Supplement to
Linear Multistep Methods for Volterra Integral and Integro-Differential Equations

By P. J. van der Houwen and H. J. J. te Riele

In these appendices we present, successively,
I conditions for the existence of a unique solution of (1.1) and (1.2);
II three tables of coefficients of forward differentiation formulas, and
   of two common LM formulas for ODEs, viz., backward differentiation
   formulas and Adams-Moulton formulas;
III two lemmas which are needed in;
IV proofs of the main results of this paper, as far as they are non-
   trivial (in the opinion of the authors).

APPENDIX I

Conditions for the existence of a unique solution \( y(t) \in C(1) \) of (1.1) with \( \theta = 1 \)
- \( K(t,T,y) \) is continuous with respect to \( t \) and \( T \), for all \( (t,T) \in S \);
- \( K \) satisfies a (uniform) Lipschitz condition with respect to \( y \), i.e.,
  \[ |K(t,T,y) - K(t,T,z)| \leq L |y-z|, \]
  for all \( (t,T) \in S \), for all finite \( y,z \in \mathbb{R} \);
- \( g(t) \in C(1) \).

Conditions for the existence of a unique solution \( y(t) \in C(1) \) of (1.1) with \( \theta = 0 \)
- \( K(t,T,y) \in C^{1}(S \times \mathbb{R}) \);
- for \( t = t \) the derivative \( 3K/3y \) is bounded away from zero;
  \[ |3K(t,T,y)/3y| \geq r_{0} > 0 \]
  for all \( t \in I, y \in \mathbb{R} \);
- \( 3K(t,T,y)/3t \) satisfies a (uniform) Lipschitz condition with respect to \( y \)
  on \( S = \mathbb{R} \);
- \( g(t) \in C^{1}(1) \) with \( g(t_{0}) = 0 \).

Conditions for the existence of a unique solution \( y(t) \in C^{1}(1) \) of (1.2),
for given initial value \( y(t_{0}) = y_{0} \)
The following three (uniform) Lipschitz conditions:
- \[ |f(t,y_{1},z) - f(t,y_{2},z)| \leq L_{1} |y_{1} - y_{2}|, \]
  for all \( t < 1 \), for all finite \( z,y_{1},y_{2} \in \mathbb{R} \);
- \(|f(t_1, y_{12}) - f(t_2, y_{22})| = t_{21}|y_{12} - y_{22}|, \) for all \( t_1, t_2 \) in \( S_1, \) for all \( x, y_{12}, y_{22} \) in \( R \);
- \(|K(t_1, y_{12}) - K(t_2, y_{22})| = t_{21}|y_{12} - y_{22}|, \) for all \( t_1, t_2 \) in \( S_1, \) for all \( x, y_{12}, y_{22} \) in \( R \).

**APPENDIX II**

Table 1  Coefficients of forward differentiation formulas

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<th>( k )</th>
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<td>-10/3</td>
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Table 2  Coefficients of the backward differentiation formulas

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Table 3  Coefficients of the Adams-Moulton formulas

\[ f(t) = g(t), \quad f_n - f_{n+1} = \sum_{i=0}^k b_i f_{n-i} \]

<table>
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**APPENDIX III**

**Lemma A.1.** Let \( x_n \geq 0 \) for \( n = 0, 1, \ldots, N, \) and suppose that
\[ x_n = hC_1 \sum_{k=0}^{n-1} k + C_2, \quad n = k, k+1, \ldots, N, \]
where \( k > 0, h > 0 \) and \( C_1 > 0 \) (i.e., 2). Suppose, moreover, that \( x_j \leq z/k \)
for \( j = 1, \ldots, k-1. \) Then
\[ x_n \leq (hC_1k^2C_2)(1+ hC_1)^{n-k}, \quad n = k, k+1, \ldots, N. \]

**Proof.** See [ ? ].

**Lemma A.2.** Consider the linear inhomogeneous difference equation with constant coefficients \( \xi_j: \)
\[ (A.1) \quad \xi^{n+k} + \xi^{n+k-1} + \cdots + \xi^{n} = b_n f_n, \quad n \geq 0, \]
where \( \{b_n\} \) is a given sequence, independent of the \( f. \)
(i) If the characteristic polynomial \( Z(s) := \sum_{i=0}^{n+k} Z_i s^i \) is simple von Neumann (cf. Section 2.3) then the solution of \((A.1)\) satisfies the inequality
\[ |y_n| \leq C \max_{0 \leq j \leq k} |y_j| + \max_{k+1 \leq j \leq n} |g_j|, \quad n \geq k, \]

where \( C \) is independent of \( n \).

