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Effective Irrationality Measures for Certain Algebraic Numbers

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Abstract. A result of Chudnovsky concerning rational approximation to certain algebraic numbers is reworked to provide a quantitative result in which all constants are explicitly given. More particularly, Padé approximants to the function $(1-x)^{1/3}$ are employed to show, for certain integers a and b, that

$$\left|(a/b)^{1/3} - p/q\right| > cq^{-\kappa} \quad \text{when } q > 0.$$

Here, c and κ are given as functions of a and b only.

In 1964 Baker [1], improving a technique used by Siegel [8], was able to obtain effective irrationality measures for the function $(1-x)^{m/n}$ evaluated at certain rational points. In particular, he was able to show that for integers p, q we have

$$(1) \qquad |2^{1/3} - p/q| > 10^{-6}q^{-2.955} \quad \text{when } q > 0.$$

The technique was further refined by Chudnovsky [2] whose results, when applied to $2^{1/3}$, imply that for any $\varepsilon > 0$ there exists a positive integer $q_0(\varepsilon)$ such that for integers p, q we have

$$(2) \qquad |2^{1/3} - p/q| > q^{-(2.429+\varepsilon)} \quad \text{when } q > q_0(\varepsilon).$$

Chudnovsky's result is effective in the sense that it is possible in principle to work through the proof and compute, for any particular value of ε, a $q_0(\varepsilon)$ for which (2) holds. However, Chudnovsky does not undertake such computations.

In this article we rework Baker's proof using Chudnovsky's refinement, together with a Chebyshev-type result for primes in arithmetical progressions due to McCurley [6], and obtain the following quantitative result:

THEOREM. *Let a, b be integers with $0 < b < a$. Define d by*

$$(3) \qquad d = \begin{cases} 0 & \text{if } 3 \nmid (a-b), \\ 1 & \text{if } 3 \| (a-b), \\ 3/2 & \text{otherwise}. \end{cases}$$

Further, define λ, κ, c and q_0 by

$$(4) \qquad \lambda = (.2328)3^d(a^{1/2} - b^{1/2})^{-2},$$

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(5) $$\kappa = 1 + \log(8.591(a+b)3^{-d})(\log\lambda)^{-1},$$

(6) $$c = 1.69 \times 10^{-2}(a+b)^{-1}\left[.9302(a^{1/2}+b^{1/2})^{-1}(ab^2)^{1/3}(a^{1/2}-b^{1/2})\right]^{\kappa-1},$$

(7) $$q_0 = \lambda^{300}\left[.9302(a^{1/2}+b^{1/2})^{-1}(ab^2)^{1/3}(a^{1/2}-b^{1/2})\right].$$

Then, assuming $\lambda > 1$, *we have for integers* p, q

(8) $$\left|(b/a)^{1/3} - p/q\right| > cq^{-\kappa} \quad \textit{when } q > q_0.$$

We remark that the Theorem yields an improvement on Liouville's Theorem provided $\kappa < 3$, which occurs when

$$(158.5)(3^{-3d})(a+b)(a^{1/2}-b^{1/2})^4 < 1.$$

As a consequence of the Theorem we are able to obtain

COROLLARY. *For the values of* α, κ, c *and* q_0 *given by the following table, we have, for integers* p, q *that*

$$|\alpha - p/q| > cq^{-\kappa} \quad \textit{when } q > q_0.$$

α	c	κ	q_0
$2^{1/3}$	2.2×10^{-8}	2.795	0
$6^{1/3}$	1.03×10^{-17}	2.405	10^{1976}
$10^{1/3}$	7.81×10^{-10}	2.619	0
$15^{1/3}$	4.5×10^{-7}	2.933	0
$17^{1/3}$	2.51×10^{-10}	2.3391	0
$19^{1/3}$	1.1×10^{-8}	2.473	0
$20^{1/3}$	3.84×10^{-10}	2.333	0
$22^{1/3}$	5.16×10^{-8}	2.482	0
$26^{1/3}$	7.8×10^{-7}	2.9099	0
$28^{1/3}$	7.59×10^{-7}	2.899	0
$37^{1/3}$	1.31×10^{-8}	2.427	0
$39^{1/3}$	1.46×10^{-11}	2.313	0
$42^{1/3}$	2.12×10^{-7}	2.766	0
$43^{1/3}$	1.94×10^{-8}	2.506	0

