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On the Numerical Solution of Singular Boundary Value Problems of Second Order by a Difference Method

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Abstract. The standard three-point discretization applied to the numerical solution of nonlinear boundary value problems for second-order systems with a singularity of the first kind is investigated. The results are extended to the boundary value problems arising in practical problems from mechanics and chemistry. A number of numerical examples illustrating the theoretical results is presented.

1. Introduction. The main purpose of this paper is to investigate the application of a finite difference scheme (with three-point discretization) to the following nonlinear boundary value problems:

$$y''(t) - \frac{A_1}{t}y'(t) - \frac{A_0}{t^2}y(t) = f(t, y(t), y'(t)), \qquad 0 < t \leqslant 1, \tag{1.1a}$$

$$B(y(0); y(1), y'(1)) = 0, \tag{1.1b}$$

and

$$y''(t) - \frac{A_1}{t}y'(t) - \frac{A_0}{t^2}y(t) = f\left(t, y(t), \frac{y(t)}{t}\right), \qquad 0 < t \leqslant 1, \tag{1.2a}$$

$$B(y(1), y'(1)) = 0, \qquad y(0) = 0. \tag{1.2b}$$

Here y and f are vector-valued functions of dimension n, A_0 and A_1 are constant $n \times n$ matrices and B in (1.1) is an m-dimensional vector-valued function, with $m \leqslant 2n$, while B in (1.2) is n-dimensional. We also assume y to be real-valued and continuously differentiable on $[0, 1]$ and its second derivative to be continuous on $(0, 1]$. The problems (1.1) and (1.2) occur often in applications from mechanics and chemistry, see for example Keller and Wolfe [5], Parter, Stein and Stein [10] and Rentrop [12], typically when transforming partial differential equations to ordinary differential equations.

The numerical solution of scalar equations of this type has been investigated by Jamet [3], Natterer [9], Russell and Shampine [13]. Brabston and Keller [1] and de Hoog and Weiss [2] have considered first-order systems with a singularity of the first kind.

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The basic analytical properties of nonlinear, second-order systems have been studied by the author in [14]. Questions of existence, smoothness and uniqueness of the analytical solutions have been discussed. In [15] the finite difference method has been applied to linear systems with variable coefficient matrices $A_0(t)$ and $A_1(t)$, the stability of the scheme has been shown, and the convergence order has been derived. The analysis in [14] and [15] is based on the transformation of the system of second order to a system of first order, cf. (3.2a), where the $2n \times 2n$ matrix $M(t)$ occurs. It has been shown in [15] that if the analytical problem is well-posed, then for every equidistant gridspacing h on $[0,1]$ there exists a $t_0(h) > 0$ such that the difference system associated with the linear boundary value problem has a unique solution on $[t_0(h), 1]$. Furthermore, the order of convergence depends on the eigenvalues of $M \equiv M(0)$ and the smoothness properties of y. It has been shown for $f \in C^2$ that the order of convergence is $h^q|\ln h|^p$, $p \geqslant 0$, where $q = 2$ if all eigenvalues of M have nonpositive real parts, and $q = \min(\sigma_+, 2)$, where σ_+ is the smallest positive real part, otherwise. In a fairly large number of computer examples (also nonlinear) it has been observed that the choice $t_0(h) = 0$ worked well, and this suggests that the restriction $t_0(h) > 0$ is rather technical (contraction arguments were used in [14]) and can be omitted. This will be shown here.

The outline of the paper follows. In Section 2 we collect the notations and preliminary results used in the subsequent analysis. In Section 3 we present the basic analytical properties of the solution of the linear problem, which we require for the discussion of the numerical method. Furthermore, we formulate assumptions for the treatment of nonlinear problems. Because in many practical nonlinear problems the solution is not unique, cf. [5], we only assume that a solution exists and is isolated. In Section 4, we formulate the numerical schemes to be considered and show existence and uniqueness of the solution of the difference systems for the linear case on the whole interval $[0,1]$. To avoid repetitions in the analysis, we state the convergence results without proofs, which can be found in [15]. The results are extended to the nonlinear case in Section 5. If the problems (1.1) and (1.2) have an isolated solution, we show that the associated discrete problems have also a unique solution in a neighborhood of the continuous solution, when the gridspacing h is sufficiently small. In the analysis of the nonlinear problems we use the results of Keller [6]. The order of convergence depends on the smoothness of the solution y as well as on the spectral properties of A_0 and A_1. Typically, one has convergence order $h^2|\ln h|^p$, $p \geqslant 0$. Finally, Section 6 contains numerical examples illustrating the theory, and Section 7 (appendix) technical details.

2. Notations and Preliminary Results. We denote by X^n the space of complex-valued vectors of dimension n and use $|\cdot|$ to denote the maximum norm on X^n,

$$|x| := \left|(x_1, x_2, \dots, x_n)^T\right| = \max_{1 \leqslant i \leqslant n} |x_i|.$$

Let Δ be an equidistant partition of the interval $[0,1]$

$$\Delta = \{t_i, i = 0(1)N \,|\, t_i = ih,\, t_N = 1\},$$

where $i = 0, 1, \dots, N$ is denoted by $i = 0(1)N$. With each partition Δ we associate the linear space X_Δ with elements

$$x_\Delta = (x_0, x_1, \dots, x_N),$$

where $x_m = (x_{m1}, x_{m2}, \ldots, x_{mn})^T \in X^n$, $m = 0(1)N$, and the norm on X_Δ is

$$\|x_\Delta\| := \max_{0 \leqslant m \leqslant N} |x_m|.$$

$C^p[0,1]$ is the space of vector-valued functions as well as the space of complex-valued matrices, whose elements are p times continuously differentiable on $[0,1]$, and $C^p(0,1]$ is defined analogously. We use $C = C[0,1] = C^0[0,1]$ and $C(0,1] = C^0(0,1]$. For each vector $y \in C$ we define the norm

$$\|y\| := \max_{0 \leqslant t \leqslant 1} |y(t)|,$$

and for each matrix $A \in C$, $\|A\|$ is the induced norm. For any continuous function y the modulus of continuity is defined by

$$\omega(y;\delta) := \max_{0 \leqslant t \leqslant 1-\delta} |y(t+\delta) - y(t)|.$$

Finally, $R_\Delta \colon C \to X_\Delta$ is the bounded linear map such that

$$R_\Delta y = \big(y(t_0), y(t_1), \ldots, y(t_N)\big), \quad R_\Delta y' = \big(y'(t_0), y'(t_1), \ldots, y'(t_N)\big).$$

We now formulate the results which are basic in our analysis.

LEMMA 2.1. *Given a complex number* $\lambda = \sigma + i\kappa$, $\sigma > 0$ *and* $\Omega(\lambda) = \{\mu \mid |\lambda - \mu| \leqslant \sigma/2\}$; *for* $\mu \in \Omega(\lambda)$ *define*

$$z_{kj}(\mu) := \begin{cases} 1, & k = j, \\ \displaystyle\prod_{l=k}^{j-1}\left(1 - \frac{\mu}{l}\right), & 1 \leqslant k < j,\ j = 2(1)N+1. \end{cases} \tag{2.1}$$

Then there exists an $\eta > 0$ *such that*

$$|z_{kj}(\mu)| \leqslant \text{const}(t_k/t_j)^\eta, \qquad k \leqslant j,\ j = 1(1)N,$$

for all $\mu \in \Omega(\lambda)$.

Proof. Let $\mu \in \Omega(\lambda)$ and $l \geqslant l_0 = \frac{3}{2}\sigma$. Then $|l - \mu| \leqslant |l - \nu|$, where $\nu = \sigma/2 - i(|\kappa| + \sigma/2)$. Let $k \geqslant l_0$. Then $|z_{kj}(\mu)| \leqslant |v_{kj}|$, where

$$v_{kj} = \prod_{l=k}^{j-1} (l - \nu)/l.$$

Using

$$\Gamma(z+n) = \left(\prod_{k=0}^{n-1} (z+k)\right)\Gamma(z)$$

and the asymptotic expansion

$$\Gamma(s+a)/\Gamma(s+b) = s^{a-b}(1 + O(1/s)), \qquad \mathrm{Re}(s) > 0,$$

cf. [7], we can rewrite v_{kj} and obtain

$$v_{kj} = \prod_{l=k}^{j-1} (l-\nu)/l = \Gamma(j-\nu)\Gamma(k)/\Gamma(k-\nu)\Gamma(j) = (k/j)^\nu(1 + O(1/k)),$$

which yields the desired result for $k \geqslant l_0$. The result for $k < l_0$ now follows, since $z_{kj}(\mu) = z_{k,l_0}(\mu) z_{l_0,j}(\mu)$ and z_{k,l_0} consists of at most l_0 terms. □

LEMMA 2.2. *Given a complex number* $\lambda = \sigma + i\kappa$, $\sigma > 0$, $\Omega_1(\lambda) = \{\mu \mid |\lambda - \mu| \leqslant \sigma/4\}$ *and* $\Omega_2(\lambda) = \{\mu \mid |\lambda - \mu| \leqslant \sigma/2\}$; *for any* $\mu \in \Omega_1(\lambda)$ *and complex number* δ *define*

$$(2.2)\quad z_{kj}(\mu,\delta) := \begin{cases} 1, & k = j, \\ \displaystyle\prod_{l=k}^{j-1}\left(1 - \frac{1}{l}\left(\mu + \frac{\delta}{l}\right)\right), & 1 \leqslant k < j,\ j = 2(1)N+1. \end{cases}$$

Then there exists an $\eta > 0$ *such that*

$$\left|z_{kj}(\mu,\delta)\right| \leqslant \text{const}\left(t_k/t_j\right)^{\eta}, \qquad k \leqslant j,\ j = 1(1)N,$$

for all $\mu \in \Omega_1(\lambda)$.

Proof. Let $\mu_l = \mu + \delta/l$. Then we can rewrite (2.2) and obtain

$$z_{kj}(\mu,\delta) = \prod_{l=k}^{j-1}\left(1 - \frac{\mu_l}{l}\right).$$

If $\mu \in \Omega_1(\lambda)$ and $l \geqslant l_1 = 4|\delta|/\sigma$, then $\mu_l \in \Omega_2(\lambda)$. Let $l \geqslant \max\{l_0, l_1\}$. Then $|l - \mu_l| \leqslant |l - \nu|$, and hence the assertion follows as in Lemma 2.1. □

LEMMA 2.3. *Let* $\lambda, \Omega_1(\lambda)$ *and* $\Omega_2(\lambda)$ *be as in Lemma* 2.2. *Let* $\delta(l,\mu)$ *be a bounded sequence of complex numbers such that* $|\delta(l,\mu)| \leqslant \delta$ *for each* $l \geqslant \rho \geqslant 1$ *and all* $\mu \in \Omega_1(\lambda)$. *For* $\mu \in \Omega_1(\lambda)$ *and* $\delta(l,\mu)$ *define*

$$(2.3)\quad z_{kj}(\mu,\delta(\cdot)) := \begin{cases} 1, & k = j, \\ \displaystyle\prod_{l=k}^{j-1}\left(1 - \frac{1}{l}\left(\mu + \frac{\delta(l,\mu)}{l}\right)\right), & 1 \leqslant k < j,\ j = 2(1)N+1. \end{cases}$$

Then there exists an $\eta > 0$ *such that*

$$\left|z_{kj}(\mu,\delta(\cdot))\right| \leqslant \text{const}\left(t_k/t_j\right)^{\eta}, \qquad k \leqslant j,\ j = 1(1)N,$$

for all $\mu \in \Omega_1(\lambda)$.

