$ has minimal polynomial

$$\begin{aligned} & x^d - x - 1 && \text{if } d \not\equiv 3, 5 \pmod{6}, \\ (x^{d+2} - x^4 - 1)/(x^2 - x + 1) && \text{if } d \equiv 3 \pmod{6}, \\ (x^{d+2} - x^2 - 1)/(x^2 - x + 1) && \text{if } d \equiv 5 \pmod{6}. \end{aligned}$$

(*N.B.* $x^5 - x^4 - 1 = (x^2 - x + 1)(x^3 - x - 1)$.)

4. The Computations. The method is based on similar principles to those used in [1], but is somewhat simpler. Given a bound $B > 1$, we wish to generate the set R of polynomials of degree d all of whose zeros are at most B in modulus. From this finite set we will eliminate the cyclotomic polynomials and the reducible polynomials. If B has been chosen sufficiently large, the remaining set will be nonempty and will contain the minimal polynomials of the extremal α for $|\alpha|$ and the smallest Perron number of degree d . For $d \geq 3$, the choice $B = (2 + 1/d)^{1/d}$ suffices, since $B^d - B - 1 > 0$ and $B^d - 2 > 0$. In practice we chose B to be a “round” number approximately equal to this value.

Let $P(x) = x^d + a_1x^{d-1} + \dots + a_d$ have zeros $\alpha_1, \dots, \alpha_d$, and let $S_k = \sum_{i=1}^d \alpha_i^k$ for $k = 1, 2, \dots$. If all $|\alpha_i| \leq B$, then, clearly,

$$(1) \quad |S_k| \leq dB^k, \quad k = 1, 2, \dots$$

In addition, we have [1, Lemma 1]

$$(2) \quad -dB^{k/2} + (2/d)S_{k/2} \leq S_k, \quad k = 2, 4, \dots$$

By Newton’s identities, the S_k and a_k are related by

$$(3) \quad S_k + a_1S_{k-1} + \dots + a_{k-1}S_1 + ka_k = 0, \quad k \geq d,$$

$$(4) \quad S_k + a_1S_{k-1} + \dots + a_dS_{k-d} = 0, \quad k > d.$$

According to (3), (S_1, \dots, S_k) is uniquely determined from (a_1, \dots, a_k) and vice versa for $k \leq d$. If the a_i are integers, then a_1, \dots, a_{k-1} determine $S_k \pmod{k}$. Hence, the number of P satisfying (1) for $k \leq d$ is approximately

$$N_d = \prod_{k=1}^d \frac{2dB^k}{k} \sim (2eB^{d/2})^d \sim (2e\sqrt{2})^d$$

if $B \sim 2^{1/d}$.

If we apply (2) for $k \leq d$, then we reduce this by a factor of approximately $(2/3)^{d/2}$. To see this, note that, e.g., the number of pairs (S_1, S_2) which satisfy (1) and (2) is approximately

$$\int_{-dB}^{dB} \left(2dB^2 - \left(\frac{2}{d} \right) x^2 \right) dx = \frac{2}{3} (2dB)(2dB^2).$$

The factor $(2/3)^{n-1}$ is not quite correct for n -tuples $(S_1, S_2, S_4, \dots, S_m)$ with $m = 2^{n-1}$. For triples (S_1, S_2, S_4) , the correct factor, for example, should be

$(2/3) \cdot (24/35)$, since

$$\int_{-dB}^{dB} dx \int_{-dB^2 + (2/d)x^2}^{dB^2} \left(2dB^4 - \left(\frac{2}{d} \right) y^2 \right) dy = \frac{2}{3} \cdot \frac{24}{35} (2dB)(2dB^2)(2dB^4).$$

However, the approximation is good enough for these purposes.

Thus, we are ultimately faced with investigating about $(4e/\sqrt{3})^d \sim (6.28)^d$ polynomials, so it is apparent that only relatively small d will be feasible.

Of course, one can use some symmetry and insist that $S_1 \geq 0$. For $d = 12$ and $B = 1.063$, the size of the set is thus predicted to be about

$$\frac{1}{2} \left(\frac{2}{3} \right)^6 N_{12} \approx 3.93 \times 10^8.$$

The exact size of the set was in fact 451 682 220.

If we use no other information than (1) and (2) for $k \leq d$, then it is clear that all polynomials which appear in this set must be investigated further. Thus the size of this set does play a critical role in determining the running time of the algorithm. However, it is clear that one should not simply solve all such P to determine whether P is in R . The inequalities (1) for $k > d$ provide further tests which should provide the same sort of information more inexpensively.

