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On the Definiteness of Gauss-Kronrod Integration Rules

By Philip Rabinowitz

Abstract. The nondefiniteness of the Kronrod extension of the Gauss-Gegenbauer integration rule with weight function $w(x; \mu) = (1 - x^2)^{\mu - 1/2}$, $0 < \mu < 1$, is shown when there are more than three abscissas.

In a recent paper, Akrivis and Förster [1] have shown that the Clenshaw-Curtis and related integration rules are nondefinite, i.e., that the error Rf cannot be expressed in the form

$$Rf = cf^{(d+1)}(\xi)$$

where d is the precision of the rule. Using their approach combined with some of our previous results [3], we shall show that the same holds for the Kronrod extension (KE) of the Gauss-Gegenbauer integration rule (GGIR) with respect to the weight function

$$w(x; \mu) = (1 - x^2)^{\mu - 1/2} \tag{1}$$

when μ satisfies $0 < \mu < 1$. In particular, the usual Gauss-Kronrod rule ($\mu = 1/2$) is nondefinite. We shall first give the results in [1] needed for our presentation. Then we shall introduce the KEGGIR. Finally, we shall prove the nondefiniteness of the KEGGIR for $\mu \in (0, 1)$. The results on the KEGGIR appear in [3] and we shall not mention this in the sequel.

Consider the open integration rule Q_n satisfying

$$\int_{-1}^{1} w(x) f(x)\, dx = Q_n f + R_n f, \tag{2}$$

where

$$Q_n f = \sum_{i=1}^{n} w_i f(x_i), \qquad -1 < x_1 < x_2 < \cdots < x_n < 1, \tag{3}$$

and $w(x)$ is a weight function which is positive for $x \in (-1, 1)$. Q_n is said to be of (exact) precision d if $R_n f = 0$ when f is a polynomial of degree $\leqslant d$ and if there exists at least one polynomial p of degree $d + 1$ for which $R_n p \neq 0$. A slight generalization of Proposition 1 in [1] states that if there exists a function $f \in C[-1, 1]$ such that $f^{(d+1)} \geqslant 0$, $f^{(d+1)} \not\equiv 0$ and $R_n f < 0$, then the open rule Q_n of precision d

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is nondefinite. We shall now introduce the KEGGIR, Q_{2n+1}, and determine a function f_k satisfying the hypotheses of this generalization. This will prove our claim that the KEGGIR's are nondefinite.

The abscissas x_i, $i = 1, \ldots, n$, of the GGIR are the zeros of the Gegenbauer polynomial $C_n^\mu(x)$ and lie in $(-1, 1)$. These polynomials are orthogonal with respect to $w(x; \mu)$ and have the following normalization:

$$\int_{-1}^{1} w(x;\mu) C_n^\mu(x) C_m^\mu(x)\, dx = \delta_{nm} h_{n\mu}, \tag{4}$$

where

$$h_{n\mu} = \pi^{1/2}\Gamma(n + 2\mu)\Gamma(\mu + 1/2)/((n + \mu)n!\Gamma(\mu)\Gamma(2\mu)). \tag{5}$$

The KEGGIR, Q_{2n+1}, is given by

$$Q_{2n+1} f = \sum_{i=1}^{n} u_i f(x_i) + \sum_{i=1}^{n+1} v_i f(y_i), \tag{6}$$

where the y_i are the zeros of the Szegö polynomial $E_{n+1,\mu}(x)$ which satisfies the orthogonality conditions

$$\int_{-1}^{1} w(x;\mu) C_n^\mu(x) E_{n+1,\mu}(x) x^k\, dx = 0, \qquad k = 0, 1, \ldots, n.$$

For $0 < \mu < 1$, the y_i lie in $(-1, 1)$ so that Q_{2n+1} is an open integration rule. The precision d of Q_{2n+1} is given by

$$d = \begin{cases} 3n + 1, & n \text{ even}, \\ 3n + 2, & n \text{ odd}, \end{cases}$$

for $0 < \mu \leqslant 2$, $\mu \neq 1$. The Szegö polynomials are given by

$$E_{n+1,\mu}(x) = \sum_{i=0}^{m-1} \lambda_{i\mu} T_{n+1-2i}(x) + \begin{cases} \lambda_{m\mu} T_1(x), & n \text{ even}, \\ \frac{1}{2}\lambda_{m\mu}, & n \text{ odd}, \end{cases} \tag{7}$$

where $m = [(n + 1)/2]$ and the $T_k(x)$ are the Chebyshev polynomials of the first kind. The $\lambda_{i\mu}$ are given by

$$\lambda_{0\mu} = 2\gamma_{n\mu}^{-1}, \qquad \sum_{i=1}^{k} f_{i,\mu}\lambda_{k-i,\mu} = 0, \qquad k = 1, 2, \ldots, \tag{8}$$

where

$$\gamma_{n\mu} = \sqrt{\pi}\,\Gamma(n + 2\mu)/\Gamma(n + \mu + 1), \qquad f_{0\mu} = 1,$$
$$f_{j\mu} = (1 - \mu/j)(1 - \mu/(n + \mu + j)) f_{j-1,\mu},$$

and we have not shown the dependence on n of the $f_{j\mu}$ and the $\lambda_{i\mu}$. For $0 < \mu < 1$, the sequence $\{\lambda_{i\mu}: i = 1, 2, \ldots\}$ is strictly monotonic increasing.*