Proof. On setting $\mu_{l,\delta} = \mu + \delta(l,\mu)/l$, we have

$$z_{kj}(\mu,\delta(\cdot)) = \prod_{l=k}^{j-1}\left(1 - \frac{\mu_{l,\delta}}{l}\right).$$

Let $\mu \in \Omega_1(\lambda)$. Then $\mu_{l,\delta} \in \Omega_2(\lambda)$ for each $l \geqslant l_1 = \max(\rho, 4\delta/\sigma)$, and the same arguments as in the proof of Lemma 2.2 apply. □

LEMMA 2.4. *For every* $k > j \geqslant 1$ *and* $\gamma \in \mathbf{R}$,

$$\sum_{l=j}^{k-1} h t_l^{\gamma-1} \leqslant \begin{cases} \text{const}\left|t_k^{\gamma} - t_j^{\gamma}\right|, & \gamma \neq 0, \\ \text{const}\ln\left(t_k/t_j\right), & \gamma = 0. \end{cases}$$

Proof. See [15, Lemma 2.2]. □

Finally, for the matrix A and an analytic function $\xi(\lambda)$, we write the matrix function $\xi(A)$ as

$$(2.4)\qquad \xi(A) = \frac{1}{2\pi i}\int_{\Gamma}\xi(\lambda)(\lambda I - A)^{-1}\,d\lambda,$$

where Γ is a closed curve containing all eigenvalues of A.

3. Analytical Problems. In this section we recall the formulation of the linear problem given in [14], some analytical properties of the solutions of this problem and the assumptions made for the nonlinear case. For technical details, proofs, etc., see [14] and [15].

Consider the linear boundary value problem

$$y''(t) - \frac{A_1(t)}{t}y'(t) - \frac{A_0(t)}{t^2}y(t) = f(t), \qquad 0 < t \leqslant 1, \tag{3.1a}$$

$$B_0Y(0) + B_1Y(1) = \beta, \qquad Y(t) = \big(y(t), y'(t)\big)^T, \tag{3.1b}$$

where y, f are n-vectors, A_0, A_1 are $n \times n$ matrices, B_0, B_1 are $m \times 2n$ constant matrices and β is an m-vector, with $m \leqslant 2n$. By the linear transformation $z(t) = (z_1(t), z_2(t))^T = (y(t), ty'(t))^T$, the system (3.1a) can be reduced to a first-order system of the form

$$z'(t) = \frac{1}{t}\begin{bmatrix} 0 & I \\ A_0(t) & I + A_1(t) \end{bmatrix} z(t) + t\begin{bmatrix} 0 \\ f(t) \end{bmatrix} \equiv \frac{1}{t}M(t)z(t) + t\mathring{f}(t), \tag{3.2a}$$

$$B_0Y(0) + B_1Y(1) = \beta. \tag{3.2b}$$

Let $f \in C$ and assume that $A_0(t)$ and $A_1(t)$ are

$$A_0(t) = A_0 + t^{\nu}C_0(t), \qquad A_1(t) = A_1 + t^{\nu}C_1(t), \qquad \nu \geqslant 1, \tag{3.3}$$

where A_0, A_1 are constant matrices and C_0, $C_1 \in C$. Then we can rewrite (3.2) and obtain

$$\begin{aligned} z'(t) &= \frac{1}{t}\begin{bmatrix} 0 & I \\ A_0 & I + A_1 \end{bmatrix} z(t) + t^{\nu-1}\begin{bmatrix} 0 & 0 \\ C_0(t) & C_1(t) \end{bmatrix} z(t) + t\begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ &\equiv \frac{1}{t}Mz(t) + t^{\nu-1}\mathring{C}(t)z(t) + t\mathring{f}(t), \qquad 0 < t \leqslant 1, \end{aligned} \tag{3.4a}$$

$$B_0Y(0) + B_1Y(1) = \beta. \tag{3.4b}$$

In order to formulate the existence result for the solution of (3.4a), we have to introduce the following notations. We denote by R the spectral projection onto the eigenspace of $M \equiv M(0)$ corresponding to the eigenvalue $\lambda = 0$, and by S the spectral projection onto the invariant subspace of M corresponding to the eigenvalues with positive real parts. We also set

$$P = R + S, \qquad Q = I - P,$$

and have the following result.

THEOREM 3.1. *Let* $Qz(0) = 0$; *then for every* $f \in C$ *and constant* $2n$*-vector* γ *there exists a unique, continuous solution* $z(t)$ *of* (3.4a) *satisfying* $Pz(1) = P\gamma$. *Since* $y(t) = z_1(t)$, *we obtain a solution* $y(t)$ *of* (3.1a), *and* $y \in C \cap C^2(0, 1]$.

The question as to whether this solution satisfies the boundary value problem (3.1) can be answered if we substitute the solution y and its derivative y' into (3.1b). On noting that $Pz(1) = PY(1) = P\gamma$ we see that we need $m = \text{rank}[P]$ conditions to make the solution y unique and these conditions have to be given by (3.1b). It has been shown in [14, Theorem 4.2] that the m constants can be uniquely determined from (3.1b) if and only if the inverse of a certain $m \times m$ matrix obtained by the above substitution exists.

For the nonlinear problem (1.1) we make the following assumptions.

N.1.1. Problem (1.1) has a solution $y \in C^1$. We define the sphere S_ρ associated with the solution y of (1.1) by

$$S_\rho(y(t)) = \{ v \in X^n | \, |v - y(t)| \leqslant \rho, \, \rho > 0 \}$$

and the sphere S_δ associated with its derivative y' by

$$S_\delta(y'(t)) = \{ w \in X^n | \, |w - y'(t)| \leqslant \delta, \, \delta > 0 \}.$$

Additionally, we set

$$T_{\rho,\delta} = \{ (t, v, w) \big| 0 \leqslant t \leqslant 1, \, v \in S_\rho(y(t)), \, w \in S_\delta(y'(t)) \}.$$

N.1.2. B: $D_B \to X^m$ is a nonlinear map and f: $D_f \to X^n$ is a nonlinear map, which is continuous on $[0,1] \times X^n \times X^n$, D_B and D_f are open sets and $m = \operatorname{rank}[P]$.

N.1.3. f is continuously differentiable with respect to v and w and $f_v(t, v, w)$, $f_w(t, v, w)$ are continuous on $T_{\rho,\delta}$; B is continuously differentiable with respect to all variables on

$$S_\rho(y(0)) \times S_\rho(y(1)) \times S_\delta(y'(1)).$$

N.1.4. If M has eigenvalues λ with positive real parts σ, then $\sigma > 1$.

N.1.5. The solution y of (1.1) is isolated. This is equivalent to the condition that the following (linear) problem

$$\text{(3.5a)} \quad L(y)v(t) \equiv v''(t) - \frac{A_1}{t} v'(t) - C_1(t)v'(t) - \frac{A_0}{t^2} v(t) - C_0(t)v(t) = 0,$$

$$\text{(3.5b)} \quad B_{00} v(0) + B_{10} v(1) + B_{11} v'(1) = 0,$$

where $f \equiv f(\cdot, v(\cdot), w(\cdot))$, $B \equiv B(u_1; u_2, u_3)$ and

$$\text{(3.6a)} \quad \begin{aligned} B_{00} &= \frac{\partial B}{\partial u_1}(y(0); y(1), y'(1)), \\ B_{10} &= \frac{\partial B}{\partial u_2}(y(0); y(1), y'(1)), \qquad B_{11} = \frac{\partial B}{\partial u_3}(y(0); y(1), y'(1)), \end{aligned}$$

$$\text{(3.6b)} \quad C_0(t) = \frac{\partial f}{\partial v}(t, y(t), y'(t)), \qquad C_1(t) = \frac{\partial f}{\partial w}(t, y(t), y'(t)),$$

has only the trivial solution $v(t) \equiv 0$.

Let $\tilde{P}$ be the $2n \times m$ matrix consisting of the linearly independent columns of P and $\hat{P}$ the unique $m \times 2n$ matrix such that $\tilde{P}\hat{P} = P$. Let I_1 be the $n \times 2n$ matrix consisting of the first rows of the identity matrix I and I_2 the $n \times 2n$ matrix consisting of the last rows of I. Then any continuous solution of (1.1) satisfies

$$\text{(3.7a)} \quad y(t) = I_1\{ (Hf(\cdot, y, y'))(t) + \phi(t)\tilde{P}\alpha \},$$

$$\text{(3.7b)} \quad y'(t) = I_2\{ (Hf(\cdot, y, y'))(t) + \phi(t)\tilde{P}\alpha \}/t,$$

$$\text{(3.7c)} \quad \alpha = \alpha - B(y(0); y(1), y'(1)),$$

where $\alpha = \hat{P}Y(1)$, $\alpha \in X^m$ and

$$\text{(3.8a)} \quad (Hf(\cdot, y, y'))(t) = t^2 \int_0^1 Q s^{-M} s\mathring{f}(ts, y(ts), y'(ts))\, ds + t^M \int_1^t P s^{-M} s\mathring{f}(s, y(s), y'(s))\, ds,$$

$$\text{(3.8b)} \quad \phi(t) = t^M P.$$

We can write (3.7) as

(3.9) $$x = N(x),$$

where $x = (y, y', \alpha)$ and N: $U_{\rho,\delta} \times X^m \to C^m$ is a compact nonlinear operator, $U_{\rho,\delta} = \{u \in C^1 \mid u(t) \in S_\rho(y(t)),\ u'(t) \in S_\delta(y'(t)),\ 0 \leqslant t \leqslant 1\}$ and $C^m = C^1 \times X^m$. A simple modification of contraction arguments given in [14, Section 5] yields the stability of the solution of (1.1), i.e., the continuous dependence of the solution y on small perturbations in the right-hand side of the differential equations and boundary conditions.

The extension of this result to the problem (1.2) is rather straightforward, if we change assumptions N.1.1–N.1.5 properly.

N.2.1. Problem (1.2) is well-posed and has a solution $y \in C^1$, $y(t)/t \in C$. We define the following sphere for the solution y

$$S_\varepsilon(y(t)/t) = \left\{ v \in X^n \left| \left| \frac{v}{t} - \frac{y(t)}{t} \right| \leqslant \varepsilon,\ \varepsilon > 0 \right. \right\}$$

and note that if $v \in S_\rho(y(t)/t)$ then $v \in S_\rho(y(t))$. We also set

$$T_{\rho,\varepsilon} = \left\{ (t, v, w) \,\middle|\, 0 \leqslant t \leqslant 1,\ v \in S_\rho(y(t)),\ w \in S_\varepsilon(y(t)/t) \right\}.$$

N.2.2. B: $D_B \to X^n$ is a nonlinear map and D_B is an open set.

N.2.3. B is continuously differentiable on $S_\rho(y(1)) \times S_\delta(y'(1))$.