Let us denote by R_d the set of P satisfying $S_1 \geq 0$ and (1) and (2) for $k \leq d$. For $n > d$, let R_n denote the set of P in R_d satisfying $a_d \neq 0$ and (1) for $k \leq n$. Clearly, the R_n are nested, and their intersection is R , since

$$\limsup (\log |S_k|/k) = \log |\bar{\alpha}|.$$

Thus for sufficiently large N , the set R_N is not much larger than R , and we can afford simply to solve all P in R_N . The optimal choice of N depends on the rate of decay $|R_n|$ and on the time t_1 for applying the test (1) for a given $k = n$ relative to the time t_2 for solving P . Clearly, $t_1 \ll t_2$. Of course, since we naturally generate the P 's one at a time without storing them, we do not know the values of $|R_n|$ until after the computation is complete. Thus, optimizing the choice of N is not feasible, but $N = 3d$ worked well in practice.

As a sample of the numbers involved, for $d = 12$, $B = 1.063$ we have

$$\begin{aligned} |R_{12}| &= 451,682,220, & |R_{23}| &= 37,019, & |R_{35}| &= 4931, \\ |R_{13}| &= 23,746,503, & |R_{24}| &= 28,277, & |R_{36}| &= 4435. \\ |R_{14}| &= 4,987,914, \end{aligned}$$

In fact, $|R| = 867$, of which 811 are cyclotomic, 26 are reducible, and 30 are irreducible.

The algorithm then is simply to generate each P in R_d and apply the sequence of tests (1) sequentially for $k = d + 1, \dots, N$. The surviving P are in R_N . We then test for small cyclotomic factors (of order 7 or less) and then solve P using the QR -algorithm. Using the ideas in [2], one can get a priori lower bounds on $|\bar{\alpha}|$ for noncyclotomic P , so we can reject any P which have $|\bar{\alpha}| \leq 1.0005$ or $|\bar{\alpha}| > B$. The remaining P are generally irreducible, but reducibility is easily checked, since we apply the algorithm in order of increasing d , so we have a list of possible factors.

To save time in generating R_d , the bounds in (1) and (2) are precomputed so that, for example, the test (2) simply requires testing $S_k \geq C(k/2, S_{k/2})$, where $C(i, j)$ is a precomputed array. Thus, only integer arithmetic is required when applying (1) and (2).

For reciprocal P of even degree, since $a_{d-k} = a_k$, P is determined completely by $S_1, \dots, S_{d/2}$. Writing $m = d/2$, we see that the initial set R_m contains approximately

$$\frac{1}{2} \frac{(4m)^m B^{m^2/2}}{m!} \left(\frac{2}{3}\right)^{m/2}$$

polynomials. For $d = 16$ and $B = 1.09$, this is about 4.25×10^7 . The actual number generated was 46,345,943.

The same choice $N = 3d$ was made and the same procedure followed in processing the set R_N . In this case, a number of reducible polynomials of the form QQ^* appeared, where $Q^*(x) = \pm x^{d/2}Q(x^{-1})$. These correspond to α of degree $d/2$ with $|\alpha^{-1}| \leq |\alpha|$.

The running time was essentially proportional to the size of the initial set of polynomials. For example, the case $d = 12$, $B = 1.063$ required 4.69 hours of CPU time on an Amdahl 470 V7A.

5. The Tables. Tables 3 and 4 appear as an appendix in the supplements section of this issue. If $P_1(x) = Q(x^s)$ and $P_2(x) = \pm Q(-x^s)$ for some $s \geq 1$, then we say P_1 and P_2 are equivalent. Since $|\bar{\alpha}|$ is the same for P_1 and P_2 , only one of such a pair is listed in the tables. Generally, it is the one in which the first nonvanishing a_i is positive, except when an α_i attaining $|\bar{\alpha}|$ is real, in which case we choose the sign so $\alpha_i > 0$.

All the tables exhibit $\alpha_i = |\bar{\alpha}|e^{i\phi}$, where ϕ is given in degrees and chosen minimally so that $0 \leq \phi < 180$. The minimal polynomial of α is exhibited as a vector $a_1 \cdots a_d$ except in Table 1.

Table 1 gives a list of extrema for $|\bar{\alpha}|$ for degrees $1 \leq d \leq 12$. Table 2 gives the corresponding list for reciprocal polynomials of even degrees $2 \leq d \leq 16$.

Table 3 gives a complete list of inequivalent α of degree d with $|\bar{\alpha}|$ smaller than the given bound B . Perron numbers are indicated by a "P" in the column preceding v . Table 4 gives the corresponding lists for reciprocal polynomials. (Perron numbers are not marked.)