(ii) If \( f(s) \) is Schur (cf. Section 2.1) then the solution of (A.1) satisfies the inequality

\[ |y_n| \leq C \max_{0 \leq j \leq k} |y_j| + \max_{j + k + 1 \leq j \leq n} |g_j|, \quad n \geq k, \]

where \( C \) is independent of \( n \).

**Proof.** See [7].

**Appendix IV**

**Proof of Theorem 2.2.1.** Taylor expansion of \( Y(t_{n+j}, t_{n-1}) \) around \( (t_n, t_n) \) yields

\[
Y_n(Y) = \sum_{i=0}^{k} \frac{p}{q_0} \int_{0}^{\pi} h^q \left( \begin{array}{c} a \times \frac{3}{3k} - \frac{3}{3k} \\ b \times \frac{3}{3k} - \frac{3}{3k} \\ \end{array} \right) Y(t, s) \]

\[ + \sum_{j=k}^{n} \left( \sum_{i=0}^{k} \frac{p}{q_0} \int_{0}^{\pi} h^q \left( \begin{array}{c} a \times \frac{3}{3k} - \frac{3}{3k} \\ b \times \frac{3}{3k} - \frac{3}{3k} \\ \end{array} \right) Y(t, s) \right) (t_n, t_n) \]

\[ + O(h^{p+1}) \quad \text{as} \quad h \to 0. \]

Writing this formula in the form

\[
Y_n(Y) = \sum_{i=0}^{k} \frac{p}{q_0} \int_{0}^{\pi} h^q \left( \begin{array}{c} a \times \frac{3}{3k} - \frac{3}{3k} \\ b \times \frac{3}{3k} - \frac{3}{3k} \\ \end{array} \right) Y(t, s) \]

\[ + O(h^{p+1}) \]

and expanding the differential operator \( D_q \) by the binomial theorem we find

\[
D_q = \sum_{i=0}^{k} \frac{p}{q_0} \int_{0}^{\pi} h^q \left( \begin{array}{c} a \times \frac{3}{3k} - \frac{3}{3k} \\ b \times \frac{3}{3k} - \frac{3}{3k} \\ \end{array} \right) Y(t, s) \]

\[ + \sum_{j=k}^{n} \left( \sum_{i=0}^{k} \frac{p}{q_0} \int_{0}^{\pi} h^q \left( \begin{array}{c} a \times \frac{3}{3k} - \frac{3}{3k} \\ b \times \frac{3}{3k} - \frac{3}{3k} \\ \end{array} \right) Y(t, s) \right) (t_n, t_n) \]

\[ + O(h^{p+1}) \quad \text{as} \quad h \to 0. \]

where \((-i)^{p+1} \xi \) is assumed to be zero for \( i = \xi \). Equating to zero all terms in the \( n^{\text{th}} \) order yields the order equations (2.2.3) and at the same time \( L_n(Y) = O(h^{p+1}) \) as required in Definition 2.2.1.

**Supplement**

**Proof of Theorem 2.2.2.** Taylor expansion of \( Y(t_{n+j}, t_{n-1}) \) around \( (t_n, t_n) \) yields

\[
Y(t_{n+j}, t_{n-1}) = \sum_{i=0}^{k} \frac{p}{q_0} \int_{0}^{\pi} h^q \left( \begin{array}{c} a \times \frac{3}{3k} - \frac{3}{3k} \\ b \times \frac{3}{3k} - \frac{3}{3k} \\ \end{array} \right) Y(t, s) \]

\[ + O(h^{p+1}) \quad \text{as} \quad h \to 0. \]

In order to exploit the fact that \( Y(t, t) = 0 \) (see definition 2.2.1), we introduce the variables \( u = t + s \) and \( v = t - s \) and write

\[
Y(t, s) = Y_{\frac{u+v}{2}, \frac{u-v}{2}} \equiv Z(u, v). \]

The identity \( Y(t, t) = 0 \) implies that \( Z \) and all its derivatives with respect to \( u \) vanish for \( u = 2t \) and \( v = 0 \). In the following we use the notation

\[
Z(u, v) = \frac{\partial^m}{2u^m} (2t, 0). \]