N.2.4. If the smallest real part of the eigenvalues of M is 1, then $\lambda = 1$ and the Jordan box associated with $\lambda = 1$ is diagonal.

N.2.5. The solution y of (1.2) is isolated, i.e., the problem

(3.10a) $$v''(t) - \frac{A_1}{t} v'(t) - \frac{A_0}{t^2} v(t) - C_0(t) v(t) = 0,$$

(3.10b) $$B_{10} v(1) + B_{11} v'(1) = 0,$$

where

$$C_0(t) = \frac{\partial f}{\partial v}\left(t, y(t), \frac{y(t)}{t}\right) + \frac{\partial f}{\partial w}\left(t, y(t), \frac{y(t)}{t}\right)/t,$$

has only the trivial solution.

All other assumptions remain valid with respect to the above changes and new definitions.

4. Numerical schemes. We consider the partition Δ of Section 2 and the nonlinear problem (1.1). Then the three-point discretization yields

(4.1a) $$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{A_1}{t_i}\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) - \frac{A_0}{t_i^2} y_i = f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right), \qquad i = 1(1)N,$$

(4.1b) $$B\left(y_0; y_N, \frac{y_{N+1} - y_{N-1}}{2h}\right) = 0.$$

Without loss of generality we assume that the boundary conditions, which are to be posed at $t = 0$ for the continuity of the solution, i.e., $Qz(0) = 0$, are given by

(4.1c) $$\tilde{Q} Y_0 \equiv \tilde{Q} \begin{bmatrix} y_0 \\ (y_1 - y_0)/h - hA^{-1} f(0, y_0, 0)/2 \end{bmatrix} = 0,$$

where $\tilde{Q}$ is a constant $q \times 2n$ matrix and $q = \text{rank}[Q]$. To see that the lower expression in (4.1c) is an approximation for $y'(0)$ we assume $y \in C^2$ and apply Taylor's theorem to (1.1a). Then we have

$$Ay''(0) \equiv (I - A_1 - A_0/2)y''(0) = f(0, y(0), y'(0))$$

if and only if $(A_1 + A_0)y'(0) = 0$ and $A_0 y(0) = 0$. By N.1.4 and N.1.2 we have immediately $y'(0) = 0$ (cf. [14, Lemmas 3.1, 3.2 and 3.3]), and provided that A^{-1} exists, the result follows from

$$y''(0) := (y_{-1} - 2y_0 + y_1)/h^2, \qquad y'(0) := (y_1 - y_{-1})/2h.$$

The difference scheme associated with (1.2) is

$$\text{(4.2a)} \quad \begin{aligned} &\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{A_1}{t_i}\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) - \frac{A_0}{t_i^2} y_i \\ &= f\left(t_i, y_i, \frac{y_i}{t_i}\right), \qquad i = 1(1)N, \end{aligned}$$

$$\text{(4.2b)} \quad B\left(y_N, \frac{y_{N+1} - y_{N-1}}{2h}\right) = 0,$$

$$\text{(4.2c)} \quad y_0 = 0.$$

In the following subsection we consider the linear case and show how the stability results from [15] can be extended to the whole interval $0 \leqslant t \leqslant 1$. The convergence results are repeated, because their proofs do not change.

Numerical Results for Linear Problems. We first formulate the difference scheme for the case when the coefficient matrices $A_0(t)$ and $A_1(t)$ are constant, i.e., $C_0(t) = C_1(t) \equiv 0$, and then transform the second-order system to the first-order one. For (3.1) we have

$$\text{(4.3a)} \quad \left(I - \frac{A_1}{2i}\right)y_{i+1} - \left(2I + \frac{A_0}{i^2}\right)y_i + \left(I + \frac{A_1}{2i}\right)y_{i-1} = h^2 f(t_i), \qquad i = 1(1)N,$$

$$\text{(4.3b)} \quad B_0\begin{bmatrix} y_0 \\ (y_1 - y_0)/h - hA^{-1}f(0)/2 \end{bmatrix} + B_1\begin{bmatrix} y_N \\ (y_{N+1} - y_{N-1})/2h \end{bmatrix} = \beta.$$

We define

$$\text{(4.4)} \quad u_{1,i} = y_i, \qquad u_{2,i} = (i+1)(y_{i+1} - y_i), \qquad i = 0(1)N,$$

and obtain immediately

$$\text{(4.5)} \quad u_{1,i} = u_{1,i-1} + \frac{1}{i}u_{2,i-1}, \qquad i = 1(1)N.$$

The application of (4.4) to (4.3a) yields

$$\text{(4.6)} \quad \begin{aligned} u_{2,i} = &\left(\frac{A_0}{i} + \frac{1}{i^2}\Theta_1(i)\right)u_{1,i-1} + \left(I + \frac{1}{i}(I + A_1) + \frac{1}{i^2}\Theta_2(i)\right)u_{2,i-1} \\ &+ \Theta_3(i)t_{i+1}hf(t_i), \qquad i = 1(1)N, \end{aligned}$$

where

$$\Theta_1(i) = \left(I - \frac{A_1}{2i}\right)^{-1}\left(I + \frac{A_1}{2}\right)A_0,$$

$$\Theta_2(i) = \left(I - \frac{A_1}{2i}\right)^{-1}\left[\left(1 + \frac{1}{i}\right)A_0 + \left(I + \frac{A_1}{2}\right)A_1\right],$$

$$\Theta_3(i) = \left(I - \frac{A_1}{2i}\right)^{-1}.$$

Clearly, we have to assume that for $1 \leqslant i \leqslant |A_1|/2$, $(I - A_1/2i)^{-1}$ exist. From (4.5) and (4.6) we have for $u_i = (u_{1,i}, u_{2,i})^T$

$$u_i = u_{i-1} + \frac{1}{i}Mu_{i-1} + \frac{1}{i^2}\Theta_4(i)u_{i-1} + t_{i+1}h\Theta_5(i)\mathring{f}(t_i), \qquad i = 1(1)N, \tag{4.7}$$

where Θ_4 and Θ_5 are appropriate $2n \times 2n$ matrices. Since $J = E^{-1}ME$, $v_i = E^{-1}u_i$ solves

$$v_i = v_{i-1} + \frac{1}{i}Jv_{i-1} + \frac{1}{i^2}\Theta(i)v_{i-1} + t_{i+1}h\psi(i)g_i, \tag{4.8}$$

where

$$\Theta(i) = E^{-1}\Theta_4(i)E, \quad \psi(i) = E^{-1}\Theta_5(i)E, \quad g_i = E^{-1}\mathring{f}(t_i), \qquad i = 1(1)N.$$

For (4.8) we write

$$v_i - v_{i-1} - \frac{1}{i}Jv_{i-1} - \frac{1}{i^2}\Theta(i)v_{i-1} = t_{i+1}h\psi(i)g_i, \qquad i = 1(1)N, \tag{4.9}$$

and this is formally

$$GV = R, \tag{4.10}$$

where $G\colon X_\Delta \dot{\to} X_\Delta$. In [15] we proceeded similarly, but there the expression on the left-hand side of (4.9) was $v_i - v_{i-1} - (1/i)Jv_{i-1}$. In order to show that G has an inverse bounded independently of h, we assume that J consists of one Jordan box

$$J = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}, \qquad \lambda = \sigma + i\kappa,$$

and consider the following three cases $\sigma < 0$, $\lambda = 0$ and $\sigma > 0$ separately. Let us write (4.8) as

$$v_i = v_{i-1} + \frac{1}{i}Jv_{i-1} + \frac{1}{i^2}\Theta(i)v_{i-1} + r_i, \qquad i = 1(1)N, \tag{4.11}$$

and consider

Case 1. $\sigma < 0$. Let $v_0 = \gamma \in \mathbf{R}^{2n}$. Then

$$\begin{aligned} v_i &= \prod_{l=i}^{1}\left(I + \frac{1}{l}J + \frac{1}{l^2}\Theta(l)\right)\gamma + \sum_{k=1}^{i-1}\left(\prod_{l=i}^{k+1}\left(I + \frac{1}{l}J + \frac{1}{l^2}\Theta(l)\right)\right)r_k + r_i, \\ &\equiv \tilde{Z}_{i,1}\gamma + \sum_{k=1}^{i-1}\tilde{Z}_{i,k+1}r_k + r_i, \qquad i = 1(1)N. \end{aligned} \tag{4.12}$$

Let

$$Z_{j,k+1} = \prod_{l=j}^{k+1}\left(I + \frac{1}{l}J + \frac{1}{l^2}\Delta(l,J)\right)$$
$$= \frac{1}{2\pi i}\int_\Gamma z_{k+1,j+1}(-\lambda, -\delta(\cdot))(\lambda I - J)^{-1}\,d\lambda,$$

where $\Gamma = \{\mu \mid |\lambda - \mu| = -\sigma/4\}$, $\Delta(l, J)$ is a diagonal matrix, $\Delta(l, J) = \operatorname{diag}(\delta(l, \lambda))$ and $\delta(l, \lambda)$ is a complex number such that

$$\operatorname{Re}(\delta(l,\lambda)) = \operatorname{sign}\left(\operatorname{Re}\left(1 + \frac{1}{l}\lambda\right)\right)|\Theta(l)|, \qquad \operatorname{Im}(\delta(l,\lambda)) = \operatorname{sign}(\operatorname{Im}(\lambda))|\Theta(l)|$$

for each $1 \leqslant l \leqslant N$. We notice that if $|A_1| \equiv a_1$ and $|A_0| \equiv a_0$, then for each $l \geqslant \rho > a_1/2$

$$|\Theta(l)| \leqslant |\Theta(\rho)| \leqslant \text{const}[(1 + a_1/2)(a_0 + a_1) + 2a_0]/(1 - a_1/2\rho) \equiv \Theta$$

and $|\Theta(l)| \leqslant$ const otherwise. Using results shown in the appendix and Lemma 2.3, we now have

$$|\tilde{Z}_{j,k+1}| \leqslant |Z_{j,k+1}| \leqslant \text{const}(t_{k+1}/t_{j+1})^\eta, \qquad k < j,$$

and

$$|v_i| \leqslant \text{const}\left\{|\gamma| + \sum_{k=1}^{i}(t_{k+1}/t_{i+1})^\eta |r_k|\right\}, \qquad i = 1(1)N.$$

Using Lemma 2.4, we obtain for the solution of (4.8)

$$|v_i| \leqslant \text{const}\{|\gamma| + t_i^2\|f_\Delta\|\}, \qquad i = 1(1)N. \tag{4.13}$$

Case 2. $\lambda = 0$. Before studying (4.8),we investigate the growth of solutions of the following system

$$w_i = \left(I + \frac{1}{i}J + \frac{1}{i^2}\tilde{\Theta}(i)\right)w_{i-1} + r_i, \qquad i = 1(1)N, \tag{4.14}$$

where $\tilde{\Theta}(i)$ is a diagonal matrix and each diagonal element is equal to $|\Theta(i)| \equiv n(i)$. Let $w_{0r} = \delta_{0r}$, $r = 2(1)2n$ and $w_{N1} = \delta_{N1}$, and let us consider the system (4.14) componentwise. For the kth component we have

$$w_{ik} = w_{i-1,k} + \frac{1}{i}w_{i-1,k+1} + \frac{1}{i^2}n(i)w_{i-1,k} + r_{ik}, \qquad k = 1(1)2n - 1,$$
$$w_{i,2n} = w_{i-1,2n} + \frac{1}{i^2}n(i)w_{i-1,2n} + r_{i,2n}.$$