It should be emphasized that in our proof certain choices must be made, which essentially correspond to fixing a value for ε in (2). Unfortunately, decreasing the size of ε, and hence of κ, causes the value of q_0, as given by (7), to increase. Moreover, since in some of the estimates we use, we employ bounds which are not sharp, we are not able, in our proof, to take ε to be arbitrarily small. For example, the smallest value of κ which our proof can be made to yield in the case of $2^{1/3}$ is $\kappa = 2.4862\ldots$; here $q_0 = 10^{9\times10^5}$ and $c = 10^{-2}$. It was our aim in making the choices we did, to obtain as small a value for κ as possible while keeping q_0 sufficiently small that it is practical, at least for most of the values of α given in the Corollary, to compute and employ continued fraction expansions to remove the restriction the Theorem places on the size of q. The continued fractions were computed at the University of Waterloo on a Honeywell DPS 8/49 using a program written in MAPLE.

Lastly, we remark that while we have here restricted our attention to cubic irrationalities, our proof can easily be modified so that, by employing McCurley [5, Theorem 1.2], we are able to obtain results similar to our Theorem for any function of the form $(1 - x)^{m/n}$, where m and n are coprime integers with $1 \leqslant m < n$, $n \geqslant 10$ and n not "exceptional" as defined in [5].

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Preliminary Results. The hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$$ {}_2F_1(a,b;\ c;\ z) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)z^n}{c(c+1)\cdots(c+n-1)n!}. $$

When c and a are negative integers with $c < a$, the coefficients of z^n for $n > |a|$ are understood to be zero. For r a positive integer we define $X_r(z)$, $Y_r(z)$ and $R_r(z)$ by

$$ (9) \qquad X_r(z) = \frac{(r+1)\cdots(2r)}{(2/3)(5/3)\cdots(r-1/3)}\ {}_2F_1(-r, -r-1/3;\ -2r;\ 1-z), $$

$$ (10) \qquad Y_r(z) = z^r X_r(z^{-1}), $$

$$ (11) \qquad R_r(z) = \frac{(1/3)(4/3)\cdots(r+1/3)}{(r+1)(r+2)\cdots(2r+1)}\ {}_2F_1(r+2/3, r+1;\ 2r+2;\ 1-z). $$

We shall employ the following Lemmas:

LEMMA 1. *Let r be a positive integer. Then for any real number z with $0 < z < 1$,*

$$ (12) \qquad z^{1/3}X_r(z) - Y_r(z) = (z-1)^{2r+1}R_r(z). $$

Proof. We obtain (12) from (4.2) of [2] upon noting that with $\nu = 1/3$, (9) agrees with $X_r(z)$ in (4.4) of [2], (10) agrees with $Y_r(z)$ in (4.1) of [2] and (11) agrees with (4.3) of [2].

LEMMA 2. *Let r be a positive integer, and define Δ_r to be the smallest positive integer such that $\Delta_r X_r(z)$ is a polynomial with integer coefficients. Then $3 \nmid \Delta_r$.*

Further, let a, b be integers with $0 < b < a$, and suppose d is as defined in (3). *Define d_0 by*

$$ d_0 = \begin{cases} 3/2 + \log r/\log 3 & \text{if } d = 3/2, \\ 0 & \text{otherwise}. \end{cases} $$

Then $\Delta_r a^r 3^{d_0 - dr} X_r(b/a)$ and $\Delta_r a^r 3^{d_0 - dr} Y_r(b/a)$ are integers.

Proof. From (4.1) of [2], with $\nu = 1/3$ we have

$$ (13) \qquad X_r(z) = \sum_{l=0}^{r} \binom{r}{l} \frac{(3r+1)(3r+4)\cdots(3(r-l+1)+1)}{2\cdot 5\cdots(3l-1)} z^{r-l}, $$

hence $3 \nmid \Delta_r$.

Our proof of the second half of Lemma 2 is based on the proof of Proposition 5.1 of [2].

We first note that if $3 \nmid (a - b)$, the result follows from the definition of Δ_r together with the observation that $X_r(z)$ and $Y_r(z)$ are both polynomials in z of degree r.