By means of the binomial theorem we have

\[
Y(t_{n+j}, t_{n-1}) = \sum_{i=0}^{k} \frac{p}{q_0} \int_{0}^{\pi} h^q \left( \begin{array}{c} a \times \frac{3}{3k} - \frac{3}{3k} \\ b \times \frac{3}{3k} - \frac{3}{3k} \\ \end{array} \right) Y(u, v) \quad (2t, 0) \quad O(h^{p+1}) \]

\[ + \sum_{j=k}^{n} \left( \sum_{i=0}^{k} \frac{p}{q_0} \int_{0}^{\pi} h^q \left( \begin{array}{c} a \times \frac{3}{3k} - \frac{3}{3k} \\ b \times \frac{3}{3k} - \frac{3}{3k} \\ \end{array} \right) Y(u, v) \right) (2t, 0) \quad O(h^{p+1}) \quad \text{as} \quad h \to 0. \]

and
\[(A.3) \quad h_\nu(t_{n+j},t_{n-i}) = \sum_{q=0}^{p} \sum_{\lambda=0}^{q} \frac{1}{\lambda!} h^{(\lambda)}(-i,j) q^{\lambda} \left( z_{n+j}^2 \right)^{\lambda} q^{-\lambda} \left( z_{n-i}^2 \right)^{\lambda} + 0(h^{p+1}) \]

+ 0(h^{p+1}) as \( n \to 0. \)

Substitution of (A.2) and (A.3) into \( I_n[Y] \) and using \( z(0,0) = 0 \) yields

\[I_n[Y] = \sum_{q=0}^{p} \frac{1}{q!} h^{(q)}(-i,j) q^{q} B_{q} \left( z_{n+j}^2 \right) + 0(h^{p+1}) \text{ as } n \to 0 \]

where \( B_{q} \) is defined in (2.2.4). This proves the theorem. \( \square \)

**PROOF OF THEOREM 2.3.1.**

**PROOF.** Taylor expansion in a fixed point \( t = t_n \) yields, respectively,

\[Y(t_{n+j}) = \sum_{q=0}^{p} \frac{1}{q!} (-i) q \left( z_{n+j}^2 \right)^{q} + 0(h^{p+1}) \]

\[Y(t_{n-i}) = \sum_{q=0}^{p} \frac{1}{q!} (-i) q \left( z_{n-i}^2 \right)^{q} + 0(h^{p+1}) \]

From these expansions it is immediate that the VLM formula (2.1.4) satisfies the relation

\[(A.4) \quad I_n[Y(t_{n+j}) + I_n[Y(t_{n-i}) - h_{ij} K(t_{n+j})]] \]

\[= \sum_{q=0}^{p} \frac{1}{q!} q \left( z_{n+j}^2 \right)^{q} + 0(h^{p+1}) \]

where \( A_q \) and \( C_q \) are defined by (2.3.2) and (2.2.3), respectively. Under the conditions of the theorem it is easily verified that this equation leads to (2.3.3). Furthermore, (2.3.3) is obviously the \( m \)-times differentiated form of equation (1.1). \( \square \)

**PROOF OF THEOREM 2.3.2.** Let \( Y(t,s) \) be given by (1.6) where \( y(t) \) is the exact solution of (1.1), then we may write for \( n \to k \)

\[I_n[Y] = I_k[Y] = \sum_{q=0}^{p} \frac{1}{q!} (-i) q \left( z_{n+i}^2 \right)^{q} + 0(h^{p+1}) \]

Substitution of the functions \( Y(t,s) \) and \( Y_n(t) \) and using (2.1.3) and (2.3.6b) leads to
\begin{equation}
L_n(Y) = \frac{1}{k} \sum_{i=0}^{k} \left( t_i \epsilon_{n-i} + \frac{1}{k} \left[ \delta_{ij} \left( \sum_{i=0}^{n} \frac{1}{h^2} \sum_{j=0}^{n} (t_{n+j}^2 - t_j^2) \gamma(t_{n+j}, t_j) \right) + \epsilon_{n-i} \right] \right).
\end{equation}

Thus, we have found for the errors \( \epsilon_n \) the relation

\begin{equation}
a_n = \frac{1}{k} \sum_{i=0}^{k} \sum_{j=0}^{n-i} \left[ h \delta_{ij} \left( \sum_{i=0}^{n} \frac{1}{h^2} \sum_{j=0}^{n} (t_{n+j}^2 - t_j^2) \gamma(t_{n+j}, t_j) \right) + \epsilon_{n-i} \right],
\end{equation}

We now proceed with the two cases (a) and (b) separately.

