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Estimation of the Error in the Reduced Basis Method Solution of Nonlinear Equations*

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Abstract. The reduced basis method is a projection technique for approximating the solution curve of a finite system of nonlinear algebraic equations by the solution curve of a related system that is typically of much lower dimension. In this paper, the reduced basis error is shown to be dominated by an approximation error. This, in turn, leads to error estimates for projection onto specific subspaces; for example, subspaces related to Taylor, Lagrange and discrete least-squares approximation.

1. Introduction. Consider the equation

$$F^*(\phi) = 0, \tag{1.1}$$

where F^*: $R^n \to R^n$. The reduced basis method is a technique for approximating solutions of (1.1) by way of solutions of a related equation,

$$F_R(z) = 0, \tag{1.2}$$

where F_R: $R^m \to R^m$ and $m \leqslant n$. The relationship between (1.1) and (1.2) is such that if a solution of (1.2) is known, then the n-dimensional approximation of ϕ, say ϕ_R, follows in a trivial manner. The power of the method derives from the fact that for many systems (1.1) of practical interest, mappings F_R can be constructed that provide highly-accurate approximations ϕ_R when $m \ll n$.

The method has been applied to a variety of structural shell problems [1], [2], [6]–[9] and, more recently, to a problem in steady fluid flow [10]. Although the effectiveness of the reduced basis method is numerically established in these papers, no error estimates are given.

In the papers [3], [4], Fink and Rheinboldt address the error analysis question for nonlinear equations in a Banach space setting. They consider one-parameter families of finite-dimensional problems and show that as the parameter tends to zero, solution segments of these problems converge to a solution segment of the infinite-dimensional problem. However, no estimate of the order of the error is given for any fixed finite-dimensional segment in the family. Indeed, the existence of a finite-dimensional solution segment is assured only if the dimension of the associated

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approximating subspace is sufficiently large.** In contrast to this, the finite-dimensional case considered here allows order estimates of the errors resulting from approximations of any fixed lower dimension.

The reduced basis method proceeds from the notion of an imbedding of the map F^* into a family of maps. The imbedding produces a manifold of solutions in place of the point solutions of (1.1). This is the same idea that lies at the heart of continuation methods (see, for example, [11]). Accordingly, in the next section we define the *manifold problem* and use it in Section 3 to formulate the reduced basis problem (1.2). To develop (1.2) we employ a projection of R^n onto an m-dimensional subspace $\mathscr{S}_R$. Thus, the reduced basis method may be regarded as nothing more than a composite projection-continuation method.

In Section 3, we establish the existence of solutions of (1.2) and then develop an estimate of the error $\phi - \phi_R$ in terms of an approximation error in $\mathscr{S}_R$. This allows us, in Section 4, to obtain error estimates for specific choices of the subspace $\mathscr{S}_R$. In particular, we consider subspaces related to Taylor, Lagrange, and discrete least-squares approximation. Interestingly, the order of the error is the same for each of these seemingly disparate subspaces. Finally, in Section 5, we present some performance data derived from previous applications of the method.

2. The Manifold Problem. Suppose that the mapping F^* of (1.1) is obtained as a restriction of a one-parameter family of maps. That is, suppose that we are given a map F: $R^n \times R \to R^n$ such that $F^*(\phi) = F(\phi, \xi_0)$ for some fixed ξ_0. Solving (1.1) is then equivalent to the following problem. Given ξ_0, find a point $(\phi, \xi) \in R^n \times R$, such that

$$F(\phi, \xi) = 0, \tag{2.1}$$

$$\xi = \xi_0. \tag{2.2}$$

In attempting to solve problems of the type (2.1), (2.2), it is very useful to regard their solutions as particular points on a curve of solutions in R^n, the parametrization of the curve being in terms of the component ξ. Indeed, it is important to generalize this idea even further by not according any particular component the special status of a parameter; instead, one simply considers sets in R^{n+1} whose members satisfy (2.1). Obviously, without further hypothesis, this so-called *manifold problem* admits completely general solution sets.