If $d \geqslant 1$, we write $a - b = 3^h g$, where $\gcd(3, g) = 1$. It follows from (9) that

$$\Delta_r X_r(b/a) = \frac{\Delta_r \cdot r!}{(2/3)(5/3)\cdots(r-1/3)}\binom{2r}{r} \times {}_2F_1(-r, -r-1/3; -2r; 3^h g/a)$$

$$(14) \qquad = \frac{\Delta_r 3^r r!}{2 \cdot 5 \cdots (3r-1)} \times \sum_{i=0}^{r}\binom{2r-i}{r}\left(\prod_{k=r-i+1}^{r}(3k+1)\right)(i!)^{-1}(-g/a)^i 3^{(h-1)i}.$$

If $d = 1$, we observe that $h = 1$ and that

$$r! \sum_{i=0}^{r}\binom{2r-i}{r}\left(\prod_{k=r-i+1}^{r}(3k+1)\right)(i!)^{-1}x^i$$

is a polynomial of degree r with integer coefficients. Hence, since $3 \nmid \Delta_r$, $3^{-r}a^r\Delta_r X_r(b/a)$ is an integer.

If $d = 3/2$, we note that $h \geqslant 2$ and apply Lemma 4.1 of [2] with $n = 3$, $s = 1$ and see that

$$\sum_{i=0}^{r}\binom{2r-i}{r}\left(\prod_{k=r-i+1}^{r}(3k+1)\right)(i!)^{-1}3^{[i/2]}x^i$$

is a polynomial of degree r with integer coefficients. Hence the sum on the right side of (14) is a polynomial in $(-g/a)$ with integer coefficients. Thus, since $3 \nmid \Delta_r$, and since the exponent to which 3 divides $r!$ is given by

$$[r/3] + [r/9] + [r/27] + \cdots \geqslant \frac{r}{2} - \left(\frac{\log r}{\log 3} + \frac{3}{2}\right),$$

we see that $3^{d_0 - d r}a^r\Delta_r X_r(b/a)$ is an integer.

We conclude the proof by noting that the above argument shows that $b^r\Delta_r 3^{d_0 - dr}X_r(a/b)$ is an integer, and hence that

$$a^r\Delta_r 3^{d_0-dr}Y_r(b/a) = \Delta_r 3^{d_0-dr}a^r\left(\frac{b}{a}\right)^r X_r(a/b)$$

is an integer.

LEMMA 3. *Suppose r is a positive integer. Then*

$$(15) \qquad \Delta_r \left| \frac{2 \cdot 5 \cdots (3r-1)}{r!} 3^{[r/2]}\right..$$

Further, if $\Delta_{r,l}$ denotes the contribution to Δ_r of all primes $p > (3r)^{1/2}$, then

$$(16) \qquad \Delta_{r,l} < \exp\left\{r \sum_{A \geqslant 0} \sum{}' \log p\right\},$$

where the inner sum is taken over all primes $p \equiv 2 \bmod 3$ satisfying

$$r/(A+1/3) \geqslant p > r/(A+2/3).$$

Proof. We verify (15) by noting that from Lemma 4.1 of [2] with $n = 3$, $a = 1$, $s = -1$, $2 \cdot 5 \cdots (3r-1)3^{[r/2]}(r!)^{-1}$ is an integer, and moreover, from Lemma 4.2 of [2] with $n = 3$, $s = 1$, $2 \cdot 5 \cdots (3r-1)3^{[r/2]}(r!)^{-1}X_r(z)$ is a polynomial with integer coefficients.

To verify (16) we turn to Theorem 4.3 of [2]. In the proof of this theorem, Chudnovsky considers $\Delta_r^{(2)}$, the contribution to Δ_r of all primes $p > 3r^{1/2}$. Putting $n = 3$, and $s = 1$ in [2], we see that if $p|\Delta_r^{(2)}$ we must have $p \equiv 2 \bmod 3$, as is clear from the remarks made following (4.22). Moreover, the remarks made just prior to (4.20) show that $p^2 \nmid \Delta_r^{(2)}$; and from (4.22) we see that for some integer A, we must have $r/(A + 1/3) \geqslant p > r/(A + 2/3)$. This suffices to show (16) with $\Delta_r^{(2)}$ in place of $\Delta_{r,l}$. Our result follows upon observing that Chudnovsky's arguments are not affected by considering primes in the extended range $p > (3r)^{1/2}$.

LEMMA 4. *Let r be a positive integer. If* $\pi(r)$ *denotes the number of primes less than* r, *we have*

$$\pi(r) < (1.001)r(\log r)^{-1}. \tag{17}$$

Further, if we put $\theta(r,3,2) = \sum_{p \equiv 2 \bmod 3;\, p \leqslant r} \log p$, *we have*

$$(.4075)r < \theta(r,3,2) < (.5094)r \quad \textit{for } r \geqslant 47 \tag{18}$$

and

$$(.4539)r < \theta(r,3,2) < (.5094)r \quad \textit{for } r \geqslant 233. \tag{19}$$

Proof. We obtain (17) from (5.1) of [7]. The right-hand inequalities of (18) and (19) follow from Theorem 5.1 of [6], while the left-hand inequalities follow from Theorem 5.3 of [6].