From the last equation we obtain

$$w_{i,2n} = \prod_{m=1}^{i}\left(1 + \frac{1}{m^2}n(m)\right)\delta_{0,2n} + \sum_{j=1}^{i-1}\left(\prod_{l=j+1}^{i}\left(1 + \frac{1}{l^2}n(l)\right)\right)r_{j,2n} + r_{i,2n}$$

and it follows immediately from Lemma 2.4 that

$$|w_{i,2n}| \leqslant \text{const}\{|\delta_{0,2n}| + t_i^2\|f_\Delta\|\}, \qquad i = 1(1)N. \tag{4.15}$$

For $k = 2n - 1$ we have

$$w_{i,2n-1} = \prod_{m=1}^{i}\left(1 + \frac{1}{m^2}n(m)\right)\delta_{0,2n-1}$$
$$+ \sum_{j=1}^{i-1}\left(\prod_{l=j+1}^{i}\left(1 + \frac{1}{l^2}n(l)\right)\right)\left(\frac{1}{j}w_{j-1,2n} + r_{j,2n-1}\right) + \frac{1}{i}w_{i-1,2n} + r_{i,2n-1}$$

and the following estimate can be obtained using (4.15) and Lemma 2.4,

$$|w_{i,2n-1}| \leqslant \text{const}\{|\delta_{0,2n-1}| + |\delta_{0,2n}|\,|\ln h| + t_i^2\|f_\Delta\|\}.$$

Clearly,

$$(4.16)\qquad |w_{ir}| \leqslant \text{const}\left\{\sum_{k=r}^{2n}|\delta_{0k}|\,|\ln h|^{k-r} + t_i^2\|f_\Delta\|\right\}, \qquad i = 1(1)N,\ r = 2(1)2n.$$

Finally, the first component is

$$w_{i1} = \prod_{m=i+1}^{N}\left(1 + \frac{1}{m^2}n(m)\right)^{-1}\delta_{N1}$$
$$- \sum_{j=i+1}^{N}\left(\sum_{l=i+1}^{j}\left(1 + \frac{1}{l^2}n(l)\right)^{-1}\right)\left(\frac{1}{j}w_{j-1,2} + r_{j1}\right), \qquad i = 1(1)N,$$

and the estimate for w_{i1} follows in a very similar way,

$$(4.17)\qquad |w_{i1}| \leqslant \text{const}\left\{|\delta_{N1}| + \sum_{k=2}^{2n}|\delta_{0k}|\,|\ln h|^{k-1} + \|f_\Delta\|\right\}, \qquad i = 1(1)N.$$

Let v_Δ be a solution of (4.8) and $v_{N1} = \delta_1$, $v_{0r} = \delta_r$, $r = 2(1)2n$. Since

$$\left|I + \frac{1}{i}J + \frac{1}{i^2}\Theta(i)\right| \leqslant \left|I + \frac{1}{i}J + \frac{1}{i^2}\tilde{\Theta}(i)\right|, \qquad i = 1(1)N,$$

it follows from (4.16) and (4.17) that

$$(4.18)\qquad |v_i| \leqslant \text{const}\left\{|\delta_1| + \max_{2\leqslant r\leqslant 2n}|\delta_r|\,|\ln h|^{2n-1} + \|f_\Delta\|\right\}.$$

Remark 4.1. Consider the case when the Jordan box associated with $\lambda = 0$ is diagonal; this is equivalent to $J \equiv 0$. In addition, let $v_0 = \delta_0$ be the initial condition; then it follows immediately from (4.15) that in this case

$$(4.19)\qquad |v_i| \leqslant \text{const}\{|\delta_0| + t_i^2\|f_\Delta\|\}, \qquad i = 1(1)N.$$

Remark 4.2. It follows also from [14, Lemma 3.2] that the case when the initial condition is of the form $v_0 = \delta_0$ is of interest. Analytically, this condition is equivalent to $Qz(0) = 0$ and $Rz(0) = R\gamma$. From (4.16) it is easy to see that the estimate for v_i is now

$$(4.20)\qquad |v_i| \leqslant \text{const}\left\{|\delta_{01}| + \max_{2\leqslant r\leqslant 2n}|\delta_{0r}|\,|\ln h|^{2n-1} + t_i^2\|f_\Delta\|\right\}.$$

Case 3. $\sigma > 0$. Let $v_N = \gamma \in \mathbf{R}^{2n}$. Then,

$$(4.21)\quad \begin{aligned} v_i &= \prod_{k=i+1}^{N}\left(I + \frac{1}{k}J + \frac{1}{k^2}\Theta(k)\right)^{-1}\gamma \\ &\quad - \sum_{k=i+1}^{N}\left(\prod_{l=i+1}^{k}\left(I + \frac{1}{l}J + \frac{1}{l^2}\Theta(l)\right)^{-1}\right)r_k \\ &\equiv \tilde{Z}_{i+1,N}\gamma - \sum_{k=i+1}^{N}\tilde{Z}_{i+1,k}r_k, \qquad i = 0(1)N-1. \end{aligned}$$

Before considering Case 3 in detail, let us briefly discuss the question of the existence of the matrix

$$(4.22)\quad M_{i+1,k} = \prod_{l=i+1}^{k}\left(I + \frac{1}{l}M + \frac{1}{l^2}\Theta_4(l)\right)^{-1}, \qquad k \geqslant i+1,\ i = 0(1)N-1.$$

Clearly, there exists an index i_0 such that for each $i \geqslant i_0$

$$\left|\frac{1}{i^2}\Theta_4(i)\right| < \left|\left(I + \frac{1}{i}M\right)^{-1}\right| \leqslant \text{const},$$

and hence $M_{i+1,k}$ exists for each $i \geqslant i_0$. It is not so simple to solve this problem for $1 \leqslant i < i_0$ in the most general case, without any additional assumptions on the matrices A_0 and A_1. Therefore, we assume that $M_{i+1,k}$ exists for each $i = 0(1)N-1$ and show that this assumption holds in some special cases that are important for applications, cf. [5], [10], [12].

Scalar case. The eigenvalues λ_1, λ_2 of the matrix M are solutions of the following equation,

$$\lambda^2 - \lambda(1 + a_1) - a_0 = 0; \qquad a_0, a_1 \in \mathbf{R},$$

and hence if $\mathrm{Re}(\lambda_i) > 0$, $i = 1, 2$, then it follows immediately that

$$(4.23)\quad a_1 > -1.$$

Assume that an index $i \geqslant 1$ exists such that the matrix

$$(4.24)\quad I + \frac{1}{i}M + \frac{1}{i^2}\Theta_4(i)$$

is singular, or equivalently,

$$1 + \frac{1}{i}(1 + a_1) + \frac{1}{i^2}\Theta_2(i) = \frac{1}{i^2}a_0 + \frac{1}{i^3}\Theta_1(i).$$

Then we have from the last equation

$$2i^2 + (2 + a_1)i + a_1 = 0$$

and $a_1 = -2i$, which contradicts (4.23).

A_0, A_1 are real diagonal matrices. As in the scalar case, if λ_i, $i = 1(1)n$, are the eigenvalues of the matrix M and $\mathrm{Re}(\lambda_i) > 0$, then

$$a_{1,k} > -1, \qquad k = 1(1)n,$$

where $a_{1,k}$ denotes the kth element of the matrix A_1. If the matrix (4.24) is singular, then

$$\det\left[-\left(\frac{1}{i^2}A_0 + \frac{1}{i^3}\Theta_1(i)\right) + I + \frac{1}{i}(I + A_1) + \frac{1}{i^2}\Theta_2(i)\right] = 0,$$

or equivalently,

$$\det\left[2i^2 I + (2I + A_1)i + A_1\right] = 0,$$

and the contradiction is obvious.

A_1 has real eigenvalues, $A_0 = a_0 I$, $a_0 \in \mathbf{R}$. The result can be shown as in the previous case, via the eigenvalues of A_1.

Both A_1 and A_0 have real eigenvalues, A_0 and A_1 are semisimple and $A_0 A_1 = A_1 A_0$. The result follows on noting that in this case A_0, A_1 have a complete set of eigenvectors in common.

We now can estimate v_i from (4.21). A simple modification of Lemma 2.3 and the existence of a matrix function

$$\begin{aligned} Z_{j+1,k} &= \prod_{l=j+1}^{k} \left(I + \frac{1}{l} J + \frac{1}{l^2} \Delta(l, J) \right)^{-1} \\ &= \frac{1}{2\pi i} \int_\Gamma z_{j+1,k+1}(\lambda, \delta(\cdot))(\lambda I - J)^{-1} d\lambda, \end{aligned}$$

where $\Gamma = \{\mu \mid |\mu - \lambda| = \sigma/4\}$ and $|\tilde{Z}_{j+1,k}| \leqslant \text{const}|Z_{j+1,k}|$ for each $k > j$, yield the existence of a constant $0 < \eta < 1$ such that

$$|\tilde{Z}_{j+1,k}| \leqslant \text{const}(t_{j+1}/t_{k+1})^\eta$$

and finally,

$$(4.25) \qquad |v_i| \leqslant \text{const}\left\{ t_i^\eta |\gamma| + \sum_{k=i+1}^{N} (t_{i+1}/t_{k+1})^\eta |r_k| \right\}, \qquad i = 1(1)N.$$

Using (4.25) and Lemma 2.4, we have for the solution of (4.8)

$$(4.26) \qquad |v_i| \leqslant \text{const}\left\{ t_i^\eta \left(|\gamma| + \|f_\Delta\| \right) \right\}, \qquad i = 1(1)N.$$

Technical details may be found in the appendix.

For $u_i = Ev_i$, $1 \leqslant i \leqslant N$, we can now extend the results of Cases 1, 2 and 3 to the general situation when the matrix M has different eigenvalues. Let d_0 be the dimension of the largest Jordan box associated with $\lambda = 0$ and d_+ the dimension of the largest Jordan box associated with $\lambda_+ = \sigma_+ + i\kappa_+$, where σ_+ is the smallest positive real part. Then we have the following

LEMMA 4.1. *The system* (4.7) *has a unique solution for any $Qu_0 = \delta_0$, $Pu_N = \delta_N$ and f_Δ. Furthermore*

(i) $\|u_\Delta\| \leqslant \text{const}\{|\delta_0| |\ln h|^{d_0 - 1} + |\delta_N| + \|f_\Delta\|\}$.

(ii) *Let y_Δ be a solution of* (4.3a) *and $f \in C^2$. Then*

$$\|y_\Delta - R_\Delta y\| \leqslant \begin{cases} \text{const}\, h^{\sigma_+} |\ln h|^{d_+ - 1}, & 0 < \sigma_+ < 2, \\ \text{const}\, h^2 \left(|\ln h|^{d_+} + |\ln h|^{d_0 - 1} \right), & \sigma_+ = 2, \\ \text{const}\, h^2 |\ln h|^{d_0 - 1}, & \sigma_+ > 2 \text{ or } S = 0. \end{cases}$$

The constants appearing in (i) *and* (ii) *are independent of h.*

Proof. (i) The proof follows from the results of Cases 1–3.

