The Error Norm of Certain Gaussian Quadrature Formulae

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Abstract. We consider Gauss quadrature formulae $Q_n$, $n \in \mathbb{N}$, approximating the integral $I(f) := \int_{-1}^{1} w(x)f(x) \, dx$, $w = W/p_i$, $i = 1, 2$, with $W(x) = (1 - x)\alpha(1 + x)\beta$, $\alpha, \beta = \pm 1/2$ and $p_i(x) = 1 + a_i^2 + 2ax$, $p_2(x) = (2b + 1)x^2 + b^2$, $b > 0$. In certain spaces of analytic functions the error functional $R_n := I - Q_n$ is continuous. In [1] and [2] estimates for $\|R_n\|$ are given for a wide class of weight functions. Here, for a restricted class of weight functions, we calculate the norm of $R_n$ explicitly.

1. Introduction. Consider the integral $I$, 

$$I(f) = \int_{-1}^{1} w(x)f(x) \, dx,$$

approximated by the Gaussian quadrature formula $Q_n$,

$$Q_n(f) = \sum_{i=1}^{n} w_i f(x_i).$$

Let $P_k$, $P_k(x) = \alpha_k x^k + \beta_k x^{k-1} + \cdots$, $\alpha_k > 0$, $k \in \mathbb{N}_0$, be the orthonormal polynomials corresponding to the weight function $w$, i.e.,

$$\int_{-1}^{1} w(x) P_i(x) P_j(x) \, dx = \delta_{ij}.$$

The following classical representation for the error term $R_n(f) := I(f) - Q_n(f)$ can be found, e.g., in [4, p. 75],

$$\bigwedge_{f \in C^2[-1,1]} \bigvee_{\xi \in (-1,1)} R_n(f) = \frac{1}{(2n)!\alpha^2} f^{(2n)}(\xi).$$

The estimate

$$|R_n(f)| \leq \frac{1}{(2n)!\alpha^2} \|f^{(2n)}\|_{\infty},$$

following immediately from (1.1), is often unsatisfactory, since bounds for higher derivatives are required, and, in addition, the calculation usually has to be repeated for different values of $n$.

For analytic functions Hämmerlin [8] suggested the following method for obtaining derivative-free error estimates: Let $q_\kappa(x) := x^\kappa$, $\kappa \in \mathbb{N}_0$, $r > 1$ and $C_r := \{ z \in \mathbb{C} : |z| < r \}$. For a function $f$ holomorphic in $C_r$,

$$f(z) = \sum_{\kappa=0}^{\infty} a_{\kappa} z^\kappa,$$

$z \in C_r$. 

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define
\[ |f|_r := \sup \{ |\alpha^r| : \alpha \in \mathbb{N}_0 \text{ and } R_n(q_\kappa) \neq 0 \}. \]

In the space
\[ X_r := \{ f : f \text{ holomorphic in } C_r \text{ and } |f|_r < \infty \} \]

\(| \cdot |_r \) is a seminorm. The error functional \( R_n \) is continuous in \( (X_r, | \cdot |_r) \), and for the error norm
\[ \| R_n \| := \sup \left\{ \frac{|R_n(f)|}{|f|_r} : f \in X_r, |f|_r \neq 0 \right\} \]

the relation
\[ (1.5) \quad \| R_n \| = \sum_{\kappa=0}^{\infty} \frac{|R_n(q_\kappa)|}{r^{\kappa}} \]

holds (see [8], [1], [2]).

For the weight functions considered here, either the condition
\[ (1.6) \quad \frac{w(\cdot)}{w(-\cdot)} \text{ is nondecreasing} \]
or the condition
\[ (1.7) \quad \frac{w(\cdot)}{w(-\cdot)} \text{ is nonincreasing} \]
is valid.

Condition (1.6) implies
\[ (1.8) \quad R_n(q_\kappa) \geq 0, \quad \kappa \in \mathbb{N}_0 \]
(see [5]). Thus, from (1.5) there follows
\[ \| R_n \| = \sum_{\kappa=0}^{\infty} \frac{R_n(q_\kappa)}{r^{\kappa}} = R_n \left( \sum_{\kappa=0}^{\infty} \frac{q_\kappa}{r^{\kappa}} \right), \]
i.e.,
\[ (1.9) \quad \| R_n \| = r R_n(\varphi) \quad \text{with } \varphi(x) := 1/(r - x). \]