(a) \( \alpha(z) \equiv \alpha_0 e^{z} \), \( \alpha_0 \neq 0 \).

We want to apply the discrete Gronwall inequality stated in Lemma A.1 in order to derive an upper bound for the solution of this linear difference equation, and therefore we need an upper bound for \( |v_n| \). A straightforward calculation yields

\begin{equation}
|v_n| \leq T(h) + \frac{1}{k} \sum_{i=0}^{k} \left( h \delta_{ij} \left( \sum_{i=0}^{n} \frac{1}{h^2} \sum_{j=0}^{n} |\epsilon_j| + C_l h |\epsilon_{n-i}| \right) + \beta E(h) \right)
\end{equation}

\begin{equation}
\leq C_0 h \sum_{i=0}^{n} |\epsilon_i| + C_1 E(h) + T(h),
\end{equation}

where \( C_0 \) and \( C_1 \) are constants independent of \( h \) and \( n \) (in the following all constants \( C_j \) will be independent of \( h \) and \( n \)). From (A.6) it follows that

\begin{equation}
|\epsilon_0| + |\epsilon_n| \leq C_0 h \sum_{i=0}^{n} |\epsilon_i| + C_1 E(h) + T(h),
\end{equation}

so that for \( h \) sufficiently small

\begin{equation}
|\epsilon_n| \leq \frac{1}{k \beta_0 - C_0 h} \left( \sum_{i=0}^{n} |\epsilon_i| + C_1 E(h) + T(h) \right)
\end{equation}

\begin{equation}
\leq C_2 h \sum_{i=0}^{n} |\epsilon_i| + C_3 (E(h) + T(h)),
\end{equation}

Application of Lemma A.1 (with \( z = k^\beta(h) \)) yields

\begin{equation}
|\epsilon_n| \leq (1+C_2 h) |\epsilon_0| + C_3 (E(h) + T(h))
\end{equation}

\begin{equation}
\leq n \geq \beta^\beta \ldots \beta^n.
\end{equation}

Since \( nh \leq \tau - \tau_0 \), part (a) of the theorem is immediate.

(b) \( \alpha(z) \) is simple von Neumann, \( \beta(z) \equiv 0 \).

Instead of directly applying Lemma A.1 to the inequality (obtained from (A.6))

\begin{equation}
\frac{1}{k} \sum_{i=0}^{k} \left( h \delta_{ij} \left( \sum_{i=0}^{n} \frac{1}{h^2} \sum_{j=0}^{n} |\epsilon_j| + C_l h |\epsilon_{n-i}| \right) + \beta E(h) \right)
\end{equation}

\begin{equation}
\leq C_0 h \sum_{i=0}^{n} |\epsilon_i| + C_1 E(h) + T(h),
\end{equation}

we first apply Lemma A.2 (i) to obtain the “sharper” inequality

\begin{equation}
|\epsilon_n| \leq C_0 \left| \epsilon_0 \right| + \frac{\beta}{\sum_{j=0}^{n} |\epsilon_j|}, \quad n \geq \beta^\beta.
\end{equation}

Unfortunately, if we use the upper bound (A.7) for \( |v_j| \) and then apply Lemma A.1, we cannot prove convergence. However, by using the property \( \beta(z) \equiv 0 \), that is \( \beta_i = \sum_{j=0}^{n} \beta_{ij} = 0 \), a sharper upper bound than (A.7) can be derived. To that end we write
\[ \left| \sum_{j=k}^b \beta_{ij} \Delta K(t_{n+i}^j, t_k^j, y(t_{n+i}^j), y(t_k^j)) \right| = \sum_{j=k}^b \beta_{ij} \left( \Delta K(t_{n+i}^j, t_k^j, y(t_{n+i}^j), y(t_k^j)) \right. \\
\left. + \Delta K(t_{n+i}^j, t_k^j, y(t_{n+i}^j), y(t_k^j)) - \Delta K(t_{n+i}^j, t_k^j, y(t_{n+i}^j), y(t_k^j)) \right) \leq bh \sum_{j=k}^b |\epsilon_j|, \]

and, similarly,

\[ \left| \sum_{j=k}^b \beta_{ij} E_{n-1}^j (h; \epsilon_{n+i}^j) \right| \leq bh \sum_{j=k}^b |\Delta E(h)|. \]