A particularly simple situation results when the solution manifold is again a *curve*; this time in R^{n+1}. To describe this situation, we set $(\phi, \xi) = x \in R^{n+1}$ and write $F(\phi, \xi) = F(x)$. Then the regularity set of F: $R^{n+1} \to R^n$ is

$$\mathscr{R}(F) \equiv \{x \in R^{n+1} \mid \operatorname{rank} DF(x) = n\},$$

where $DF(\cdot)$ denotes the Jacobian matrix of F. With regard to solution manifolds that are curves, Rheinboldt [11] has proven the following existence theorem.

****Note added in proof.** In more recent work (Technical Report ICMA-84-70, University of Pittsburgh, Pittsburgh, PA, 1984) Fink and Rheinboldt remove this existence question by assuming that a certain finite-dimensional linear operator is nonsingular. In the proof of Theorem 3.2 of this paper we show that when the original problem is finite-dimensional, such an operator always exists.

THEOREM 2.1. *Suppose that F is at least twice continuously differentiable on R^{n+1} and that $x^0 \in \mathscr{R}(F)$ satisfies $F(x^0) = 0$. Then there exists an open interval J and a unique, simple, C' curve $x: J \to \mathscr{R}(F)$ such that*

(i) $F(x(s)) = 0, s \in J$,

(ii) *x passes through x^0,*

(iii) *x has no endpoint in $\mathscr{R}(F)$, and*

(iv) $dx/ds \neq 0, s \in J$.

This result does not require that the parameter s coincide with one of the components of x. Indeed, to avoid the possible occurrence of turning points, it is essential that such an identification be avoided. However, if D_jF denotes the $n \times n$ submatrix obtained from DF by deleting its jth column, then since $x \in \mathscr{R}(F)$, for each $s_0 \in J$, it is always possible to find an index j such that $D_jF(x(s_0))$ is nonsingular. It follows from the implicit function theorem that at each point on the solution curve, it is possible to give a *local* parametrization of the curve in terms of one of its components. Thus, we assume that at $x^0 = x(s_0)$ an index j has been chosen for which $D_jF(x^0)$ is nonsingular. Defining the change of variables, T: $R^{n+1} \to R^n \times R$, $T = (y, \lambda)$, $T(x) = (x_1 - x_1^0, \ldots, x_{j-1} - x_{j-1}^0, x_{j+1} - x_{j+1}^0, \ldots, x_{n+1} - x_{n+1}^0, x_j - x_j^0)$ and the mapping G: $R^n \times R \to R^n$,

$$G(y, \lambda) = F(T^{-1}(y, \lambda)), \tag{2.3}$$

we see that for some $\lambda_1 > 0$ there exists an interval $\Lambda_1 = [-\lambda_1, \lambda_1]$ and a C' curve: y: $\Lambda_1 \to R^n$ such that

$$G(y(\lambda), \lambda) = 0, \qquad \lambda \in \Lambda_1, \tag{2.4}$$

$$y(0) = 0. \tag{2.5}$$

Furthermore, it is clear that the solution curve $x(\lambda)$ may be recovered from $y(\lambda)$ by the trivial inversion $x(\lambda) = T^{-1}(y(\lambda), \lambda)$.

3. The Reduced Basis Approximation and Error. In view of the preceding, we assume that we are given a mapping G: $R^n \times R \to R^n$, $G = (g_1, \ldots, g_n)^T$, continuously differentiable for each $(y, \lambda) \in R^n \times R$, and satisfying $G(0, 0) = 0$. We also assume that $D_yG(0, 0)$ is nonsingular, where D_yG is the $n \times n$ Jacobian matrix having $\partial g_i/\partial y_j$ as the element in its ith row and jth column.