Let the polynomial \( \pi_{n-1} \) of degree less than \( n \) interpolate the function \( \varphi \) at the abscissae \( x_1, \ldots, x_n \) of \( Q_n \). Since \( Q_n \) integrates \( \pi_{n-1} \) exactly, \( R_n(\varphi) = R_n(\varphi - \pi_{n-1}) \) holds. Setting \( \Pi_n(x) := (x - x_1) \cdots (x - x_n) \), we obtain
\[ \varphi(x) - \pi_{n-1}(x) = \gamma_n \Pi_n(x)/(r - x), \]
where \( \gamma_n \) is a constant, because the function on the left-hand side vanishes at \( x_1, \ldots, x_n \). Multiplying by \( r - x \) and taking the limit as \( x \to r \) we obtain \( \gamma_n = 1/\Pi_n(r) \) (see [3, pp. 71-72]). Thus, from (1.9) we get the representation
\[ (1.10) \quad \| R_n \| = \frac{r}{\Pi_n(r)} \int_{-1}^{1} w(x) \frac{\Pi_n(x)}{r - x} dx \quad \text{with } \Pi_n(x) = \prod_{i=1}^{n} (x - x_i), \]
for weight functions satisfying (1.6).

If \( w \) satisfies (1.7),
\[ (1.11) \quad (-1)^* R_n(q_\kappa) \geq 0 \]
holds (see [5]), and we obtain similarly
\[ (1.12) \quad \| R_n \| = r R_n(\psi), \quad \psi(x) := 1/(r + x), \]
and

\begin{equation}
\|R_n\| = \frac{r}{\Pi_n(-r)} \int_{-1}^{1} w(x) \frac{\Pi_n(x)}{r + x} \, dx \quad \text{with } \Pi_n(x) = \prod_{i=1}^{n} (x - x_i).
\end{equation}

In [1] and [2], estimates for \(\|R_n\|\) were derived for weight functions satisfying (1.6) or (1.7), and \(\|R_n\|\) was given for \(w = W\). Starting from (1.10) or (1.13) respectively, in the next section we calculate the norm of \(R_n\) for weight functions \(w\) with

\[w = \frac{W}{p_i}, \quad i = 1, 2,\]

\[W(x) = (1 - x)^{\alpha}(1 + x)^{\beta}, \quad \alpha, \beta = \pm \frac{1}{2},\]

\[p_1(x) = 1 + a^2 + 2ax,\]

\[p_2(x) = (2b + 1)x^2 + b^2, \quad b > 0.\]

Two numerical examples conclude the paper.

**Remark.** For even weight functions, (1.4) can be written as \(|f| = \sup_{\kappa \geq 2n}(|\alpha_{2k}| r^{2k})\) (cf. [1]). If \(w(-\cdot)/w(-\cdot)\) is strictly monotone, then \(R_n(q_k) \neq 0\) for \(k \geq 2n\) (see [5]), and \(\|\cdot\|\) can be equivalently defined by \(|f| := \sup_{\kappa \geq 2n}(|\alpha_{2k}| r^{2k})\).

**2. The Norm of the Error Functional.**

a. \(p_1(x) = 1 + a^2 + 2ax\). The case \(a = 0, \pm 1\) is treated in [1], [2] if \(w\) remains integrable. For \(|a| < 1, a \neq 0\), put \(d = 1/a\) to obtain \(p_1(x) = a^2(1 + d^2 + 2dx), \quad |d| > 1\). Therefore we only consider the case \(|a| > 1\).

We first summarize some results of Kumar [9] which are important for the subsequent development.

**Lemma 1.** Let \(p_1(x) = 1 + a^2 + 2ax, \quad |a| > 1, \quad W(x) = (1 - x)^{\alpha}(1 + x)^{\beta}\) and \(w = W/p_1\). Let \(T_i\) and \(U_i\) be the Chebyshev polynomials of the first and second kind, respectively. Then the abscissae \(x_1, \ldots, x_n\) of the Gauss quadrature formula \(Q_n\) corresponding to \(w\) are the zeros of

(i) \(aT_n + T_{n-1}\) if \(\alpha = \beta = -1/2,\)

(ii) \(aU_n + U_{n-1}\) if \(\alpha = \beta = 1/2,\)

(iii) \(aU_n + (1 + a)U_{n-1} + U_{n-2}\) if \(\alpha = -\beta = 1/2\) and \(n > 1\).

**Remark.** For \(\alpha = \beta = \pm 1/2\) the condition (1.6) is satisfied if \(a < -1\), the condition (1.7) if \(a > 1\). For \(\alpha = -\beta = -1/2, (1.6)\) holds, for \(\alpha = -\beta = 1/2\) we have (1.7).