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Supplement to The Maximal Modulus of an Algebraic Integer

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Table 3

All inequivalent α of degree d with $|\bar{\alpha}|$ less than the given bound. The table gives the minimal polynomial as a vector of integers, the number of conjugates outside the unit circle, $|\alpha|$ and $\arg \alpha$ in degrees, where α is a conjugate which attains $|\bar{\alpha}|$. The "P" preceding certain v denotes that α is a Perron number.

Degree = 2 Bound = 1.800

1 0-2		2	1.4142135624	0.0
1 1 2		2	1.4142135624	110.7048110546
1 2 2		2	1.4142135624	135.0000000000
1-1-1		P 1	1.6180339887	0.0
1 0-3		2	1.7320508076	0.0
1 1 3		2	1.7320508076	106.7786548810
1 2 3		2	1.7320508076	125.2643896828
1 3 3		2	1.7320508076	150.0000000000

Degree = 3 Bound = 1.360

1 1 0-1		2	1.1509639253	139.6719231917
1 0 1 1		2	1.2106077944	73.6316148171
1 0 0-2		3	1.2599210499	0.0
1 1-1-2		3	1.2880089603	148.8918227959
1 0-1-1		P 1	1.3247179572	0.0
1 1 2 1		2	1.3247179572	99.3438463835
1-1 1-2		P 3	1.3532099642	0.0
1 1 1-1		2	1.3562030656	124.6889973915

Degree = 4	Bound = 1.250	Degree = 7	Bound = 1.113
1 0 0 1	2 1.1837518186	1 1 0 0 1 0 -1 -1	4 1.0928455996
1 0 0 1	2 1.1837518186	1 1 1 0 0 0 1 1	4 1.1015405971
1 0 0 2	4 1.1892071150	1 1 1 0 0 0 0 -1	4 1.1033553700
1 0 2 0	4 1.1892071150	1 0 0 0 0 0 0 -2	7 1.1040895137
1 0 0 -1	3 1.2207440846	1 0 1 0 1 0 0 1	6 1.1045243113
2 3 3 2	4 1.2279486431	1 0 0 0 0 0 1 1	6 1.1076510547
1 1 0 1 2	4 1.2403674040	1 0 -1 0 0 1 1 -1	4 1.1098777988
1 1 0 1	2 1.2474409390	1 1 1 1 1 0 0 -1	6 1.1103320258
1 0 1 1 1	2 1.2474409390	1 0 0 0 0 0 -1 -1	P 5 1.1112775683
1 0 -1 1 -1	4 1.1216451786	1 1 0 0 1 0 -1 0 1	6 1.0756204773
1 1 0 -1 -1	3 1.1237328210	1 0 -1 0 1 1 -1 -1 1	6 1.0837043264
1 0 1 0 1	4 1.1392502979	1 0 1 0 0 0 0 1 1	4 1.0880035931
1 1 1 2 1	4 1.1415099712	1 0 0 0 0 1 0 1 1	4 1.0880035931
1 0 0 0 -2	5 1.1486983550	1 0 0 0 0 0 0 -2	8 1.0905077327
1 1 0 0 -1	2 1.1617728283	1 0 0 1 0 0 0 2	8 1.0905077327
1 0 0 -1 -1	3 1.1673039783	1 0 0 2 0 0 0 2	8 1.0905077327
1 0 1 0 -1	5 1.1679895315	1 1 1 0 0 0 0 1	6 1.0905405495
1 0 -1 0 -2	5 1.170599325	1 0 0 0 0 0 -1 -1	6 1.0969815578
1 0 -1 0 -2	5 1.1752469656	1 0 0 0 0 0 0 -1	P 5 1.0969815578
1 1 1 0 -1	4 1.1777393416	1 1 1 0 -1 -1 -2 -1	7 1.0972950360
1 0 1 0 0 1	2 1.1871121409		
1 1 0 0 0 -1	2 1.1873801922		
1 0 0 -1 0 -1	P 3 1.1938591113		