We are interested in approximating the solution curve y that satisfies (2.4) and (2.5) by a curve y_R lying in an m-dimensional subspace $\mathscr{S}_R$ of R^n. The curve y_R is defined by a projection method. Specifically, we let P denote the projector from R^n onto $\mathscr{S}_R$ relative to some complement of $\mathscr{S}_R$. Then, we seek an interval $\Lambda_R = [-\lambda_R, \lambda_R]$, $\lambda_R > 0$, and a curve y_R: $\Lambda_R \to \mathscr{S}_R$, such that

$$PG(y_R(\lambda), \lambda) = 0, \qquad \lambda \in \Lambda_R, \tag{3.1}$$

$$y_R(0) = 0. \tag{3.2}$$

Problem (3.1), (3.2) is called the reduced basis problem, and we now show that for each m, $1 \leqslant m \leqslant n$, it has a unique solution.

We begin with a lemma on the existence of a simultaneous complement of two subspaces.

LEMMA 3.1. *Let $\mathscr{S}$ and $\mathscr{T}$ be m-dimensional subspaces of R^n. Then there is a subspace $\mathscr{U}$ that is a complement of both $\mathscr{S}$ and $\mathscr{T}$.*

Proof. Let the columns of S_1 and T_1 be respectively bases for $\mathscr{S}$ and $\mathscr{T}$, and let U and V be $n \times n$ nonsingular matrices such that

$$US_1 = \begin{bmatrix} S_{11} \\ 0 \end{bmatrix}, \qquad VT_1 = \begin{bmatrix} T_{11} \\ 0 \end{bmatrix},$$

where S_{11} and T_{11} are $m \times m$ and nonsingular. Then the $n - m$ columns of S_2 and T_2 are respectively bases for complements of $\mathscr{S}$ and $\mathscr{T}$ if and only if

$$U[S_1 S_2] = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad \text{and} \quad V[T_1 T_2] = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},$$

where S_{22} and T_{22} are nonsingular. It follows that if

$$\begin{bmatrix} S_{12} \\ S_{22} \end{bmatrix} = UV^{-1} \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix},$$

and T_{12}, T_{22} can be chosen so that T_{22} and S_{22} are nonsingular, then the columns of S_2 form a basis for a common complement of $\mathscr{S}$ and $\mathscr{T}$.

If we write

$$UV^{-1} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},$$

where W_{22} is $(n - m) \times (n - m)$, then the matrix $[W_{21}\ W_{22}]$ has a set of $n - m$ linearly independent columns, say columns $k_1, \ldots, k_{n-m}$. Let e_i denote the ith unit coordinate vector in R^n and set

$$\begin{bmatrix} T_{12} \\ C \end{bmatrix} = (e_{k_1}, \ldots, e_{k_{n-m}}).$$

The matrix C is singular only by virtue of certain zero rows, $i_1, \ldots, i_s$, and columns, $j_1, \ldots, j_s$. If the index sets $\{i_\nu\}$, $\{j_\nu\}$ are not void, let $E(\varepsilon)$ denote the $(n - m) \times (n - m)$ matrix with ε in row i_ν, column j_ν, $\nu = 1, \ldots, s$ and zeros elsewhere. Now set $T_{22} = C + E(\varepsilon)$. Since $S_{22} = W_{21}T_{12} + W_{22}(C + E(\varepsilon))$ is nonsingular when $\varepsilon = 0$, and since T_{22} is nonsingular for all $\varepsilon \neq 0$, by continuity both matrices are nonsingular for some $\varepsilon \neq 0$. Q.E.D.

The existence of a solution of the reduced basis problem may now be established.

THEOREM 3.2. *For each subspace $\mathscr{S}_R$ of dimension m, $m = 1, 2, \ldots, n$, there is a projector P: $R^n \to \mathscr{S}_R$ such that problem* (3.1)–(3.2) *has a unique solution.*