We now establish the first of our results.

**Theorem 1.** Consider \(p_1(x) = 1 + a^2 + 2ax, \quad |a| > 1, \quad W(x) = (1 - x)^{\alpha}(1 + x)^{\beta}\), \(w = W/p_1\). Let \(\tau := r - \sqrt{r^2 - 1}\). For the norm of the error functional \(R_n\) the following is true:

\begin{equation}
\|R_n\| = \frac{2\pi r \tau^{2n}}{[(\tau + a)[(1 + \tau^{2n-2}) + a(1 + \tau^{2n})]\sqrt{r^2 - 1}}
\end{equation}

for \(\alpha = \beta = -1/2\) and \(a < -1,\)

\begin{equation}
\|R_n\| = \frac{2\pi r \tau^{2n+2} \sqrt{r^2 - 1}}{[(\tau + a)[(1 - \tau^{2n}) + a(1 - \tau^{2n+2})]}
\end{equation}
for $\alpha = \beta = 1/2$ and $a < -1$,

$$\|R_n\| = \frac{2\pi r^{2n+1}}{(\tau - a)[\tau(1 + \tau^{2n-1}) - a(1 + \tau^{2n+1})]} \left(\frac{r + 1}{r - 1}\right)^{1/2}$$

for $\alpha = -\beta = 1/2$ and $n > 1$.

**Proof.** First, let us verify the identity (2.1). The weight function $w$ satisfies condition (1.6) for $\alpha = \beta = -1/2$ and $a < -1$. Thus, by Lemma 1 (i) and (1.10),

$$aT_n(x) + T_{n-1}(x)$$

holds. Let the integral on the right-hand side of (2.4) be denoted by $I_n(a, r)$. Substituting $x = \cos y$ we obtain

$$I_n(a, r) = \int_0^\pi \frac{a \cos(ny) + \cos[(n - 1)y]}{(r - \cos y)(1 + a^2 + 2a \cos y)} dy.$$

Set

$$C_n(a) := 2a \int_0^\pi \frac{a \cos(ny) + \cos[(n - 1)y]}{1 + a^2 + 2a \cos y} dy$$

to obtain

$$I_n(a, r) = \frac{1}{1 + a^2 + 2ar} \left\{ \int_0^\pi \frac{a \cos(ny) + \cos[(n - 1)y]}{r - \cos y} dy + C_n(a) \right\}.$$

Since

$$\int_0^\pi \frac{\cos(my)}{r - \cos y} dy = \frac{\pi r^m}{\sqrt{r^2 - 1}}$$

(cf., e.g., [7, p. 112]), we have

$$I_n(a, r) = \frac{1}{1 + a^2 + 2ar} \left\{ \frac{\pi r^{n-1}(a + 1)}{\sqrt{r^2 - 1}} + C_n(a) \right\}.$$\]

By (1.5), $\|R_n\| = O(r^{-2n})$ holds for $r \to \infty$, and (2.4) yields $I_n(a, r) = O(r^{-n-1})$ for $r \to \infty$. Therefore $C_n(a) = 0$, which can also be established by straightforward calculation. Thus,

$$I_n(a, r) = \frac{\pi r^n}{(\tau + a)\sqrt{r^2 - 1}}.$$\]

Combining this with $T_m(r) = [(r - \sqrt{r^2 - 1})^m + (r + \sqrt{r^2 - 1})^m]/2$ (see [11, p. 5]), the relation (2.1) follows from (2.4).

(2.2) can be proved in a similar way. To prove (2.3), use the relation

$$(1 - x)\left[U_m(x) + U_{m-1}(x)\right] = T_m(x) - T_{m+1}(x),$$

which immediately follows from well-known identities for Chebyshev polynomials (cf., e.g., [11, p. 9]).