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*Professor H. Brass has pointed out a gap in the proof in [3, p. 1279] that the sequence $\{\lambda_{i\mu} = \lambda_{i0}\alpha_{i\mu}: i = 1, 2, \ldots\}$ is strictly monotonic increasing, since it does not follow that if a sequence $\{f_j: j = 0, 1, \ldots\}$ is strictly completely monotonic and $\sum_{i=0}^{\infty}\alpha_i u^i = \{\sum_{j=0}^{\infty} f_j u^j\}^{-1}$, then the sequence $\{-\alpha_i: i = 1, 2, \ldots\}$ is strictly completely monotonic. All that we can say is that it is completely monotonic. The following sequence provides a counterexample: $f_j = 2^{-j}$, $\alpha_0 = 1$, $\alpha_1 = -1/2$, $\alpha_i = 0$, $i > 1$. Professor Brass has also shown how to close this gap in our case, since if we did not have strict monotonicity, then $\lambda_{i\mu} = \lambda_{i+1,\mu}$ for some integer $i = i_0$. Hence by complete monotonicity, $\lambda_{i\mu} = \lambda_{i+1,\mu}$ for all $i \geqslant i_0$ which would imply that $F(u) = \sum_{j=0}^{\infty} f_{j\mu} u^j$ is a rational function. However, this is not the case since $F(u)$ is the hypergeometric function ${}_2F_1(1 - \mu, n + 1; n + \mu + 1; u)$ and $0 < \mu < 1$.

The Gegenbauer polynomials and the Szegö polynomials are related by the following equality:

(9) $$C_n^\mu(x)E_{n+1,\mu}(x) = \sum_{i=0}^{n} c_i C_{n+1+i}^\mu(x),$$

where the $c_i = c_i(\mu, n)$ are certain constants. For our purpose, the values of c_0 and c_1 are important. They are given by

(10) $$c_0 = \begin{cases} \dfrac{\gamma_{n\mu}}{2\gamma_{n+1,\mu}}(\lambda_{m\mu} - \lambda_{m+1,\mu}), & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases}$$

(11) $$c_1 = \begin{cases} \dfrac{\gamma_{n\mu}}{2\gamma_{n+2,\mu}}(\lambda_{m-1,\mu} - \lambda_{m+1,\mu}), & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

By the monotonicity of the sequence $\{\lambda_{i\mu}\}$, it follows that for $0 < \mu < 1$, c_0 is negative for $m \geqslant 1$, i.e., for $n \geqslant 2$ and c_1, for $m \geqslant 2$, i.e., for $n \geqslant 3$. We now define

(12) $$f_k(x) = C_n^\mu(x)E_{n+1,\mu}(x)C_{n+1+k}^\mu(x), \qquad k = 0, 1.$$

Then, since $Q_{2n+1}f_k = 0$,

(13) $$R_{2n+1}f_k = \int_{-1}^{1} w(x;\mu)f_k(x)\,dx = c_k h_{n+1+k,\mu}, \qquad k = 0, 1.$$

Furthermore, $f_k^{(3n+2+k)} > 0$. If n is even, $c_0 \neq 0$ and we choose $k = 0$ so that $3n + 2 + k = d + 1$. If n is odd, $c_0 = 0$ but $c_1 \neq 0$ and we choose $k = 1$ so that again $3n + 2 + k = d + 1$. In either case, $R_{2n+1}f_k < 0$ for $n \geqslant 2$ which implies that Q_{2n+1} is nondefinite. For $n = 1$, Q_{2n+1} is the 3-point GGIR,which is definite.

For $\mu = 0$, Q_{2n+1} is a Lobatto-Chebyshev rule of the first kind [2, p. 104] and hence is definite. Similarly, for $\mu = 1$, Q_{2n+1} is a Gauss-Chebyshev rule of the second kind [2, p. 98], which is also definite. For $1 < \mu \leqslant 2$, in which range KEGGIR's exist, the question of definiteness is still not settled. The same holds for the KE of the Lobatto-Gegenbauer integation rules except for that of the Lobatto-Chebyshev rule of the first kind,which is itself a Lobatto-Chebyshev rule and hence is definite.

Department of Applied Mathematics
The Weizmann Institute of Science
Rehovot 76100, Israel

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