LEMMA 5. *Let a, b and r be positive integers with* $0 < b < a$. *Then if* $X_r(z)$, $Y_r(z)$ *and* $R_r(z)$ *are given by* (9), (10) *and* (11), *respectively,*

$$X_r(b/a)Y_{r+1}(b/a) \neq X_{r+1}(b/a)Y_r(b/a), \tag{20}$$

$$R_r(b/a) = \frac{(1/3)(4/3)\cdots(r+1/3)}{r!} \times \int_0^1 t^r(1-t)^r(1-t(a-b)/a)^{-r-2/3}\,dt. \tag{21}$$

Proof. The proof of (20) is standard; see for instance the proof of (16) in [1]. We obtain (21) from (11) and (1.6.6) of [9].

Technical Lemmas. In this section we establish several estimates which we shall employ in the proof of the Theorem.

LEMMA 6. *Let r be an integer with* $r \geqslant 300$. *Then*

$$\Delta_r < \exp\{(1.4266)r\}. \tag{22}$$

Proof. The proof is divided into two parts. First, we estimate the contribution to Δ_r of those primes $p \leqslant (3r)^{1/2}$. We then estimate the contribution of those primes $p > (3r)^{1/2}$.

To obtain the first estimate, we begin by recalling from Lemma 2 that $3 \nmid \Delta_r$.

We now proceed as Chudnovsky does in obtaining his upper bound for $\Delta_r^{(1)}$ in the proof of Theorem 4.3 of [2]. First, we note that from (15), if $p \leqslant (3r)^{1/2}$, p can contribute to Δ_r at most

$$p^{[\log 3r/\log p]} \leqslant 3r.$$

Hence, if we denote the contribution to Δ_r of those primes $p \leqslant (3r)^{1/2}$ by $\Delta_{r,s}$, we have

$$\Delta_{r,s} \leqslant (3r)^{\pi((3r)^{1/2})}.$$

Thus, from (17),

$$\Delta_{r,s} < \exp\{2.002(3r)^{1/2}\},$$

and since $r \geqslant 300$, we have

$$\Delta_{r,s} < \exp\{.2002r\}. \tag{23}$$

Denote, as in Lemma 3, the contribution to Δ_r of all primes $p > (3r)^{1/2}$ by $\Delta_{r,l}$. We have from (16) that

$$\begin{aligned}\Delta_{r,l} &< \exp\left\{\sum_{A=0}^{\infty} \theta(r/(A+1/3),3,2) - \theta(r/(A+2/3),3,2)\right\}\\ &< \exp\left\{\sum_{A=0}^{5} (\theta(r/(A+1/3),3,2) - \theta(r/(A+2/3),3,2))\right.\\ &\qquad\qquad \left. + \theta(r/(6+1/3),3,2)\right\}.\end{aligned}$$

Hence, since $3r/2 > 233$, we have from (18) and (19) that

$$\begin{aligned}\Delta_{r,l} &\leqslant \exp\left\{(.5094)\left(\sum_{A=0}^{6} 3r/(3A+1)\right)\right.\\ &\qquad \left. -(.4539)(3r/2) - (.4075)\sum_{A=1}^{5} 3r/(3A+2)\right\}\\ &< \exp\{(1.2264)r\}.\end{aligned} \tag{24}$$

Finally, from (23) and (24),

$$\Delta_r = \Delta_{r,s}\Delta_{r,l} < \exp\{(1.4266)r\}.$$

LEMMA 7. *Let a, b and r be integers with $0 < b < a$ and $r \geqslant 300$. Let d be given by* (3), *and let d_0 and Δ_r be as defined in Lemma* 2. *Put*

$$q_r = \Delta_r a^r 3^{d_0 - dr} X_r(b/a); \qquad p_r = \Delta_r a^r 3^{d_0 - dr} Y_r(b/a). \tag{25}$$

Then p_r and q_r are integers with

$$0 < q_r < 3.434(8.591 \cdot 3^{-d}(a+b))^r. \tag{26}$$

Proof. From Lemma 2, p_r and q_r are both integers.