Proof. Define the m-dimensional subspace

$$\mathscr{T} = \left\{ D_y G(0, 0) w \mid w \in \mathscr{S}_R \right\}.$$

By Lemma 3.1, $\mathscr{S}_R$ and $\mathscr{T}$ have a common complement $\mathscr{U}$. Let P be the projector onto $\mathscr{S}_R$ relative to $\mathscr{U}$. It is known [5] that P has a representation

$$P = YU^T, \tag{3.3}$$

where the columns of Y form a basis for $\mathscr{S}_R$ and the columns of U are biorthogonal to those of Y. Thus, the reduced basis problem (3.1)–(3.2) is equivalent to finding an interval $\Lambda_R = [-\lambda_R, \lambda_R]$, $\lambda_R > 0$, and a curve z: $\Lambda_R \to R^m$ such that

$$U^T G(Yz(\lambda), \lambda) = 0, \qquad \lambda \in \Lambda_R, \tag{3.4}$$

$$z(0) = 0.$$

Let H: $R^m \times R \to R^m$,

$$H(z, \lambda) = U^T G(Yz, \lambda).$$

Then solution of (3.4) is tantamount to solving the equation $H(z, \lambda) = 0$ for z as a function of λ. But H is continuously differentiable on $R^m \times R$, and $H(0,0) = U^T G(0,0) = 0$. Furthermore, $D_z H(0,0) = U^T D_y G(0,0) Y$. We assert that this matrix is nonsingular; in which case an application of the implicit function theorem concludes the proof.

To see that $D_z H(0,0)$ is nonsingular, observe that $D_y G(0,0) Yz \in \mathscr{T}$ for any $z \in R^m$. If $U^T D_y G(0,0) Yz = 0$, then $P D_y G(0,0) Yz = 0$, and hence also $D_y G(0,0) Yz \in \mathscr{U}$. But $\mathscr{U} \cap \mathscr{T}$ consists only of the zero vector. Thus, $D_y G(0,0) Yz = 0$; that is, $z = 0$. Q.E.D.

Having established the existence of solutions y and y_R of the manifold problem (2.4), (2.5) and the reduced basis problem (3.1), (3.2) on the interval $\Lambda_{\mathbf{R}} \equiv \Lambda_1 \cap \Lambda_R$, we turn to an examination of the error $y - y_R$. Let $\|\cdot\|$ be a vector norm on R^n. We have the following theorem relating the reduced basis error to the projection error.

THEOREM 3.3. *Let $\mathscr{S}_R$ and P be related by* (3.3). *Then there is an interval $\Lambda_* = [-\lambda_*, \lambda_*]$, $\lambda_* > 0$, and a constant C such that*

$$\|y(\lambda) - y_R(\lambda)\| \leqslant C\|Py(\lambda) - y(\lambda)\|, \qquad \lambda \in \Lambda_*. \tag{3.5}$$

Proof. From (2.4) and (3.1) it follows that for $\lambda \in \Lambda_{\mathbf{R}}$,

$$PG(y_R(\lambda), \lambda) = 0 = PG(y(\lambda), \lambda),$$

$$P[G(Py(\lambda), \lambda) - G(y_R(\lambda), \lambda)] = P[G(Py(\lambda), \lambda) - G(y(\lambda), \lambda)].$$

Since P has the representation (3.3), this equation may be written

$$U^T[G(YU^T y(\lambda), \lambda) - G(y_R(\lambda), \lambda)] = U^T[G(Py(\lambda), \lambda) - G(y(\lambda), \lambda)].$$

Letting $w(\lambda) = U^T y(\lambda)$ and recalling that $y_R(\lambda) = Yz(\lambda)$, we see that

$$U^T G(Yw(\lambda), \lambda) - U^T G(Yz(\lambda), \lambda) = U^T[G(Py(\lambda), \lambda) - G(y(\lambda), \lambda)]. \tag{3.6}$$

For each $\lambda \in \Lambda_{\mathbf{R}}$, we now define the mapping H_λ: $R^m \to R^m$, $H_\lambda(u) = U^T G(Yu, \lambda)$. Clearly, H_λ is continuously differentiable on $R^n \times R$. Moreover, $D_u H_0(0) = U^T D_y G(0,0) Y$, and this is nonsingular. Therefore, by the inverse function theorem there is an interval $\Lambda_H = [-\lambda_H, \lambda_H]$, $\lambda_H > 0$, a positive number ρ, and a ball $B_\rho = \{u \in R^m \mid \|u\| \leqslant \rho\}$ such that for each $\lambda \in \Lambda_H$, H_λ maps B_ρ in a 1-to-1 manner onto a closed neighborhood of $H_\lambda(0)$. Moreover, regarding the inverse mapping H_λ^{-1}, there is a constant C_1 such that $\|D_u H_\lambda^{-1}(u)\| \leqslant C_1$, $(u, \lambda) \in B_\rho \times \Lambda_H$. If we let $u(\lambda) = H_\lambda(w(\lambda))$, and $v(\lambda) = H_\lambda(z(\lambda))$, then (3.6) reads