In this way we obtain instead of (A.7) the upper bound

\[ |v_n| \leq T_n(h) + h \sum_{i=0}^k \beta_{ii} \left( |\epsilon_i| + h \sum_{j=0}^i |\epsilon_j| + C_2 h^2 |\Delta E(h)| + T(h) \right), \]

(A.9)

Substitution into (A.8) yields the inequality

\[ |c_n| \leq C_3 \left( |\epsilon_n^{m-1}h| + h \sum_{j=0}^k |\epsilon_j| + h \sum_{j=0}^k |\epsilon_j| + h^2 |\Delta E(h)| + h^2 |T(h)| \right). \]

It is easily verified that

\[ \sum_{j=0}^k \sum_{i=0}^k |\epsilon_j^{m-1}| \leq (k+1) \sum_{j=0}^k |\epsilon_j|. \]

Hence,

\[ |c_n| \leq C_4 \left( |\epsilon_n| + nh |\Delta E(h)| + nh |T(h)| \right). \]

Since nh \leq T we find for h sufficiently small

\[ |c_n| \leq C_5 h^{n-1} \sum_{i=0}^k |\epsilon_i| + C_6 h^{n-1} |\Delta E(h)| + T(h). \]

Finally, by applying Lemma A.1 we arrive at the estimate

\[ |c_n| \leq (1+C_3 h)^{-1} \left( h^2 + C_6 h^{n-1} (|\Delta E(h)| + T(h)) \right), \]

from which part (b) of the theorem follows. \( \square \)

**PROOF OF THEOREM 2.3.4.** Following the first lines of the proof of Theorem 2.3.2 we obtain the following relation, analogous to (A.5), where

\[ K_{i,j} = K(\epsilon_{i,j}) \]

(A.10)

\[ K_{i,j} = \sum_{i=0}^b \sum_{j=0}^b \beta_{ij} \gamma_{i,j}^{n+i,j} h^{n+i} \sum_{i=0}^k \sum_{j=0}^k \beta_{ij} \gamma_{i,j}^{n+i,j} \sum_{i=0}^k \beta_{ij} \gamma_{i,j}^{n+i,j} h^{n+i} (h; \epsilon_{n+i,j}) \]

\[ - h^{-1} |\epsilon_{n+i,j}^m|, \quad n \geq k. \]

Now we write \( K_{i,j}^{n+i,n-i} = K_{i,j}^{n+i,n-i} (K_{n+i,j}^{n+i,n-i}) \) and \( K_{i,j}^{n+i,n-i} = K_{n+i,j}^{n+i,n-i} (K_{n+i,j}^{n+i,n-i}) \) and rewrite (A.10) to obtain

\[ K_{i,j}^{n+i,n-i} = v_n^{n+i,n-i} \]

(A.11)

where

\[ K_{n+i,n-i}^{n+i,n-i} = h \sum_{i=0}^k \sum_{j=0}^k \beta_{ij} \gamma_{i,j}^{n+i,j} \sum_{i=0}^k \beta_{ij} \gamma_{i,j}^{n+i,j} h^{n+i} (h; \epsilon_{n+i,j}) \]

\[ + h^{-1} \sum_{i=0}^k \sum_{j=0}^k \beta_{ij} \gamma_{i,j}^{n+i,j} (h; \epsilon_{n+i,j}) - h^{-1} |\epsilon_{n+i,j}^m|, \quad n \geq k. \]
Since \( \gamma(z) \) is Schur, we may apply Lemma A.2 (ii) to (A.11) and find

\[(A.12) \quad |c_n| \leq C(h) + \max_{k \leq j \leq n} |v_j|, \quad n \geq k^*.\]

where \( C \) (and all subsequent \( C_k \)) is independent of \( h \) and \( n \). So we have to find bounds on \( |v_j| \). Using the conditions of the theorem, we find

\[|v_r| \leq C_1 h \sum_{i,j} |v_{ij}| (j + 1) |\xi_{r-1}| + h \sum_{r=0}^{k^*} \sum_{\alpha \in \mathcal{E}} \sum_{x \in \mathcal{E}} \sum_{x'} |\xi_0| \quad r \leq k^*,\]

\[+ C_2 h \sum_{i,j} |v_{ij}| |\xi_0| + h \sum_{r=0}^{k^*} \sum_{\alpha \in \mathcal{E}} \sum_{x \in \mathcal{E}} \sum_{x'} |\xi_0| |\lambda_0(y)|, \quad r > k^*.