The proof we shall give of (26) is essentially the proof of Lemma 3 of [1]. We begin by noting that from (13) we have

$$\begin{aligned}a^r X_r(b/a) &= a^r \sum_{l=0}^{r} \binom{r}{l} \frac{(r+1/3)\cdots(r-l+4/3)}{(2/3)(5/3)\cdots(l-1/3)}\left(\frac{b}{a}\right)^{r-l}\\ &= \prod_{k=1}^{r}(k-1/3)^{-1} \sum_{l=0}^{r}\binom{r}{l} \prod_{k=r-l+1}^{r}(k+1/3)\prod_{k=l+1}^{r}(k-1/3)(a^l b^{r-l}).\end{aligned}$$

This, together with (25), gives the left-hand inequality of (26). Using the estimates

$$\prod_{k=r-l+1}^{r}(k+1/3)\prod_{k=l+1}^{r}(k-1/3) \leqslant \prod_{k=r-l+1}^{r}(k+1)\prod_{k=l+1}^{r}k = r!\binom{r+1}{l} \leqslant r!2^{r+1},$$

we have

$$\begin{aligned} a^r X_r(b/a) &\leqslant r!\left(\prod_{k=1}^{r}(k-1/3)\right)^{-1} 2^{r+1}\sum_{l=0}^{r}\binom{r}{l}a^l b^{r-l} \\ &\leqslant 2(r!)\left(\prod_{k=1}^{r}(k-1/3)\right)^{-1}(2(a+b))^r. \end{aligned} \tag{27}$$

Now

$$\begin{aligned} r!\left(\prod_{k=1}^{r}(k-1/3)\right)^{-1} &= \frac{3}{2}\prod_{k=2}^{r}\frac{3k}{3k-1} = \frac{3}{2}\exp\left\{\sum_{k=2}^{r}\log\left(1+\frac{1}{3k-1}\right)\right\} \\ &< \frac{3}{2}\exp\left\{\sum_{k=2}^{r}\frac{1}{3k-1}\right\} < \frac{3}{2}\exp\left\{\int_1^r \frac{1}{3x-1}\,dx\right\} \\ &= \frac{3}{2}\exp\left\{\frac{1}{3}\log(3r-1)-\frac{1}{3}\log 2\right\} < 1.717r^{1/3}. \end{aligned}$$

Since $r \geqslant 300$,

$$r!\left(\prod_{k=1}^{r}(k-1/3)\right)^{-1} < 1.717(1.0064)^r. \tag{28}$$

Further, since $r \geqslant 300$, $d_0 = 3/2 + \log r/\log 3 < (.02231)r$ and

$$3^{d_0-dr} \leqslant 3^{(.02231-d)r}. \tag{29}$$

The result follows from (22), (27), (28) and (29).

LEMMA 8. *Let a, b and r be integers with $0 < b < a$ and $r \geqslant 300$. Then,*

$$0 < \left|(b/a)^{1/3} - p_r/q_r\right| < \frac{(.4445)(a-b)}{(ab^2)^{1/3}q_r}\left\{\frac{4.296}{3^d}(a^{1/2}-b^{1/2})^2\right\}^r \tag{30}$$

and

$$p_r q_{r+1} \neq p_{r+1}q_r. \tag{31}$$

Proof. It is clear that (31) follows from (25) and (20). To verify (30), we first substitute $z = b/a$ in (12). Since from (26) $q_r \neq 0$, we have from (12), (21) and (25) that

$$\begin{aligned} \left|(b/a)^{1/3} - p_r/q_r\right| &= \frac{\Delta_r a^r 3^{d_0-dr}}{q_r}\left(1-\frac{b}{a}\right)^{2r+1}\frac{(1/3)(4/3)\cdots(r+1/3)}{r!} \\ &\quad\times\left|\int_0^1 t^r(1-t)^r\left(1-\frac{(a-b)}{a}t\right)^{-r-2/3}dt\right|. \end{aligned} \tag{32}$$

Now the left-hand inequality of (30) follows upon observing that the integrand on the right side of (32) is positive for $0 < t < 1$. To obtain the right-hand inequality of (30) we first note that $(1 - t(a-b)/a)^{-2/3} \leqslant (a/b)^{2/3}$ if $0 \leqslant t \leqslant 1$. Moreover, the function $t(1-t)(1-t(a-b)/a)^{-1}$ obtains a maximum value of $a(a^{1/2}+b^{1/2})^{-2}$ on the range $0 \leqslant t \leqslant 1$. Hence

$$(33)\qquad \left|\int_0^1 t^r(1-t)^r(1-t(a-b)/a)^{-r-2/3}\,dt\right| \leqslant (a/b)^{2/3}\left(a(a^{1/2}+b^{1/2})^{-2}\right)^r.$$

Further, in the same way as we obtained (28), we find that

$$(34)\qquad \begin{aligned} \frac{(1/3)(4/3)\cdots(r+1/3)}{r!} &= 4/9\prod_{k=2}^{r}\frac{k+1/3}{k} \\ &< 4/9\exp\left\{1/3\int_1^r\frac{dx}{x}\right\} < 4/9(1.0064)^r. \end{aligned}$$

This, together with (29), (32), (33), and (22), implies (30).

