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Long Arithmetic Progressions of Primes: Some Old, Some New

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Abstract. The results are reported of an extensive search with a computer for "long" arithmetic progressions of primes. Such progressions with minimum last term are now known for all lengths up to and including nineteen.

A long-standing conjecture of Hardy and Littlewood [5]—the "prime k-tuples conjecture"—is that if $a_1, \ldots, a_k$ do not form a complete residue system for any prime p, then there are infinitely many values of x such that $x + a_1, \ldots, x + a_k$ are all prime. The conjecture is actually given in [5] in a much stronger form, viz., an asymptotic formula for the number of such values of x not exceeding a given bound. Two interesting implications of the prime k-tuples conjecture are that there are infinitely many twin primes, and that for all $n > 0$ there are infinitely many arithmetic progressions of n primes (denoted *PAPs* in the sequel). We are concerned herein with the second of these.

Very little about our subject has been proved. Roughly speaking, the present state of knowledge is that n can be $3\frac{1}{2}$. More precisely, Chowla [1] showed that there are infinitely many PAPs of length 3, and, more recently, Grosswald [2] established the validity of the Hardy-Littlewood asymptotic formula in that case, and Heath-Brown [6] has shown that there are infinitely many arithmetic progressions consisting of three primes and an "almost prime" (a number with at most two prime factors).

With the aid of computers, PAPs have been discovered that are substantially longer than those guaranteed to exist by these or other available theorems. Before our work, the longest known PAP had length 17 [13]. In [11] we announced a PAP of length 18 discovered by a search that was then still in progress, and undertook to report again on completion of the search. This paper contains the promised report and also the results of other, shorter computations that fill in some gaps left in earlier work. The highlight is a PAP of length 19. Note that we do not consider PAPs with so-called "negative primes", unlike, e.g., [8].

The new results reported herein were discovered by programs whose design is presented in detail in [10]. Our major computation was a search for all PAPs, of

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length at least 17, that have a common difference divisible by 9699690—the product of the primes $\leqslant 19$—and no term exceeding 30001×9699690. (A PAP of length 19 must have a common difference of this form or else have first term 19; the latter case is unlikely and was checked separately.) Our search ran in the background on two Digital Equipment Corporation VAX-11/780s in the Department of Computer Science at Cornell University, from 6 October 1982 to 18 March 1984, and consumed almost 14000 hours of computer time.

Table 1 is an update of Table 2 of [3]. It lists for each m, $1 < m \leqslant 19$, the PAP of length m with minimum last term. For the heuristic argument leading to the estimates in the last column, see [3]. The PAPs for $m = 14, 15, 18, 19$ have been discovered since [3] appeared. Also new is the knowledge that each PAP listed is indeed the one with minimum last term; previously, this was known only for $m \leqslant 10$.

TABLE 1

The known PAPs with minimum last term

m	PAP of length m with minimal last term ($k = 0, 1, 2, \ldots, m-1$)	last term	estimated last term
2	2, 3	3	2
3	3, 5, 7	7	2
4	5, 11, 17, 23	23	2
5	5, 11, 17, 23, 29	29	29
6	$7 + 30k$	157	92
7	$7 + 150k$	907	497
8	$199 + 210k$	1669	1406
9	$199 + 210k$	1879	5086
10	$199 + 210k$	2089	24310
11	$110437 + 13860k$	249037	177300
12	$110437 + 13860k$	262897	829800
13	$4943 + 60060k$	725663	5582000
14	$31385539 + 420420k$	36850999	2.332×10^7
15	$115453391 + 4144140k$	173471351	1.137×10^8
16	$53297929 + 9699690k$	198793279	6.793×10^8
17	$3430751869 + 87297210k$	4827507229	5.774×10^9
18	$4808316343 + 717777060k$	17010526363	3.303×10^{10}
19	$8297644387 + 4180566390k$	83547839407	2.564×10^{11}
20	?	?	1.261×10^{12}

Sierpiński defines $g(x)$ to be the maximum number of terms in an arithmetic progression of primes not greater than x. The least x, $l(x)$, for which $g(x)$ takes the values $0, 1, \ldots, 19$ can be read off from Table 1, thereby correcting and extending the information in [4].