**Remark.** $I_n(a, r)$ is also calculated by Kumar [9] by means of the generating function for the polynomials $aT_n + T_{n-1}$.
**Corollary 1.** Let \( p_1(x) = 1 + a_2 + 2ax, |a| > 1, \) \( W(x) = (1 - x)^a(1 + x)^b \) and \( w = W/p_1. \) Then the norm of \( R_n \) can be expressed as

\[
\|R_n\| = \frac{2\pi r^{2n}}{(\tau - a)^2 \left[ \tau(1 + \tau^{2n-2}) - a(1 + \tau^{2n}) \right] \sqrt{r^2 - 1}}
\]

if \( \alpha = \beta = -1/2 \) and \( a > 1, \) and as

\[
\|R_n\| = \frac{2\pi r^{2n+2} \sqrt{r^2 - 1}}{(\tau - a)^2 \left[ \tau(1 - \tau^{2n} - a(1 - \tau^{2n+2}) \right]}
\]

if \( \alpha = \beta = 1/2 \) and \( a > 1, \) and as

\[
\|R_n\| = \frac{2\pi r^{2n+1}}{(\tau + a)^2 \left[ \tau(1 + \tau^{2n-1}) + a(1 + \tau^{2n+1}) \right] \left( \frac{r + 1}{r - 1} \right)^{1/2}}
\]

if \( \alpha = -\beta = -1/2 \) and \( n > 1. \)

**Proof.** Let \( R_n \) and \( R_n^* \) be the error functionals corresponding to the weight functions \( w \) and \( w(-\cdot), \) respectively. Then obviously \( R_n(q_k) = (-1)^k R_n^*(q_k) \) holds, and thus \( \|R_n\| = \|R_n^*\|. \) Hence, the corollary immediately follows from Theorem 1.

b. \( p_2(x) = (2b + 1)x^2 + b^2, b > 0. \) We first summarize some results of Kumar [10] which are needed in the sequel.

**Lemma 2.** Let \( p_2(x) = (2b + 1)x^2 + b^2, b > 0, \) \( W(x) = (1 - x)^a(1 + x)^b \) and \( w = W/p_2. \) The abscissae \( x_1, \ldots, x_n \) of the Gauss quadrature formula \( Q_n \) corresponding to \( w \) are the zeros of

(i) \( (2b + 1)T_n + T_{n-2} \) if \( \alpha = \beta = -1/2 \) and \( n > 1, \)
(ii) \( (2b + 1)U_n + U_{n-2} \) if \( \alpha = \beta = 1/2 \) and \( n > 1, \)
(iii) \( (2b + 1)(U_n + U_{n-1}) + U_{n-2} + U_{n-3} \) if \( \alpha = -\beta = 1/2 \) and \( n > 2. \)

Our second result is presented in the following theorem.

**Theorem 2.** Let \( p_2(x) = (2b + 1)x^2 + b^2, b > 0, \) \( W(x) = (1 - x)^a(1 + x)^b \) and \( w = W/p_2. \) For the norm of the error functional we have:

\[
\|R_n\| = \frac{4\pi r^{2n}}{(b + r\tau)^2 \left( (2b + 1)(1 + \tau^{2n}) + \tau^2(1 + \tau^{2n-4}) \right) \sqrt{r^2 - 1}}
\]

for \( \alpha = \beta = -1/2, n > 1, \)

\[
\|R_n\| = \frac{4\pi r^{2n+2} \sqrt{r^2 - 1}}{(b + r\tau)^2 \left( (2b + 1)(1 - \tau^{2n+2}) + \tau^2(1 - \tau^{2n-2}) \right)}
\]

for \( \alpha = \beta = 1/2, n > 1, \) and

\[
\|R_n\| = \frac{4\pi r^{2n+1}}{(b + r\tau)^2 \left( (2b + 1)(1 + \tau^{2n+1}) + \tau^2(1 + \tau^{2n-1}) \right) \left( \frac{r + 1}{r - 1} \right)^{1/2}}
\]

for \( \alpha = -\beta = 1/2, n > 2. \)

**Proof.** In this case (1.7) holds, and the results follow from (1.13) using Lemma 2.

Symmetry arguments yield the following corollary.

**Corollary 2.** Let \( w(x) = ((1 + x)/(1 - x))^{1/2}/[(2b + 1)x^2 + b^2], b > 0. \) The norm of the error functional corresponding to \( w \) is then given by (2.10) also.
Remark. Let \( K_n(z) := R_n(\varphi_z), \varphi_z(x) := 1/(z - x), |z| = r \). If \( f \) is holomorphic in a region \( B \) including \( C_r \), the representation
\[
R_n(f) = \frac{1}{2\pi i} \int_{C_r} K_n(z) f(z) \, dz
\]
holds. Gautschi and Varga [6] showed that for weight functions satisfying either (1.6) or (1.7)

\[
\max_{|z|=r} |K_n(z)| = \max\{K_n(r), |K_n(-r)|\} = \sum_{\kappa=0}^{\infty} \frac{|R_n(\varphi_{\kappa})|}{r^{\kappa+1}}
\]
holds. Therefore, we have \( \max_{|z|=r} |K_n(z)| = |R_n|/r \), and for the weight functions considered here \( \max_{|z|=r} |K_n(z)| \) has also been determined.