$$u(\lambda) - v(\lambda) = U^T[G(Py(\lambda), \lambda) - G(y(\lambda), \lambda)]. \tag{3.7}$$

Since $u(0) = v(0) = 0$, by continuity there is an interval $\Lambda_0 = [-\lambda_0, \lambda_0]$, $\lambda_0 > 0$, such that $u(\lambda), v(\lambda) \in B_\rho$ for all $\lambda \in \Lambda_0$. If $\Lambda_* \equiv \Lambda_H \cap \Lambda_0$, then by (3.7) and the mean value theorem,

$$\begin{aligned} \|w(\lambda) - z(\lambda)\| &= \|H_\lambda^{-1}(u(\lambda)) - H_\lambda^{-1}(v(\lambda))\| \leqslant C_1\|u(\lambda) - v(\lambda)\| \\ &\leqslant C_1\|U^T\| \, \|G(Py(\lambda), \lambda) - G(y(\lambda), \lambda)\|, \qquad \lambda \in \Lambda_*. \end{aligned} \tag{3.8}$$

Furthermore, since $Py(\lambda)$ and $y(\lambda)$ are contained in some finite ball B_r, there is a constant C_2 such that $\|D_u G(y, \lambda)\| \leqslant C_2$ for all $(y, \lambda) \in B_r \times \Lambda_*$. It follows from (3.8) that for $\lambda \in \Lambda_*$,

$$\begin{aligned} \|Py(\lambda) - y_R(\lambda)\| &= \|Y(w(\lambda) - z(\lambda))\| \leqslant \|Y\| \, \|w(\lambda) - z(\lambda)\| \\ &\leqslant \|Y\| \, \|U^T\| C_1 C_2 \|Py(\lambda) - y(\lambda)\| \equiv \kappa\|Py(\lambda) - y(\lambda)\|. \end{aligned}$$

But then

$$\|y(\lambda) - y_R(\lambda)\| \leqslant \|y(\lambda) - Py(\lambda)\| + \|Py(\lambda) - y_R(\lambda)\|$$
$$\leqslant (1 + \kappa)\|y(\lambda) - Py(\lambda)\| \equiv C\|y(\lambda) - Py(\lambda)\|. \quad \text{Q.E.D.}$$

Although (3.5) relates the reduced basis error to the projection error, it does not directly yield the order of the approximation under various choices of the subspace, $\mathscr{S}_R$. For this purpose it is necessary to supplement (3.5) by a majorization of the projection error in terms of an approximation error in $\mathscr{S}_R$. If P is the orthogonal projection onto $\mathscr{S}_R$, i.e., the subspace $\mathscr{U}$ in Theorem 3.2 is the orthogonal complement of $\mathscr{S}_R$, and $w(\lambda)$ is any curve in $\mathscr{S}_R$, then in terms of the Euclidean norm, $\|\cdot\|_2$, we have the simple majorization

$$\|Py(\lambda) - y(\lambda)\|_2 \leqslant \|y(\lambda) - w(\lambda)\|_2.$$

For other projections and norms the following lemma generalizes this inequality.