Degree = 5	Bound = 1.200	Degree = 8	Bound = 1.098
1 0 -1 1 -1	4 1.1216451786	1 1 0 0 1 0 -1 0 1	6 1.0756204773
1 1 0 -1 -1	3 1.1237328210	1 0 -1 0 1 1 -1 -1 1	6 1.0837043264
1 0 1 0 1	4 1.1392502979	1 0 1 0 0 0 0 1 1	4 1.0880035931
1 1 1 2 1	4 1.1415099712	1 0 0 0 0 1 0 1 1	4 1.0880035931
1 0 0 0 -2	5 1.1486983550	1 0 0 0 0 0 0 -2	8 1.0905077327
1 1 0 0 -1	2 1.1617728283	1 0 0 1 0 0 0 2	8 1.0905077327
1 0 0 -1 -1	3 1.1673039783	1 0 0 2 0 0 0 2	8 1.0905077327
1 0 1 0 -1	5 1.1679895315	1 1 1 0 0 0 0 1	6 1.0905405495
1 0 -1 0 -2	5 1.170599325	1 0 0 0 0 0 -1 -1	6 1.0969815578
1 0 -1 0 -2	5 1.1752469656	1 0 0 0 0 0 0 -1	P 5 1.0969815578
1 1 1 0 -1	4 1.1777393416	1 1 1 0 -1 -1 -2 -1	7 1.0972950360
1 0 1 0 0 1	2 1.1871121409		
1 1 0 0 0 -1	2 1.1873801922		
1 0 0 -1 0 -1	P 3 1.1938591113		

Degree = 6	Bound = 1.135	Degree = 9	Bound = 1.086
1 0 1 0 0 0 -1	4 1.0728298678	1 0 0 1 0 0 0 0 0 -1	6 1.0479821944
1 1 0 -1 0 1	4 1.0937316919	1 0 0 0 0 1 0 0 1	6 1.0657806276
1 0 1 0 0 1	4 1.1002762355	1 1 0 -1 -1 0 1 1 0 -1	6 1.0671508605
1 1 0 0 0 1	4 1.1181386638	1 0 0 0 0 0 0 0 -2	9 1.0800597389
1 0 0 0 -2	6 1.1224620483	1 1 0 -1 -1 0 1 0 -1	P 5 1.0815290867
1 0 0 1 0 0 2	6 1.124620483	1 1 1 1 0 0 0 0 1 -1	6 1.0823669349
1 0 2 0 0 2	6 1.124620483	1 0 0 0 0 0 1 1	6 1.082680366
1 0 0 1 0 -1	4 1.1232292282	1 0 1 0 1 0 1 1	8 1.0829693977
1 0 0 0 0 1	4 1.1261126550	1 1 1 1 0 0 0 -1 -1	6 1.083751364
1 0 0 0 -1	P 3 1.1347241384	1 0 0 0 0 0 0 -1 -1	P 5 1.0850702455
1 0 0 0 -1 -1	6 1.1349048244		

Degree = 7	Bound = 1.113	Degree = 10	Bound = 1.082
1 0 0 1 1	149.4007468584	1 0 1 0 1 0 -1 0 1	6 1.0479821944
1 0 0 1	52.1015419454	1 0 0 0 1 0 0 1	6 1.0657806276
1 0 0 2	0.0	1 1 0 -1 -1 0 1 1 0 -1	6 1.0671508605
1 0 2 0	55.3524055273	1 0 0 0 0 0 0 0 -2	9 1.0800597389
1 0 0 -1	67.5000000000	1 1 0 -1 -1 0 1 0 -1	P 5 1.0815290867
2 3 3 2	92.0215046526	1 1 1 1 0 0 0 0 1 -1	6 1.0823669349
1 1 0 1 2	155.5017558164	1 0 0 0 0 0 1 1	6 1.082680366
1 1 0 1	133.06766639483	1 0 1 0 1 0 1 1	8 1.0829693977
1 0 1 1 1	63.9695632716	1 1 1 1 0 0 0 -1 -1	6 1.083751364
		1 0 0 0 0 0 0 -1 -1	P 5 1.0850702455
1 0 -1 1 -1	158.8014602860	1 1 0 0 1 0 -1 0 1	6 1.0756204773
1 1 0 -1 -1	0.0	1 0 -1 0 1 1 -1 -1 1	6 1.0837043264
1 0 1 0 1	109.9807757154	1 0 1 0 0 0 0 1 1	4 1.0880035931
1 1 1 2 1	59.2975705341	1 0 0 0 0 1 0 1 1	4 1.0880035931
1 0 0 0 -2	0.0	1 0 0 0 0 0 0 -2	8 1.0905077327
1 1 0 0 -1	97.8592379479	1 0 0 1 0 0 0 2	8 1.0905077327
1 0 0 -1 -1	66.5439914899	1 0 0 2 0 0 0 2	8 1.0905077327
1 0 1 0 -1	83.2273664596	1 1 1 0 0 0 0 1	6 1.0905405495
1 0 -1 0 -2	0.0	1 0 0 0 0 0 -1 -1	6 1.0969815578
1 1 1 0 -1	84.5869747031	1 0 0 0 0 0 0 -1	P 5 1.0969815578
1 0 1 0 0 1	100.5745824076	1 1 1 0 -1 -1 -2 -1	7 1.0972950360
1 1 0 0 0 -1	155.2589589625		
1 0 0 -1 0 -1	0.0		