(ii) [15, Lemma 4.4]; we note that if zero is not an eigenvalue of M or its Jordan box is diagonal, then we have to replace $|\ln h|^{d_0 - 1}$ by 1 in (ii). The convergence results for the case when $f \in C$ can be found in [15, Lemma 4.4]. □

We now consider the numerical scheme defined in (4.3a), applied to the problem (3.1a). Let $A_0(t)$ and $A_1(t)$ be as in (3.3); then we have

$$\Theta_1(i,h) = \left(I - \frac{A_1 + t_i^{\nu}C_1(t_i)}{2i}\right)^{-1}\left(I + \frac{A_1(t_i)}{2}\right)A_0(t_i), \tag{4.27a}$$

$$\Theta_2(i,h) = \left(I - \frac{A_1 + t_i^{\nu}C_1(t_i)}{2i}\right)^{-1}\left[\left(1 + \frac{1}{i}\right)A_0(t_i) + \left(I + \frac{A_1(t_i)}{2}\right)A_1(t_i)\right], \tag{4.27b}$$

$$\Theta_3(i,h) = \left(I - \frac{A_1 + t_i^{\nu}C_1(t_i)}{2i}\right)^{-1}, \tag{4.27c}$$

and assuming that for $1 \leqslant i \leqslant \max_{0 \leqslant t \leqslant 1}|A_1(t)|/2$, $\Theta_j(i,h)$, $j = 1,2,3$ exist, we obtain the following system for u_i,

$$\begin{aligned} u_i = u_{i-1} &+ \frac{1}{i}Mu_{i-1} + \frac{t_i^{\nu}}{i}\mathring{C}(t_i)u_{i-1} + \frac{1}{i^2}\Theta_4(i,h)u_{i-1} \\ &+ t_{i+1}\Theta_5(i,h)\mathring{f}_i, \qquad i = 1(1)N_0,\ N_0 \leqslant N, \end{aligned} \tag{4.28a}$$

when Θ_4 and Θ_5 are defined as before with respect to the new definitions of Θ_1, Θ_2 and Θ_3. To show the existence and uniqueness of the solution of (4.28a), subject to the boundary conditions

$$Qu_0 = \delta_0, \qquad Pu_{N_0} = \delta_{N_0}, \tag{4.28b}$$

we use a contraction argument. For that purpose we consider the following iteration scheme

$$\begin{aligned} u_i^{(k+1)} - u_{i-1}^{(k+1)} - \frac{1}{i}Mu_{i-1}^{(k+1)} - \frac{1}{i^2}\Theta_4(i,h)u_{i-1}^{(k+1)} \\ = \frac{t_i^{\nu}}{i}\mathring{C}(t_i)u_{i-1}^{(k)} + t_{i+1}\Theta_5(i,h)\mathring{f}_i, \qquad i = 1(1)N_0, \\ Qu_0^{(k+1)} = \delta_0, \qquad Pu_{N_0}^{(k+1)} = \delta_{N_0}, \end{aligned}$$

which can be formally (cf. (4.10)) written as

$$GU^{(k+1)} = CU^{(k)} + F, \tag{4.29}$$

where G, C: $X_\Delta \to X_\Delta$ are linear maps. Since the existence of a bounded inverse of G follows from Lemma 4.1, it remains to show that $(I - G^{-1}C)^{-1}$ exists. This holds if $\|G^{-1}C\| < 1$. It has been shown in [15, Section 4] that the latter condition is satisfied if $\tau = hN_0$ is sufficiently small. The standard contraction argument now yields the existence and uniqueness of the solution of (4.28) on $[0,\tau]$. This solution can be uniquely continued to $t = 1$. We formulate this result as

LEMMA 4.2. *Let $Qu_0 = \delta_0$ and $Pu_N = \delta_N$. Then for each f_Δ there exists a unique solution u_Δ of* (4.28a), *i.e., the existence of the solution y_Δ of the associated second-order difference equation, and the following estimate holds,*

$$\|u_\Delta\| \leqslant \text{const}\left\{|\delta_0|\,|\ln h|^{d_0-1} + |\delta_N| + \|f_\Delta\|\right\}.$$

Finally, for the linear boundary value problem consisting of (4.28a) and the boundary conditions (4.3b) we have the following result corresponding to Lemma 4.2, cf. [15, Remark (i) and Lemma 4.5].

LEMMA 4.3. *Let us assume that the homogeneous boundary value problem* (3.1) *has only the trivial solution. Then the difference system* (4.28a), *subject to the boundary conditions* (4.3b) *and*

$$\tilde{Q}\begin{bmatrix} y_0 \\ (y_1 - y_0)/h - hA^{-1}f(0)/2 \end{bmatrix} = \delta, \tag{4.30}$$

has a unique solution for each f_Δ and β if h is sufficiently small and the following estimate holds,

(i) $\|y_\Delta\| \leqslant \text{const}\{|\delta|\,|\ln h|^{d_0-1} + |\beta| + \|f_\Delta\|\}$.

Clearly, if $\delta = 0$, then we have

(ii) $\|y_\Delta\| \leqslant \text{const}\{|\beta| + \|f_\Delta\|\}$.

(iii) *Let y_Δ be a solution of the difference system* (4.28a) *and $f, C_1 \in C^2$, $C_0 \in C^3$. Then*

$$\|y_\Delta - R_\Delta y\| \leqslant \begin{cases} \text{const}\, h^{\sigma_+}|\ln h|^{d_+-1}, & 0 < \sigma_+ < 2, \\ \text{const}\, h^2\left(|\ln h|^{d_+} + |\ln h|^{d_0-1}\right), & \sigma_+ = 2, \\ \text{const}\, h^2|\ln h|^{d_0-1}, & \sigma_+ > 2 \text{ or } S = 0. \end{cases}$$

5. The Nonlinear Problems. We now consider the nonlinear problem (1.1) and write it first as the following nonlinear operator equation

$$Ny(t) \equiv y''(t) - \frac{A_1}{t}y'(t) - \frac{A_0}{t^2}y(t) - f(t, y(t), y'(t)) = 0, \tag{5.1a}$$

$$B(y(0); y(1), y'(1)) = 0, \tag{5.1b}$$

$$\tilde{Q}Y(0) = 0. \tag{5.1c}$$

The discrete problem related to (5.1) is of the form (cf. (4.1))

$$\begin{aligned} N_\Delta v_i \equiv \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} - \frac{A_1}{t_i}\left(\frac{v_{i+1} - v_{i-1}}{2h}\right) - \frac{A_0}{t_i^2}v_i \\ -f\left(t_i, v_i, \frac{v_{i+1} - v_{i-1}}{2h}\right) = 0, \qquad i = 1(1)N, \end{aligned} \tag{5.2a}$$

$$B\left(v_0; v_N, \frac{v_{N+1} - v_{N-1}}{2h}\right) = 0, \tag{5.2b}$$

$$\tilde{Q}V_0 = 0. \tag{5.2c}$$

Before proceeding, the following additional notations are required:

$$B_{00}^\Delta = \frac{\partial B}{\partial u_1}(y(0); y(1), (y(1+h) - y(1-h))/2h),$$

$$B_{10}^\Delta = \frac{\partial B}{\partial u_2}(y(0); y(1), (y(1+h) - y(1-h))/2h),$$

$$B_{11}^\Delta = \frac{\partial B}{\partial u_3}(y(0); y(1), (y(1+h) - y(1-h))/2h),$$

$$\left.\begin{aligned} C_0^\Delta(t_i) &= \frac{\partial f}{\partial v}(t_i, y(t_i), (y(t_{i+1}) - y(t_{i-1}))/2h), \\ C_1^\Delta(t_i) &= \frac{\partial f}{\partial w}(t_i, y(t_i), (y(t_{i+1}) - y(t_{i-1}))/2h). \end{aligned}\right\} \quad i = 1(1)N,$$

The linearization of the nonlinear difference operator defined by (5.2) (at the solution $y(t)$) and applied to v_Δ is

$$L_\Delta(y_\Delta)v_i \equiv \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}$$

(5.3a)
$$-\left(\frac{A_1}{t_i} + \frac{\partial f}{\partial y'}(t_i, y(t_i), (y(t_{i+1}) - y(t_{i-1}))/2h)\right)\left(\frac{v_{i+1} - v_{i-1}}{2h}\right)$$
$$-\left(\frac{A_0}{t^2} + \frac{\partial f}{\partial y}(t_i, y(t_i), (y(t_{i+1}) - y(t_{i-1}))/2h)\right)v_i, \qquad i = 1(1)N,$$

(5.3b)
$$B_\Delta(y_\Delta)v_\Delta \equiv B_{00}^\Delta v_0 + B_{10}^\Delta v_N + B_{11}^\Delta\left(\frac{v_{N+1} - v_{N-1}}{2h}\right),$$

(5.3c)
$$Q_\Delta v_\Delta \equiv \tilde{Q}\begin{bmatrix} v_0 \\ (v_1 - v_0)/h - hA^{-1}\dfrac{\partial f}{\partial y}(0, y(0), 0)v_0/2 \end{bmatrix}.$$

Let us apply the difference scheme to the linear problem (3.5) augmented by the condition $\tilde{Q}V(0) = 0$. This yields

(5.4a)
$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} - \left(\frac{A_1}{t_i} + C_1^\Delta(t_i)\right)\left(\frac{v_{i+1} - v_{i-1}}{2h}\right) - \left(\frac{A_0}{t_i^2} + C_0^\Delta(t_i)\right)v_i = 0, \qquad i = 1(1)N,$$

(5.4b)
$$B_{00}^\Delta v_0 + B_{10}^\Delta v_N + B_{11}^\Delta\left(\frac{v_{N+1} - v_{N-1}}{2h}\right) = 0.$$

To derive a discretization of $\tilde{Q}V(0) = 0$ we assume $v \in C^2$, apply Taylor's theorem to (3.5a) and obtain

$$Av''(0) \equiv (I - A_1 - A_0/2)v''(0) = \frac{\partial f}{\partial v}(0, y(0), 0)v(0)$$

if $(A_1 + A_0)v'(0) = 0$ and $A_0 v(0) = 0$. It follows that

(5.4c)
$$\tilde{Q}\begin{bmatrix} v_0 \\ (v_1 - v_0)/h - hA^{-1}\dfrac{\partial f}{\partial v}(0, y(0), 0)v_0/2 \end{bmatrix} = 0,$$

and we can see from (3.6) that (5.4) is equivalent to

(5.5)
$$L_\Delta(y_\Delta)v_i = 0, \quad 1 \leqslant i \leqslant N; \qquad B_\Delta(y_\Delta)v_\Delta = 0; \qquad Q_\Delta v_\Delta = 0.$$

The main result of this section is formulated in Theorem 5.1. The proof of this theorem follows from the theory developed by Keller [6]. Thus the proof of Theorem 5.1 reduces to verifying the hypotheses of Theorem 4.7 in [6]. For notational convenience we define $Dx_\Delta = (Dx_0, Dx_1, \ldots, Dx_N)$, where $Dx_0 = (x_1 - x_0)/h - hA^{-1}f(0, x_0, 0)/2$ and $Dx_i = (x_{i+1} - x_{i-1})/2h$, $i = 1(1)N$.