LEMMA 3.4. *Let $\mathscr{S}_R$ be a subspace of R^n and let P be the projector onto $\mathscr{S}_R$ relative to a complement $\mathscr{U}$. Then, there is a constant K such that*

$$\|y - Py\| \leqslant K\|y - w\|, \qquad w \in \mathscr{S}_R. \tag{3.9}$$

Proof. Let $u = y - Py$, $v = Py - w$. If u or v vanishes, then (3.9) holds for any $K \geqslant 1$. If u and v are nonzero, then

$$u/\|u\|_2 \in \{ f \in \mathscr{U} \mid \|f\|_2 = 1 \} \equiv B_1,$$
$$v/\|v\|_2 \in \{ g \in \mathscr{S}_R \mid \|g\|_2 = 1 \} \equiv B_2.$$

But as B_1 and B_2 are compact sets satisfying $B_1 \cap B_2 = \{0\}$, we have

$$\sup|f^T g| \equiv \beta < 1,$$

where the supremum is taken over all $f \in B_1$, $g \in B_2$. Therefore,

$$\|y - w\|_2^2 \geqslant \|u\|_2^2 + \|v\|_2^2 - 2|u^T v| \geqslant \|u\|_2^2 + \|v\|_2^2 - 2\beta\|u\|_2\|v\|_2$$
$$= \|u\|_2^2 + (\|v\|_2 - \beta\|u\|_2)^2 - \beta^2\|u\|_2^2 \geqslant (1 - \beta^2)\|u\|_2^2.$$

Consequently,

$$\|y - Py\|_2 \leqslant (1 - \beta^2)^{-1/2}\|y - w\|_2.$$

This establishes (3.9) for the Euclidean norm and the general case then follows from the norm equivalence theorem. Q.E.D.

By combining (3.5) and (3.9) we obtain the estimate

$$\|y(\lambda) - y_R(\lambda)\| \leqslant L\|y(\lambda) - w(\lambda)\|, \qquad \lambda \in \Lambda_*, \tag{3.10}$$

where $L = CK$ and $w(\lambda)$ is any curve in $\mathscr{S}_R$.

In the next section we shall use (3.10) to obtain error estimates for some specific choices of the subspace $\mathscr{S}_R$.

4. Some Reduced Basis Subspaces. Before proceeding to the definition of the subspaces, we note that in general we do not know the solution curve y for $\lambda > 0$. Indeed, the whole idea of the reduced basis method is to provide an approximation for this part of the curve. As a practical matter then, in defining the subspaces of this section, we do not require any information beyond the knowledge of a finite number

of points on that part of the solution curve corresponding to $\lambda \in [-\lambda_*, 0]$, where λ_* is the constant guaranteed by Theorem 3.3.

The Taylor Subspace. In this case, assuming that y has M derivatives at $\lambda = 0$, we take

$$\mathscr{S}_R = \operatorname{span}\left\{ u^j \mid u^j = \left.\frac{d^j y}{d\lambda^j}\right|_{\lambda=0}, j = 1, \ldots, M \right\}. \tag{4.1}$$

In other words, we form a subspace $\mathscr{S}_R$ of dimension $m \leqslant M$ from linear combinations of the first M derivatives of the solution curve at $\lambda = 0$. This subspace has been extensively used by Noor and his coworkers to solve finite-element discretizations of nonlinear structural shell problems [6]-[9]. In this work the elements u^j are referred to as "global basis vectors" or "path derivatives". Peterson [10] has also used the Taylor subspace to generate finite-element solutions of the stationary Navier-Stokes equations.

If G is sufficiently smooth, then the vectors u^j may be obtained from successive differentiations of (2.4). Thus,

$$\begin{aligned} D_y G(0,0) u^1 &= -D_\lambda G(0,0), \\ D_y G(0,0) u^2 &= -D_{yy} G(0,0) u^1 u^1 + 2 D_{\lambda y} G(0,0) u^1 + D_{\lambda\lambda} G(0,0)\big], \end{aligned} \tag{4.2}$$

etc., where D_{yy}, $D_{\lambda y}$ and $D_{\lambda\lambda}$ are the coordinate representations of the indicated second derivatives of the mapping G. We observe that each u^j may be obtained from its predecessors by solving an $n \times n$ linear system having the *same* coefficient matrix $D_y G(0,0)$. Thus, as noted in [7], only one matrix factorization is required to obtain the u^j. However, in the most general case, it is clear from (4.2) and succeeding formulas, that when $j \geqslant 2$, it will require $O(n^{j+1})$ multiplications to form the right-hand side of the linear system defining u^j! The computational efficiency of the Taylor subspace in shell and fluid dynamics problems is apparently due to the fact that in these instances each coordinate function g_i is a low-order (e.g., quadratic or cubic) polynomial in only a few of the variables y_j. Hence, the right-hand sides of (4.2) and its successors may be computed in significantly fewer multiplications than the $O(n^{j+1})$ estimate of the general case.