Now we use the condition \( \delta(z) = 0 \), i.e., \( \delta = 0 \), and (2.3.6a) to obtain (cf. the derivation of (A.9) in the proof of Theorem 2.3.2)

\[|v_r| \leq C_3 h \sum_{i,j} |v_{ij}| + h \sum_{r=0}^{k} |\xi_0| |\lambda_0(y)| + h^{-1} T(h), \quad r \leq k^*,\]

\[+ C_4 h \sum_{r=0}^{k^*} |\xi_0| + h^{-1} T(h), \quad r > k^*.

Substituting this into (A.12) we find, for \( h \) sufficiently small,

\[|c_n| \leq C_5 \left( \delta(h) + h^{-1} T(h) + h \sum_{r=0}^{k} |\xi_0| \right),\]

and application of Lemma A.1 yields the result of the theorem. \( \square \)

**Proof of Theorem 3.3.1.** Proceeding as in the proof of Theorem 2.3.2 we derive the relations

\[|\eta_n| = \frac{h}{k} \sum_{i=0}^{k} |\eta_{n-i} - \eta_n T_n h|,\]

\[(A.13) \quad |\eta_n| = \frac{k}{\sum_{i=0}^{k} |\eta_{n-i} - \eta_n T_n h| + h^{-1} \sum_{j=0}^{k} |\eta_{n-j} - \eta_n T_n h|},\]

\[+ \sum_{j=0}^{k} |\eta_{n-j} - \eta_n T_n h| + \sum_{j=0}^{k} |\eta_{n-j} - \eta_n T_n h|,\]

The first relation is written as (cf. (A.6))

\[(A.14) \quad \frac{k}{\sum_{i=0}^{k} \eta_{n-i} - \eta_n T_n h},\]

where \( \eta_n \) satisfies the inequality (using (1.3') and (1.3'))

\[|\eta_n| \leq |\eta_n T_n h| + h \sum_{i=0}^{k} |\eta_{n-i} - \frac{1}{k} T_n T_n h|,\]

\[\leq T_n h + h \sum_{i=0}^{k} |\eta_{n-i} - \frac{1}{k} T_n T_n h| + \sum_{j=0}^{k} \eta_{n-j} - \eta_n T_n h|\]

Application of Lemma A.2 (i) yields (because \( \sigma^*(z) \) is simple von Neumann)

\[(A.15) \quad |c_n| \leq C_0 \left( h \sum_{j=0}^{k} \left( |\xi_j| + |\eta_j| \right) + \delta(h) + \sum_{j=0}^{k} T_n T_n h \right),\]

where \( C_0 \) is some constant independent of \( n \) and \( h \).

For \( h \) we derive from the second relation in (A.13)

\[(A.16) \quad \frac{h}{\sum_{i=0}^{k} \eta_{n-i} - \eta_n T_n h} = \eta_n,\]

where \( \eta_n \) is defined as in (A.6).

(a) In the case where \( \sigma(z) = \sigma_0 \) we have from (A.7):

\[|c_n| \leq C_1 |\xi_n T_n h| + h \sum_{i=0}^{k} |\xi_0| + T_n h|, \quad n \geq k^*.\]
for some constant $C_1$. Substitution into (A.15) yields

$$
|e_n| \leq C_2 \left\{ h \sum_{j=0}^n \left| e_j \right| + h \sum_{j=0}^n \left| e_j \right| + E_n(h) + \right.
$$

$$
\left. + T_j(h) + h^{-1}T_j(h) + \delta(h) + h\delta(h) \right\}
$$

$$
\leq C_3 \left\{ h \sum_{j=0}^n \left| e_j \right| + E_n(h) + T_n(h) + h^{-1}T_n(h) + \delta(h) + h\delta(h) \right\}
$$

where we have used that $nh \leq T - t_0$. From Lemma A.1, part (a) of the theorem easily follows.

(b) Since $\alpha(z)$ is simple von Neumann, we apply Lemma A.2 (i) to (A.16) and use (A.9) (since $\delta(z)=0$) to find

$$
|n_n| \leq C_4 \left\{ e^*(h) + \delta E_n(h) + T_j(h) \right\}
$$

$$
\leq C_5 \left\{ h \sum_{j=0}^n \left| e_j \right| + \delta E_n(h) + h^{-1}T_n(h) \right\}.
$$

Substitution into (A.15) and applying Lemma A.1 leads to part (b) of the theorem. \( \square \)

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