Table 2 is an adaption and extension of Table 1 on p. 11 of [4]. It gives, for each n, $12 \leqslant n \leqslant 19$, the first-discovered PAP with length n and the PAP of length n with smallest last term. Note that the first-discovered PAPs of lengths 13, 17, and 19 are also those with smallest last term, and that the first-discovered PAPs of lengths 14 and 15 are initial parts of Root's PAP of length 16.

TABLE 2

Some PAPs and their discoverers

n	common difference	first term	last term	discovery
12	30030	23143	353473	V. A. Golubev, 1958 (see [8])
12	13860	110437	262897	E. Karst, 1967 ([8])
13	60060	4943	725663	V. N. Seredinskij, 1963 (see [8])
14	223092870	2236133941	5136341251	S. C. Root, 1969 (see [7])
14	420420	31385539	36850999	P. A. Pritchard, 1983
15	223092870	2236133941	5359434121	S. C. Root, 1969 (see [7])
15	4144140	115453391	173471351	P. A. Pritchard, 1983
16	223092870	2236133941	5582526991	S. C. Root, 1969 (see [7])
16	9699690	53297929	198793279	S. Weintraub, 1976 ([12])
17	87297210	3430751869	4827507229	S. Weintraub, 1977 ([13])
18	9922782870	107928278317	276615587107	P. A. Pritchard, 1982 ([11])
18	717777060	4808316343	17010526363	P. A. Pritchard, 1983
19	4180566390	8297644387	83547839407	P. A. Pritchard, 1984

We know of no PAP of length 20 (or greater). The known PAPs of length at least 18 are given in Table 3. (The two PAPs of length 18 that can be obtained from the one of length 19 are not listed.)

TABLE 3

The known PAPs of length at least 18

length	first term	common difference	last term
18	4808316343	717777060	17010526363
19	8297644387	4180566390	83547839407
18	64158606367	2735312580	110658920227
18	2518035911	7536659130	130641241121
18	115936060313	3103900800	168702373913
18	98488875263	5169934770	186377766353
18	170263333103	5063238180	256338382163
18	107928278317	9922782870	276615587107
18	51565746467	13889956080	287694999827

Grosswald [2] showed that with an additional assumption, the Hardy-Littlewood asymptotic estimate of the number of PAPs of length n with no term exceeding x could be cast in an easily computable form. Grosswald and Hagis [3] showed that, for relatively short progressions, the estimate is reasonably accurate. Our major calculation was designed to test Grosswald's estimate, using much longer PAPs than had hitherto been employed.

Before presenting the results of this test, it is necessary to adapt Grosswald's estimate so that it applies to PAPs with a restricted form of common difference. Fortunately, an estimate is implicit in Grosswald's argument. Let us define $N_{m,n}(x)$

to be the number of PAPs of length m with all terms $\leqslant x$ and common difference a multiple of the product of the primes $\leqslant n$. We have

$$N_{m,n}(x) \sim \prod_{m<p\leqslant n} \frac{1}{p+1-m} \cdot \frac{c_m x^2}{\log^m x} \tag{1}$$

where

$$c_m = \frac{1}{2(m-1)} \prod_{p>m} \left(\left(\frac{p}{p-1} \right)^{m-1} \frac{p-(m-1)}{p} \right) \prod_{p\leqslant m} \left(\frac{1}{p} \left(\frac{p}{p-1} \right)^{m-1} \right).$$

The right-hand side of (1) is actually the dominant term of an infinite series for $N_{m,n}(x)$. Grosswald [2] gives a computable expression for the next term, which contributes significantly for the numbers under consideration, and which was therefore incorporated into our calculations. The third term was not computed, mainly because the calculations are *very* involved, but also because we did not expect it to significantly alter our estimates.

Table 4 shows the actual versus estimated counts of $N_{m,n}(x)$ for our experiment, in which $x = 300001 \times 9699690$, $n = 19$, and $m = 17, 18, 19$. As reported in [3], and as is reasonable for an *asymptotic* estimate, the predicted value is most accurate when it is largest. We assure the skeptical reader that the successive powers of 10 in the second column are the exact counts of PAPs found.

TABLE 4
Actual versus predicted numbers of PAPs

length	Number of PAPs	
	actual	predicted
17	100	85.7
18	10	13.1
19	1	1.9

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