Using (3.10), it is easy to estimate the error resulting from projection onto the Taylor subspace. Note that it suffices to establish such an estimate for any particular norm. We use the ∞-norm, $\|\cdot\|_\infty$, where for $y = (y_1, \ldots, y_n)$, $\Lambda_* = [-\lambda_*, \lambda_*]$,

$$\|y\|_\infty \equiv \max_{1 \leqslant i \leqslant n} \sup_{\Lambda_*} |y_i(\lambda)|.$$

COROLLARY 4.1. *Suppose that $d^{M+1}y/d\lambda^{M+1}$ is continuous on Λ_*. If $\mathscr{S}_R$ is given by* (4.1), *then*

$$\|y - y_R\|_\infty = O(\lambda_*^{M+1}). \tag{4.3}$$

Proof. Let $u^0 = 0$ and define the Taylor polynomial,

$$y_T(\lambda) = \sum_{j=0}^{M} \frac{\lambda^j}{j!} u^j.$$

Using (3.10) and the fact that $y_T(\lambda) \in \mathscr{S}_R$, we have

$$\|y - y_R\|_\infty \leqslant L\|y(\lambda) - y_T(\lambda)\|_\infty \leqslant \frac{L\|d^{M+1}y/d\lambda^{M+1}\|_\infty}{(M+1)!}\lambda_*^{M+1}. \quad \text{Q.E.D.} \tag{4.4}$$

By Corollary 4.1, the Taylor subspace provides an approximation of y which is of order $M+1$ in λ. However, as we have seen, it may be costly to implement. We now consider two other subspaces that yield the same asymptotic order of accuracy without the need to form and solve equations such as (4.2).

The Lagrange Subspace. Suppose that for $j = 1,\dots,M$, the points $\lambda_j \in [-\lambda_*, , 0)$ are distinct and $y(\lambda_j)$ is known. The Lagrange subspace is given by

$$\mathscr{S}_R = \operatorname{span}\{u^j \mid u^j = y(\lambda_j), j = 1,\dots,M\}; \tag{4.5}$$

that is, it is the set of linear combinations of M points on the solution curve. A subspace of this type was employed by Almroth and associates in their numerical treatment of nonlinear structural shell problems [1], [2].

Concerning the error, we have the same estimate as before. To see this, set $\lambda_0 = 0$, $u^0 = 0$, and define the Lagrange interpolating polynomial

$$y_L(\lambda) = \sum_{j=0}^{M} l_j(\lambda) u^j,$$

where $l_j(\lambda) \equiv \prod_{k=0;k\neq j}^{M}(\lambda - \lambda_k)/(\lambda_j - \lambda_k)$. Then, as in the proof of Corollary 4.1,

$$\|y - y_R\|_\infty \leqslant L\left\|y(\lambda) - \sum_{j=0}^{M} l_j(\lambda) y(\lambda_j)\right\|_\infty. \tag{4.6}$$

But it is well-known that for each $\lambda \in \Lambda_*$, there is an $\eta \in \Lambda_*$, such that

$$y_i(\lambda) - \sum_{j=0}^{M} l_j(\lambda) y_i(\lambda_j) = \frac{\prod_{k=0}^{M}(\lambda - \lambda_k)}{(M+1)!} \frac{d^{M+1} y_i(\eta)}{d\lambda^{M+1}}.$$

Hence,

$$\left\|y - \sum_{j=0}^{M} l_j(\lambda) y(\lambda_j)\right\|_\infty \leqslant \frac{2^{M+1}}{(M+1)!} \|d^{M+1} y/d\lambda^{M+1}\|_\infty \lambda_*^{M+1},$$

and this combined with (4.6) establishes the estimate.