3. Numerical Results. For \( f \in X_\rho \), \( |R_n(f)| \) is bounded by \( \|R_n\| \|f\|_r, r \in (1, \rho] \). Therefore,
\[
|R_n(f)| \leq \inf_{1 < r < \rho} \left( \frac{\|R_n\| \|f\|_r}{r} \right)
\]
holds. (Although not explicitly noted, \( \|R_n\| \) is obviously a function of \( r \).) Estimating \( \|f\|_r \) by \( \|f\|_{2,r} \),
\[
\|f\|_{2,r} := \frac{1}{\sqrt{2\pi r}} \left( \int_{|z|=r} |f(z)|^2 |dz| \right)^{1/2},
\]
or by \( \max_{|z|=r} |f(z)| \), which exist at least for \( r < \rho \), we obtain
\[
|R_n(f)| \leq \inf_{1 < r < \rho} \left( \frac{\|R_n\| \|f\|_{2,r}}{r} \right)
\]
and
\[
|R_n(f)| \leq \inf_{1 < r < \rho} \left( \frac{\|R_n\| \max_{|z|=r} |f(z)|}{r} \right),
\]
respectively (see [8]). The sharpness of these estimates is demonstrated by two numerical examples.

Example 1. Let \( f(z) := \exp(z), f \in X_\rho, r > 1 (\rho = \infty) \). Approximate the integral
\[
\int_{-1}^{1} \frac{1}{(3 + 2\sqrt{2})(1 + x^2)\sqrt{1 - x^2}} f(x) \, dx
\]
by the Gaussian quadrature formula \( Q_2 \) corresponding to
\[
w(x) = \frac{1}{(3 + 2\sqrt{2})(1 + x^2)\sqrt{1 - x^2}}.
\]
The abscissae and the weights of \( Q_2 \) are given in [10]. The remainder term is \( R_2(f) \approx 2.016 \cdot 10^{-3} \). Setting \( b = 1 + \sqrt{2} \) and \( n = 2 \) in (2.8), we obtain the norm of the error functional \( R_2 \). With \( |f|_r = r^{4}/24 \) for \( 1 < r < \sqrt{30} \), \( |f|_r = r^6/720 \) for \( \sqrt{30} < r < \sqrt{56} \), and so on, and \( \max_{|z|=r} |f(z)| = \exp(r) \), (3.1) and (3.3) yield for \( |R_2(f)| \) the bounds \( 2.019 \cdot 10^{-3} (r = 5.45) \) and \( 1.073 \cdot 10^{-2} (r = 4.15) \), respectively.
Example 2. Let
\[ f(z) = \sum_{k=4}^{\infty} \left( \frac{z}{2} \right)^k = \frac{1}{8} \frac{z^4}{2 - z}, \quad f \in X_r \text{ for } r \in (1, 2] \ (\rho = 2). \]

The remainder term \( R_2(f) \) for the approximation of
\[ \int_{-1}^{1} \frac{1}{(5 + 4x)\sqrt{1 - x^2}} f(x) \, dx \]
by the Gaussian quadrature formula \( Q_2 \) corresponding to
\[ w(x) = \frac{1}{(5 + 4x)\sqrt{1 - x^2}} \]
is \( 7.18 \cdot 10^{-3} \). The abscissae of \( Q_2 \) are the zeros of \( 2T_2 + T_1 \) (Lemma 1(i), \( a = 2 \)). We have
\[ |f|_r = \frac{r^4}{16}, \quad \|f\|_2,r = \left[ \sum_{k=4}^{\infty} \left( \frac{r}{2} \right)^{2k} \right]^{1/2} = \frac{r^4}{8\sqrt{1 - r^2}} \]
(cf. [8]) and \( \max_{|z|=r} |f(z)| = r^4/(16 - 8r) \). Setting \( a = 2 \) and \( n = 2 \) in (2.5), we obtain the norm of \( R_2 \). Now, from (3.1), (3.2) and (3.3), we get for \( |R_2(f)| \) the bounds \( 1.25 \cdot 10^{-2} \) \((r = 2)\), \( 3.06 \cdot 10^{-2} \) \((r = 1.65)\) and \( 8.75 \cdot 10^{-2} \) \((r = 1.50)\), respectively.

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