THEOREM 5.1. *Let the conditions* N.1.1–N.1.5 *be satisfied. Then for some* $\varepsilon > 0$, $h_0 > 0$ *sufficiently small and all* $h \leqslant h_0$,

(i) *the difference system* (5.2) *has a unique solution* v_Δ *and* $\|v_\Delta - R_\Delta y\| \leqslant \varepsilon$;

(ii) *the difference solution can be computed by Newton's method, which converges quadratically for any initial iterate* y_Δ^0 *with* $\|y_\Delta^0 - R_\Delta y\| \leqslant \varepsilon$ *and* $\|Dy_\Delta^0 - R_\Delta y'\| \leqslant \varepsilon$ *provided that* ε $(\varepsilon \leqslant \rho, \delta)$ *is sufficiently small*;

(iii)

$$\|y_\Delta - R_\Delta y\| \leqslant \begin{cases} \text{const}\Big\{h|\ln h|^{d_+-1} + \omega\big(f(\cdot, y(\cdot), y'(\cdot)), h\big)\Big\}, & \text{if } 1 < \sigma_+ < 2, \\ \text{const}\Big\{h|\ln h|^{d_+} + \omega\big(f(\cdot, y(\cdot), y'(\cdot)), h\big)\Big\}, & \text{if } \sigma_+ = 2, \\ \text{const}\big\{h + \omega\big(f(\cdot, y(\cdot), y'(\cdot)), h\big)\big\}, & \text{if } \sigma_+ > 2 \text{ or } S = 0. \end{cases}$$

Proof. (i), (ii). The necessary conditions for (i) and (ii) to hold are

(1) stability of (5.5), provided that (3.5) has a unique solution and

(2) uniform Lipschitz continuity of the operators $L_\Delta(z_\Delta)$ and $B_\Delta(z_\Delta)$ for all $z \in U_{\rho,\delta}$, i.e., the existence of constants K_L and K_B such that

$$\|L_\Delta(z_\Delta) - L_\Delta(\xi_\Delta)\| \leqslant K_L\|z_\Delta - \xi_\Delta\|,$$
$$\|B_\Delta(z_\Delta) - B_\Delta(\xi_\Delta)\| \leqslant K_B\|z_\Delta - \xi_\Delta\|,$$

for all $z, \xi \in U_{\rho,\delta}$. We point out that the operators $L_\Delta(z_\Delta)$ and $B_\Delta(z_\Delta)$ can be represented by the coefficient matrices from (5.3a, b). Since (1) and (2) hold according to Lemma 4.3, and by the Lipschitz continuity of the derivatives f_v and f_w, see N.1.3, respectively, the assertions (i) and (ii) follow from [6, Theorem 4.7].

(iii) We consider the system (4.1). We substitute $R_\Delta y$ into the scheme and obtain

$$\text{(5.6a)} \quad \begin{aligned} &\frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} - \frac{A_1}{t_i}\left(\frac{y(t_{i+1}) - y(t_{i-1})}{2h}\right) - \frac{A_0}{t_i^2}y(t_i) \\ &\qquad -f\left(t_i, y(t_i), \frac{y(t_{i+1}) - y(t_{i-1})}{2h}\right) \equiv \delta_i, \qquad i = 1(1)N, \end{aligned}$$

$$\text{(5.6b)} \quad B\left(y(0); y(1), \frac{y(1+h) - y(1-h)}{2h}\right) \equiv \delta_B,$$

$$\text{(5.6c)} \quad \tilde{Q}\begin{bmatrix} y(0) \\ (y(h) - y(0))/h - hA^{-1}f(0, y(0), 0)/2 \end{bmatrix} \equiv \delta_0.$$

From the mean-value theorem and $y \in C^2(0,1]$ we have

$$\text{(5.7a)} \quad \begin{aligned} \delta_i &= y''(\xi_i) - y''(t_i) - \frac{A_1}{t_i}\big(y'(\eta_i) - y'(t_i)\big) \\ &\quad - \big(f\big(t_i, y(t_i), y'(\eta_i)\big) - f\big(t_i, y(t_i), y'(t_i)\big)\big) \\ &= y''(\xi_i) - y''(t_i) - \left(\frac{A_1}{t_i} + F_1(t_i)\right)\big(y'(\eta_i) - y'(t_i)\big), \qquad i = 1(1)N, \end{aligned}$$

where $t_{i-1} < \xi_i, \eta_i < t_{i+1}$,

$$F_1(t_i) = \int_0^1 \frac{\partial f}{\partial y'}\big(t_i, y(t_i), y'(\eta_i) + \tau\big(y'(t_i) - y'(\eta_i)\big)\big)\, d\tau,$$

$$\text{(5.7b)} \quad \begin{aligned} \delta_B &= B\big(y(0); y(1), y'(\eta_{N+1})\big) - B\big(y(0); y(1), y'(1)\big) \\ &= \int_0^1 \frac{\partial B}{\partial u_3}\big(y(0); y(1), y'(\eta_{N+1}) \\ &\qquad + \tau\big(y'(1) - y'(\eta_{N+1})\big)\big)\, d\tau\big(y'(\eta_{N+1}) - y'(1)\big), \end{aligned}$$

where $1 - h < \eta_{N+1} < 1 + h$, and

$$\delta_0 = \tilde{Q}\begin{bmatrix} 0 \\ y'(\eta_0) - y'(0) \end{bmatrix}, \tag{5.7c}$$

where $0 < \eta_0 < h$.

On the other hand, subtracting (4.1) from (5.6) and defining $e_i = y(t_i) - y_i$, $i = 0(1)N + 1$, we obtain the following system for e_Δ,

$$\begin{aligned} &\frac{e_{i+1} - 2e_i + e_{i-1}}{h^2} - \left(\frac{A_1}{t_i} + G_1(t_i)\right)\left(\frac{e_{i+1} - e_{i-1}}{2h}\right) \\ &\qquad - \left(\frac{A_0}{t_i^2} + G_0(t_i)\right)e_i = \delta_i, \qquad i = 1(1)N, \end{aligned} \tag{5.8a}$$

where

$$G_1(t_i) = \int_0^1 \frac{\partial f}{\partial y'}\left(t_i, y_i, \frac{y(t_{i+1}) - y(t_{i-1})}{2h} - \tau\frac{e_{i+1} - e_{i-1}}{2h}\right) d\tau,$$

$$G_0(t_i) = \int_0^1 \frac{\partial f}{\partial y}\left(t_i, y(t_i) - \tau e_i, \frac{y(t_{i+1}) - y(t_{i-1})}{2h}\right) d\tau,$$

and

$$\begin{aligned} &\int_0^1 \frac{\partial B}{\partial u_1}\left(y(0) - \tau e_0; y(1), \frac{y(1+h) - y(1-h)}{2h}\right) d\tau e_0 \\ &\quad + \int_0^1 \frac{\partial B}{\partial u_2}\left(y_0; y(1) - \tau e_N, \frac{y(1+h) - y(1-h)}{2h}\right) d\tau e_N \\ &\quad + \int_0^1 \frac{\partial B}{\partial u_3}\left(y_0; y_N, \frac{y(1+h) - y(1-h)}{2h}\right. \\ &\qquad\qquad \left. - \tau\frac{e_{N+1} - e_{N-1}}{2h}\right) d\tau \frac{e_{N+1} - e_{N-1}}{2h} = \delta_B, \end{aligned} \tag{5.8b}$$

(5.8c)

$$\begin{aligned} &\tilde{Q}\begin{bmatrix} y(0) - y_0 \\ (y(h) - y(0))/h - hA^{-1}f(0, y(0), 0)/2 - (y_1 - y_0)/h + hA^{-1}f(0, y_0, 0)/2 \end{bmatrix} \\ &\quad = \tilde{Q}\begin{bmatrix} e_0 \\ (e_1 - e_0)/h - hA^{-1}\int_0^1 \frac{\partial f}{\partial y}(0, y(0) - \tau e_0, 0)\, d\tau e_0/2 \end{bmatrix} = \delta_0. \end{aligned}$$

The result follows by virtue of Lemma 4.3 and by smoothness properties of f and y, following immediately from (3.8), see also [15, Lemma 4.4]. □

Let $f \in C^p[T_{\rho,\delta}]$ denote that $f(t, v, w)$ is p times continuously differentiable on $T_{\rho,\delta}$. We see from Theorem 5.1(iii) that if $\sigma_+ > 2$ and $f \in C^1[T_{\rho,\delta}]$, then the method converges at least as $O(h)$. This result can be improved, if we assume that $f \in C^2[T_{\rho,\delta}]$.

Theorem 5.2. *Let $f \in C^2[T_{\rho,\delta}]$. Then*

$$\|y_\Delta - R_\Delta y\| \leqslant \begin{cases} \text{const}\, h^{\sigma_+}|\ln h|^{d_+ - 1}, & 1 < \sigma_+ < 2, \\ \text{const}\, h^2\left(|\ln h|^{d_+} + |\ln h|^{d_0 - 1}\right), & \sigma_+ = 2, \\ \text{const}\, h^2|\ln h|^{d_0 - 1}, & \sigma_+ > 2 \text{ or } S = 0. \end{cases}$$

Proof. Since $y \in C^4(0,1]$, we apply Taylor's theorem to (5.6) and obtain

(5.9a) $$\delta_i = h^2 c y^{(4)}(\xi_i) - \left(\frac{A_1}{t_i} + F_1(t_i)\right) h^2 d y'''(\eta_i), \qquad i = 1(1)N,$$

where $t_{i-1} < \xi_i, \eta_i < t_{i+1}$ and

$$F_1(t_i) = \int_0^1 \frac{\partial f}{\partial y'}\left(t_i, y(t_i), y'(t_i) + \tau d h^2 y'''(\eta_i)\right) d\tau,$$

(5.9b) $$\delta_B = \int_0^1 \frac{\partial B}{\partial u_3}\left(y(0); y(1), y'(1) + \tau d h^2 y'''(\eta_{N+1})\right) d\tau h^2 d y'''(\eta_{N+1}),$$

where $1 - h < \eta_{N+1} < 1 + h$, and

(5.9c) $$\delta_0 = \tilde{Q}\begin{bmatrix} 0 \\ h^2 d y'''(\eta_0) \end{bmatrix},$$

where $0 < \zeta_0 < h$. The result follows now in a very similar way, cf. [15, Lemma 4.4]. □

The extension of the results of Theorems 5.1 and 5.2 to the problem (1.2) is straightforward. In particular (i) and (ii) of Theorem 5.1 hold by [6, Theorem 4.7], so we shall restrict our attention to deriving the associated convergence results, which we formulate in

THEOREM 5.3. *Let the assumptions* N.2.1–N.2.5 *hold. Then*

(i)

$$\|y_\Delta - R_\Delta y\| \leqslant \begin{cases} \text{const}\left\{h|\ln h|^{d_+ - 1} + \omega(f(\cdot, y(\cdot), \xi(\cdot)), h)\right\}, & \lambda = 1 \text{ or } 1 < \sigma_+ < 2, \\ \text{const}\left\{h|\ln h|^{d_+} + \omega(f(\cdot, y(\cdot), \xi(\cdot)), h)\right\}, & \sigma_+ = 2, \\ \text{const}\{h + \omega(f(\cdot, y(\cdot), \xi(\cdot)), h)\}, & \sigma_+ > 2 \text{ or } S = 0, \end{cases}$$

where $y(t)/t \equiv \xi(t)$.