As a final example, we consider a subspace related to discrete least-squares approximation.

The Discrete Least-Squares Subspace. Assume, as in the case of the Lagrange subspace, that $y(\lambda_k)$ is known at distinct points $\lambda_k \in [-\lambda_*, 0]$, $k = 1,\dots,K$. For $0 \leqslant M \leqslant K-1$, let $\{P_j, j = 0,\dots,M\}$ be a set of polynomials of degree at most M that are orthonormal on the set $\{\lambda_k\}$. That is,

$$(P_i, P_j) \equiv \sum_{k=1}^{K} P_i(\lambda_k) P_j(\lambda_k) = \delta_{ij}.$$

The discrete least-squares subspace is defined as

$$\mathscr{S}_R = \operatorname{span}\{u^j \mid u_j = (y, P_j), j = 0,\dots,M\}, \tag{4.7}$$

where $(y, P_j) \equiv ((y_1, P_j),\dots,(y_n, P_j))$.

Under the hypothesis that $y \in C^{M+1}(\Lambda_*)$, we again obtain the estimate (4.3). For a proof, we let $Q(\lambda) = \sum_{j=0}^{M} P_j(\lambda) u^j$. Then, as before,

$$\|y - y_R\|_\infty \leqslant L\|y(\lambda) - Q(\lambda)\|_\infty. \tag{4.8}$$

Writing $Q(\lambda) = (q_1(\lambda),\ldots,q_n(\lambda))$, we note that for $i = 1,\ldots,n$, q_i is the least-squares approximation of y_i on the set $\{\lambda_k\}$. But then, it is not difficult to show that q_i interpolates y_i at distinct points $\theta_j, j = 0,\ldots,M$, of $[-\lambda_*, 0]$ (see for example [12]). Therefore, as in the case of the Lagrange subspace, we have

$$y_i(\lambda) - q_i(\lambda) = \frac{\prod_{j=0}^{M}(\lambda - \theta_j)}{(M+1)!}\,\frac{d^{M+1}y_i(\eta)}{d\lambda^{M+1}}, \qquad \eta \in \Lambda_*,$$

and the result follows from this and (4.8).

5. Applications. In Table 5.1 we present a summary of some performance data that has emerged during the course of past applications of the reduced basis method. Unfortunately, only reference [2] contains reduction factors for the computation times involved when the reduced basis system (3.1) is solved instead of the full system (2.1). However, even from these few cases we see that the average reduction factor exceeds 2, and indications are that for larger problems of this type, it could be as large as 5.

In all of these applications, the systems (2.1) resulted from finite-element discretizations of the corresponding infinite-dimensional operator equations. We also note that in accordance with the error estimates presented in Section 4, the reduced basis solutions were remarkably accurate. Details of the various implementation strategies used are contained in the given references.

TABLE 5.1
Performance Data

	Ref.	Subspace	Original System Size (n)	Reduced System Size (m)	Reduction Factor
Compression of pear shaped cylinder	2	Lagrange	1300	$\leqslant 6$	2.5
Bending of long cylinder	2	Lagrange	939	$\leqslant 9$	3.1
Point force on a spherical cap	2	Lagrange	1966	$\leqslant 11$	2.0
Panel in compression	2	Lagrange	3044	$\leqslant 8$	1.4
Compression of cutout cylinder	2	Lagrange	7055	$\leqslant 7$	2.8
	7	Taylor	2996	7	—
Clamped cylindrical panel	6	Taylor	500	5 and 6	—
	7	Taylor	287	6	—
Buckling of spherical shell	6	Taylor	500	4 and 5	—
Buckling of 24-ply rectangular plate	9	Taylor	714	8	—
Buckling of 8-ply rectangular plate	9	Taylor	793	8	—
Steady driven cavity flow	10	Taylor	~ 700	5	—

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