Degree = 11	Bound = 1.082	Degree = 14	Bound = 1.078
1 0 1 0 0 0 -1	69.8359615959	1 0 0 1 0 0 0 0 0 -1	6 1.0479821944
1 1 0 -1 0 1	155.9270802610	1 0 0 0 1 0 0 1	6 1.0657806276
1 0 1 0 0 1	36.8158074085	1 1 0 -1 -1 0 1 1 0 -1	6 1.0671508605
1 1 0 0 0 1	103.5297629176	1 0 0 0 0 0 0 0 -2	9 1.0800597389
1 0 0 0 -2	0.0	1 1 0 -1 -1 0 1 0 -1	P 5 1.0815290867
1 0 0 1 0 0 2	36.9016036849	1 1 1 1 0 0 0 0 1 -1	6 1.0823669349
1 0 2 0 0 2	45.0000000000	1 0 0 0 0 0 1 1	6 1.082680366
1 0 0 1 0 -1	72.9131615477	1 0 1 0 1 0 1 1	8 1.0829693977
1 0 0 0 0 1	32.9097611644	1 1 1 1 0 0 0 -1 -1	6 1.083751364
1 0 0 0 -1	0.0	1 0 0 0 0 0 0 -1 -1	P 5 1.0850702455
1 0 0 0 -1 -1	74.4459113979		

Degree = 10 Bound = 1.200

1	0	1	1	0	1	0	1	1	0	1	74.2182865812
1	1	0	0	-1	0	0	0	1	1	2	1.1257148215
1	1	0	0	-1	0	0	0	1	1	2	1.1329529384
1	1	0	-1	0	0	0	1	1	2	1.1420874580	
1	0	0	1	1	0	0	0	1	2	1.1670300606	
1	0	0	1	0	-1	0	0	1	2	1.1700421688	
1	1	0	-1	-1	-1	0	1	1	1	1.1762808183	
1	1	2	2	3	2	2	1	1	2	1.1825533943	
1	0	2	0	3	1	3	0	2	4	1.1926596333	
1	2	1	0	-1	0	1	2	2	2	1.1941492832	
1	1	1	1	0	-1	0	1	1	2	1.1964132386	

Degree = 12 Bound = 1.135

1	1	1	0	-1	-1	-1	0	1	1	1	100.4426482045
1	1	0	0	-1	-1	0	0	1	1	2	1.1080548536
1	1	0	0	-1	-1	0	0	1	1	2	1.1185019523
1	0	1	0	-1	1	0	0	1	1	2	1.1244526912
1	1	1	1	2	3	2	1	1	1	4	1.1274207202
1	0	1	1	2	1	2	1	1	0	2	1.1281925213

Degree = 14 Bound = 1.100

1	0	0	1	1	0	1	0	1	0	0	1	127.7503248885
1	1	0	0	1	0	-1	1	0	0	1	4	1.0939016857
1	1	0	-1	0	-1	1	0	0	1	4	1.0966369673	
1	1	0	-1	1	1	1	0	-1	0	1	4	1.0987312747

Degree = 16 Bound = 1.090

1	0	0	0	1	0	1	0	1	0	0	0	1	23.7491057380
1	0	1	0	0	0	-1	0	0	1	0	1	4	1.0813339123
1	1	1	0	0	-2	-2	1	0	0	1	1	4	1.0818997649
1	1	1	0	0	-2	-2	1	0	0	1	1	4	1.0856894163

Degree = 18 Bound = 1.080

1	0	0	0	0	1	0	1	0	1	0	0	0	1	100.4426482045
1	1	0	0	-1	-1	0	0	1	1	2	1.1080548536			
1	1	0	0	-1	-1	0	0	1	1	2	1.1185019523			
1	0	1	0	-1	1	0	0	1	1	2	1.1244526912			
1	1	1	1	2	3	2	1	1	1	4	1.1274207202			
1	0	1	1	2	1	2	1	1	0	2	1.1281925213			

Degree = 20 Bound = 1.070

1	0	0	0	1	0	1	0	1	0	0	0	1	127.7503248885
1	1	0	0	1	0	-1	1	0	0	1	4	1.0939016857	
1	1	0	-1	0	-1	1	0	0	1	4	1.0966369673		
1	1	0	-1	1	1	1	0	-1	0	1	4	1.0987312747	