(ii) *If* $f \in C^2[T_{\rho,\varepsilon}]$, *then*

$$\|y_\Delta - R_\Delta y\| \leqslant \begin{cases} \text{const}\, h^{\sigma_+}|\ln h|^{d_+ - 1}, & \lambda = 1 \text{ or } 1 < \sigma_+ < 2, \\ \text{const}\, h^2|\ln h|^{d_+}, & \sigma_+ = 2, \\ \text{const}\, h^2, & \sigma_+ > 2 \text{ or } S = 0. \end{cases}$$

Proof. (i) In this case we have

(5.10a) $$\delta_i = \left(y''(\xi_i) - y''(t_i)\right) - \frac{A_1}{t_i}\left(y'(\eta_i) - y'(t_i)\right), \qquad i = 1(1)N,$$

(5.10b) $$\delta_B = \int_0^1 \frac{\partial B}{\partial u_2}\left(y(1); y'(\eta_{N+1}) + \tau\left(y'(1) - y'(\eta_{N+1})\right)\right) d\tau \left(y'(\eta_{N+1}) - y'(1)\right),$$

(5.10c) $$\delta_0 = 0.$$

Furthermore, δ_i can be estimated as in [15, Lemma 4.4] and $\delta_B = O(h)$. This completes the proof of (i).

(ii) From Taylor's theorem we have

$$\delta_i = h^2 c y^{(4)}(\xi_i) - \frac{A_1}{t_i} h^2 d y'''(\eta_i) = \begin{cases} O\left(t_i^{\sigma_+ - 4} h^2 |\ln h|^{d_+ - 1}\right), & \lambda = 1, 1 < \sigma_+ < 2, \\ O\left(t_i^{-2} h^2 |\ln h|^{d_+}\right), & \sigma_+ = 2, \\ O\left(t_i^{-2} h^2\right), & \sigma_+ > 2, \\ O(h^2), & S = 0, \end{cases}$$

and $\delta_B = O(h^2)$. The proof follows now by [15, Lemma 4.4], see also [15, formulas (4.16), (4.21), (4.23)]. □

6. Numerical Examples.

Example 1. To illustrate the results of Lemma 4.1(ii) we consider a homogeneous 2×2 system, where

$$A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix}, \qquad A_0 = \begin{bmatrix} 0 & a_0 \\ 0 & 0 \end{bmatrix}.$$

If $a_1 = 2$ and $a_0 = 1$, then $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 2$, and the boundary conditions

$$y_1(0) = 3, \quad y_2(0) = 0, \qquad y_2(1) = 4, \quad y_1'(1) = -2$$

yield the solution

$$y(t) = \begin{bmatrix} 3 - 2t^2(1 - \ln t) \\ 4t^2 \end{bmatrix}. \tag{6.1}$$

For $a_1 = 2.5$ and $a_0 = 1$ we have $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = 2.5$. We choose

$$y_1(0) = 3, \quad y_2(0) = 0, \quad y_2(1) = 5, \quad y_1'(1) = -3,$$

and get the following solution

$$y(t) = \begin{bmatrix} 3 - 2t^{2.5}(1 - \ln t) \\ 5t^{2.5} \end{bmatrix}. \tag{6.2}$$

Furthermore,

$$J = \begin{bmatrix} a_1 & 1 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and the error behavior following from Lemma 4.1(ii) is illustrated in Tables (6.1) and (6.2). We note that in this case $\delta_0 = 0$.

TABLE 6.1

h	$\Delta(h)$	$\Delta(h)/h^2\lvert\ln h\rvert$	$\Delta(h/2)/\Delta(h)$
1/10	2.133 E − 2	0.9264	*
1/20	6.199 E − 3	0.8277	0.2906
1/40	1.766 E − 3	0.7660	0.2846
1/80	4.956 E − 4	0.7240	0.2807
1/160	1.375 E − 4	0.6936	0.2774
1/320	3.775 E − 5	0.6701	0.2745
1/640	1.028 E − 5	0.6517	0.2723

$$\lim_{h \to 0} \frac{\Delta(h/2)}{\Delta(h)} = \left(\frac{1}{2}\right)^2 = 0.25.$$

Here, $\Delta(h) = \max_{0 \le i \le N} |y_1(t_i) - y_{i1}|$. The second component has been computed without errors.

TABLE 6.2

	1st component			2nd component		
h	$\Delta(h)$	$\Delta(h)/h^2$	$\Delta(h/2)/\Delta(h)$	$\Delta(h)$	$\Delta(h)/h^2$	$\Delta(h/2)/\Delta(h)$
1/10	7.546 E − 3	0.7546	*	1.210 E − 2	1.2100	*
1/20	2.477 E − 3	0.9908	0.3283	3.203 E − 3	1.2810	0.2647
1/40	7.453 E − 4	1.1925	0.3009	8.334 E − 4	1.3333	0.2602
1/80	2.124 E − 4	1.3594	0.2850	2.142 E − 4	1.3708	0.2571
1/160	5.841 E − 5	1.4953	0.2750	5.461 E − 5	1.3980	0.2549
1/320	1.566 E − 5	1.6036	0.2681	1.384 E − 5	1.4172	0.2534
1/640	4.124 E − 6	1.6892	0.2633	3.492 E − 6	1.4303	0.2525

$$\lim_{h \to 0} \frac{\Delta(h/2)}{\Delta(h)} = \left(\frac{1}{2}\right)^2 = 0.25.$$

Example 2. We now investigate the 2×2 system, where

$$A_1 = \begin{bmatrix} 2\sqrt{2} - 1 & 0 \\ 2\sqrt{2} & -1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad f(t) = \begin{bmatrix} 6 - 4\sqrt{2} \\ -10 - 4\sqrt{2} \end{bmatrix}$$

and

$$J = \begin{bmatrix} 2+\sqrt{2} & 0 & 0 & 0 \\ 0 & 2-\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The boundary conditions are

$$\begin{aligned} y_1'(0) - y_2'(0) &= 0, \\ y_1(0) + y_2(0) &= 0, \end{aligned} \qquad y_1(1) = 7, \quad y_2'(1) = \sqrt{2} - 4,$$

and the solution is

$$y(t) = \begin{bmatrix} t^{\sqrt{2}+2} + t^2 + 5 \\ t^{\sqrt{2}+2} - 3t^2 - 5 \end{bmatrix}.$$

In this case $y \in C^3[0,1]$ and therefore

(i) $$\frac{y(h) - y(0)}{h} - y'(0) = O(h),$$

(ii) $$\left(\frac{y(h) - y(0)}{h} - hA^{-1}f(0)/2\right) - y'(0) = O(h^2),$$

since $(A_1 + A_0)y'(0) = 0$ and $A_0 y(0) = 0$. We can now see that even the order h approximation of the first derivative at $t = 0$ does not influence the error behavior (cf. [14, Lemma 4.2]).

TABLE 6.3

	$y'(0) - y_0' = O(h^2)$		$y'(0) - y_0' = O(h)$	
h	$\Delta = \lVert y_\Delta - R_\Delta y \rVert$	$\Delta/h^2 \lvert\ln h\rvert$	$\Delta = \lVert y_\Delta - R_\Delta y \rVert$	$\Delta/h^2 \lvert\ln h\rvert$
1/10	3.035 E − 2	1.318	4.300 E − 2	1.867
1/20	8.588 E − 3	1.147	1.238 E − 2	1.653
1/40	2.402 E − 3	1.042	3.511 E − 3	1.523
1/80	6.652 E − 4	0.972	9.831 E − 4	1.436
1/160	1.826 E − 4	0.921	2.724 E − 4	1.374
1/320	4.975 E − 5	0.883	7.478 E − 5	1.328
1/640	1.347 E − 5	0.854	2.037 E − 5	1.291

Example 3. Finally, we consider the nonlinear scalar equation

$$y''(t) + \frac{2}{t}y'(t) = -y^5(t), \quad y'(0) = 0, \quad y(1) = \sqrt{3/4}.$$

The solution of this problem is $y(t) = 1/\sqrt{1 + t^2/3} \in C^2$. The eigenvalues of M are $\lambda_1 = 0$, $\lambda_2 = -1$, and it follows from Theorem 5.2 that the order of convergence is 2.

TABLE 6.4

h	$\Delta(h) = \lvert y_\Delta - R_\Delta y \rvert$	$\Delta(h)/h^2$	$\Delta(h/2)/\Delta(h)$
1/4	5.079 E − 3	0.08126	*
1/8	1.219 E − 3	0.07802	0.2400
1/16	3.018 E − 4	0.07726	0.2476
1/32	7.526 E − 5	0.07707	0.2494
1/64	1.880 E − 5	0.07700	0.2498
1/128	4.700 E − 6	0.07700	0.2500

$$\lim_{h \to 0} \frac{\Delta(h/2)}{\Delta(h)} = 0.25.$$

All examples were computed on a CDC Cyber 170/172 in single precision.

7. Appendix.

7.1. *Case* 1. $\sigma < 0$. In order to study the growth of the solution v_i of the difference system (4.12) we consider the scalar case first and the case when $n = 2$ afterwards. In both cases we define a matrix function

$$(7.1) \qquad Z_{i,k+1} = \prod_{l=i}^{k+1}\left(I + \frac{1}{l}J + \frac{1}{l}\Delta(l, J)\right)$$

in such a way that

$$(7.2) \qquad |\tilde{Z}_{i,k+1}| \leqslant |Z_{i,k+1}|, \qquad k < i.$$

Furthermore, we define the matrix $\Delta(l, J)$ as a diagonal matrix to make the system (4.12) decoupled.

Let $n = 1$, a_0, $a_1 \in \mathbf{R}$ and let us assume that the eigenvalues of M are $\lambda_1 = \lambda$ and $\lambda_2 = \bar{\lambda}$. It follows immediately from the form of M that

$$E = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}, \qquad E^{-1} = \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}.$$

We rewrite Θ_1 and Θ_2 as

$$\Theta_1(i) = \frac{2i}{2i - a_1}a, \qquad \Theta_2(i) = \frac{2i}{2i - a_1}b$$

and obtain

$$\Theta(i) = E^{-1}\Theta_4(i)E = \frac{2i}{2i - a_1} \cdot \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix} -(a + \lambda_1 b) & -(a + \lambda_2 b) \\ a + \lambda_1 b & a + \lambda_2 b \end{bmatrix}.$$

Finally, with $c = 2i/(2i - a_1)(\lambda_2 - \lambda_1)$ we have

$$I + \frac{1}{i}J + \frac{1}{i^2}\Theta(i) = \begin{bmatrix} 1 + \frac{1}{i}\lambda_1 - \frac{c}{i^2}(a + \lambda_1 b) & -\frac{c}{i^2}(a + \lambda_2 b) \\ \frac{c}{i^2}(a + \lambda_1 b) & 1 + \frac{1}{i}\lambda_2 + \frac{c}{i^2}(a + \lambda_2 b) \end{bmatrix}$$

$$\equiv N(i),$$

and it can easily be seen that

$$|N(i)| = \left|1 + \frac{1}{i}\lambda_1 - \frac{c}{i^2}(a + \lambda_1 b)\right| + \left|\frac{c}{i^2}(a + \lambda_2 b)\right|$$

$$= \left|\frac{c}{i^2}(a + \lambda_1 b)\right| + \left|1 + \frac{1}{i}\lambda_2 + \frac{c}{i^2}(a + \lambda_2 b)\right|.$$

Furthermore, since for any $\alpha \in \mathbf{C}$ and positive real number β, $|\alpha| + \beta \leqslant |\eta|$, where

$$\mathrm{Re}(\eta) = \mathrm{Re}(\alpha) + \mathrm{sign}(\mathrm{Re}(\alpha))\beta, \qquad \mathrm{Im}(\eta) = \mathrm{Im}(\alpha) + \mathrm{sign}(\mathrm{Im}(\alpha))\beta,$$

we have for each i

$$|N(i)| \leqslant \left|1 + \frac{1}{i}\lambda\right| + \frac{2}{i^2}|c(a + \lambda b)| = \left|1 + \frac{1}{i}\lambda\right| + \frac{1}{i^2}|\Theta(i)|$$

$$\leqslant \left|1 + \frac{1}{i}\lambda + \frac{1}{i^2}\delta(i,\lambda)\right|,$$

where

$$\mathrm{Re}(\delta(i,\lambda)) = \mathrm{sign}\left(\mathrm{Re}\left(1 + \frac{1}{i}\lambda\right)\right)|\Theta(i)|, \qquad \mathrm{Im}(\delta(i,\lambda)) = \mathrm{sign}(\mathrm{Im}(\lambda))|\Theta(i)|.$$

Clearly, we define

$$\Delta(l, J) := \begin{bmatrix} \delta(l,\lambda) & 0 \\ 0 & \delta(l,\bar{\lambda}) \end{bmatrix}. \tag{7.3}$$

We notice that although the matrix $N(i)$ is a full matrix, a certain structure is present; the norm obtained from the first row belonging to λ and the norm obtained from the second row belonging to $\bar{\lambda}$ are equal. We shall now show that this structure remains unchanged if $n = 2$.

Let $n = 2$. Let us assume that $a_i, b_i \in \mathbf{R}$, $i = 1(1)4$ and

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \end{bmatrix}, \qquad J = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \bar{\lambda} & 1 \\ 0 & 0 & 0 & \bar{\lambda} \end{bmatrix};$$

then we have the following transformation matrix E,

$$E = \begin{bmatrix} a & b & \bar{a} & \bar{b} \\ 1 & 0 & 1 & 0 \\ a\lambda & a + b\lambda & \overline{a\lambda} & \overline{a + b\lambda} \\ \lambda & 1 & \bar{\lambda} & 1 \end{bmatrix},$$

where

$$a \equiv a(\lambda) = (\lambda^2 - \lambda b_4 - a_4)/(a_3 + b_3\lambda), \quad \bar{a} = a(\bar{\lambda}),$$

$$b \equiv b(\lambda) = (2\lambda - b_4 - ab_3)/(a_3 + b_3\lambda), \quad \bar{b} = b(\bar{\lambda}).$$

Furthermore, with $d \equiv \det(E) = |b|(\lambda - \bar{\lambda})^2 - (a - \bar{a})^2 \in \mathbf{R}$,

$$E^{-1} = \frac{1}{d}\begin{bmatrix} \Big| & \Big| & b - \bar{b} & b(\overline{a + b\lambda}) - \bar{b}(a + b\lambda) \\ & & (a - \bar{a}) + \bar{b}(\bar{\lambda} - \lambda) & (\overline{a + b\lambda})(a - \bar{a}) + \bar{b}(\overline{a\lambda} - a\lambda) \\ e_1 & e_2 & -(b - \bar{b}) & -(b(\overline{a + b\lambda}) - \bar{b}(a + b\lambda)) \\ \Big| & \Big| & -(a - \bar{a}) - b(\bar{\lambda} - \lambda) & -((a + b\lambda)(a - \bar{a}) + b(\overline{a\lambda} - a\lambda)) \end{bmatrix}.$$

Finally, we calculate $\Theta(i) \equiv (m_{kj})/d,\ k, j = 1(1)4$, and by comparing the first row with the third, and the second row with the fourth, we see that

$$(7.4a)\qquad m_{11} = \overline{m_{33}},\quad m_{12} = \overline{m_{34}},\quad m_{13} = \overline{m_{31}},\quad m_{14} = \overline{m_{32}},$$

$$(7.4b)\qquad m_{21} = \overline{m_{43}},\quad m_{22} = \overline{m_{44}},\quad m_{23} = \overline{m_{41}},\quad m_{24} = \overline{m_{42}}.$$

For $N(i) = I + J/i + \Theta(i)/i^2 \equiv (n_{kj}),\ k, j = 1(1)4$, (7.4) holds and this yields

$$(7.5)\qquad \sum_{j=1}^{4} |n_{1j}| = \sum_{j=1}^{4} |n_{3j}|,\qquad \sum_{j=1}^{4} |n_{2j}| = \sum_{j=1}^{4} |n_{4j}|.$$

Motivated by (7.5), we choose for $\Delta(l, J)$ in (7.1),

$$(7.6)\qquad \Delta(l, J) := \operatorname{diag}\big(\delta(l,\lambda), \delta(l,\lambda), \delta(l,\bar\lambda), \delta(l,\bar\lambda)\big),$$

where $\delta(\cdot)$ is defined as in the scalar case. Furthermore, (7.5) justifies the "Jordan-boxwise" considerations of the sytem (4.12).

7.2. *Case* 3. $\sigma > 0$. We now restrict our attention to the scalar case. The results can be carried over to systems in a similar manner as in Subsection 7.1. Let $n = 1$ and consider $(N(i))^{-1}$,

$$(N(i))^{-1} = \frac{1}{\det(N(i))}\begin{bmatrix} 1 + \frac{1}{i}\bar\lambda + \frac{c}{i^2}(a + \bar\lambda b) & \frac{c}{i^2}(a + \bar\lambda b) \\ -\frac{c}{i^2}(a + \lambda b) & 1 + \frac{1}{i}\lambda - \frac{c}{i^2}(a + \lambda b) \end{bmatrix}.$$

Then we have

$$\left|(N(i))^{-1}\right| = 1\Big/\left|\,\left|1 + \frac{1}{i}\lambda - \frac{c}{i^2}(a + \lambda b)\right| - \left|\frac{c}{i^2}(a + \lambda b)\right|\,\right|.$$

Since there exists an index i_0 such that for all $i \geqslant i_0$, $|1 + \lambda/i| \geqslant |2c(a + \lambda b)/i^2|$, we conclude that

$$(7.7)\qquad \left|(N(i))^{-1}\right| \leqslant 1\Big/\left|\,\left|1 + \frac{1}{i}\lambda\right| - \frac{1}{i^2}|\Theta(i)|\,\right| \leqslant 1\Big/\left|1 + \frac{1}{i}\lambda + \frac{1}{i^2}\delta(i,\lambda)\right|,$$

where

$$(7.8a)\qquad \operatorname{Re}(\delta(i,\lambda)) = -|\Theta(i)|\cos\varphi_i \equiv -\operatorname{sign}\left(\operatorname{Re}\left(1 + \frac{\lambda}{i}\right)\right)|\Theta(i)|\cos\varphi_i,$$

$$(7.8b)\qquad \operatorname{Im}(\delta(i,\lambda)) = -\operatorname{sign}(\operatorname{Im}\lambda)|\Theta(i)|\sin\varphi_i,\qquad \varphi_i = \arctan\left(\frac{\kappa}{i + \sigma}\right).$$

Because of (7.7) we define

$$Z_{j+1,k} := \prod_{l=j+1}^{k}\left(I + \frac{1}{l}J + \frac{1}{l^2}\Delta(l, J)\right)^{-1},\qquad k > j,$$

where $\Delta(l, J) = \operatorname{diag}(\delta(l,\lambda), \delta(l,\bar\lambda))$ and $\delta(l,\lambda)$ is given by (7.8). Finally, we define for λ, $\Omega_1(\lambda)$ and $\Omega_2(\lambda)$ as in Lemma 2.2 and for $\delta(l,\mu)$ as in Lemma 2.3,

$$(7.9)\qquad z_{kj}(\mu, \delta(\cdot)) := \begin{cases} 1, & k = j, \\ \displaystyle\prod_{l=k}^{j-1} 1\Big/\left(1 + \frac{1}{l}\left(\mu + \frac{1}{l}\delta(l,\mu)\right)\right), & \\ & 1 \leqslant k < j,\ j = 2(1)N + 1, \end{cases}$$

and show that there exists a constant $\eta > 0$ such that

$$\left| z_{kj}(\mu, \delta(\cdot)) \right| \leqslant \operatorname{const}(t_k/t_j)^{\eta}, \qquad k \leqslant j,\ j = 1(1)N.$$

To see this, we choose an index l_0 so large that for all $l \geqslant l_0$, $|\mu + \delta(l, \mu)/l|/l < 1$. Then we have

$$\prod_{l=k}^{j-1} 1 \Big/ \left(1 + \frac{1}{l}\left(\mu + \frac{1}{l}\delta(l,\mu)\right)\right) = \prod_{l=k}^{j-1}\left(1 - \frac{1}{l}\left(\mu + \frac{1}{l}\hat{\delta}(l,\mu)\right)\right),$$

and there exists an index $\hat{\rho} \geqslant 1$ such that $|\hat{\delta}(l, \mu)| \leqslant \hat{\delta}$ for all $l \geqslant \hat{\rho}$. The result now follows from Lemma 2.3 for all $k \geqslant k_0 \equiv \max\{i_0, l_0, \hat{\rho}, 4\hat{\delta}/\sigma\}$. Since for $k \leqslant k_0$, $z_{kk_0}(\mu, \delta(\cdot))$ consists of at most k_0 terms, the result holds for all $k \geqslant 1$. Since

$$Z_{j+1,k} = \frac{1}{2\pi i}\int_\Gamma z_{j+1,k+1}(\lambda, \delta(\cdot))(\lambda I - J)^{-1} d\lambda, \qquad \Gamma = \{\mu \mid |\lambda - \mu| \leqslant \sigma/4\},$$

and $|\tilde{Z}_{j+1,k}| \leqslant |Z_{j+1,k}|$ for all $k > j$, the estimate (4.25) holds. It should be mentioned that in the case when $n \geqslant 2$ we have

$$\left|\left(I + \frac{1}{l}J + \frac{1}{l^2}\Theta(l)\right)^{-1}\right| \leqslant m(l,n)\left|\left(I + \frac{1}{l}J + \frac{1}{l^2}\Delta(l,J)\right)^{-1}\right|$$

for $1 \leqslant l \leqslant N$, and $m(l, n)$ is uniformly bounded with respect to l and n.

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