Proof of Theorem. Let λ be given by (4) and let p, q be integers with q satisfying

$$(35)\qquad \lambda^r \leqslant \frac{(1.076)(a^{1/2}+b^{1/2})q}{(ab^2)^{1/3}(a^{1/2}-b^{1/2})} < \lambda^{r+1}$$

for some integer $r \geqslant 300$. Choose $R = r$ or $r+1$ so that $pq_R \neq p_Rq$, as is possible in light of (31). Further, note that from (5)

$$(36)\qquad \lambda^{\kappa-1} = (8.591)3^{-d}(a+b).$$

This, together with (26) and the left-hand inequality of (35), yields

$$(37)\qquad \begin{aligned} q_R &< 3.434\left((8.591)3^{-d}(a+b)\right)^{r+1} \leqslant 3.434\lambda^{(\kappa-1)(r+1)} \\ &\leqslant 3.434\left(\frac{1.076(a^{1/2}+b^{1/2})\lambda}{(ab^2)^{1/3}(a^{1/2}-b^{1/2})}\right)^{\kappa-1} q^{\kappa-1}. \end{aligned}$$

From the right side of (35), together with (4) and (30), we have

$$\begin{aligned} 0 < \left|(b/a)^{1/3} - p_R/q_R\right| &< \frac{(.4445)(a-b)}{(ab^2)^{1/3}q_R\lambda^r} \\ &< (.4131)\frac{(a-b)\lambda(a^{1/2}-b^{1/2})}{(a^{1/2}+b^{1/2})qq_R} < \frac{(.0962)3^d}{qq_R}. \end{aligned}$$

Since $d \leqslant 3/2$, we have

$$(38)\qquad \left|(b/a)^{1/3} - p_R/q_R\right| < \frac{1}{2qq_R}.$$

From (37) and (38) we have

$$\begin{aligned} \left|(b/a)^{1/3} - p/q\right| &\geqslant |p/q - p_R/q_R| - \left|(b/a)^{1/3} - p_R/q_R\right| \\ &\geqslant \frac{1}{qq_R} - \frac{1}{2qq_R} = \frac{1}{2qq_R} \\ &\geqslant \frac{1}{q^\kappa}\left\{.1456\left(\frac{(ab^2)^{1/3}(a^{1/2}-b^{1/2})}{1.076(a^{1/2}+b^{1/2})\lambda}\right)^{\kappa-1}\right\}. \end{aligned}$$

Hence, from (36) and (6),

$$\left|(b/a)^{1/3} - p/q\right| > \frac{1}{q^{\kappa}}\left\{1.69 \times 10^{-2}(a+b)^{-1}\left(\frac{(ab^2)^{1/3}(a^{1/2} - b^{1/2})}{1.076(a^{1/2} + b^{1/2})}\right)^{\kappa - 1}\right\}$$
$$= \frac{c}{q^{\kappa}}.$$

Proof of Corollary. To prove the corollary for $\alpha = 2^{1/3}$, we apply the Theorem with $a = 128$, $b = 125$ to rationals of the form $5q/(4p)$ to obtain

$$\frac{5}{4}\left|2^{-1/3} - q/p\right| > \frac{3.4 \times 10^{-5}}{(4p)^{2.795}} \quad \text{when } 4p > 10^{478}.$$

Since it suffices to consider q/p in the range $1 < p/q < 1.3$, we have

$$\left|2^{1/3} - p/q\right| > \frac{3.4 \times 10^{-7}}{q^{2.795}} \quad \text{when } q > 10^{478}. \tag{39}$$

To remove the restriction on q_0, we utilize the first 2000 terms in the continued fraction expansion for $2^{1/3}$. We begin by supposing that q_i is the denominator of the ith convergent to $2^{1/3}$, and that for some integers p, q with $q_i \leqslant q < q_{i+1}$,

$$\left|2^{1/3} - p/q\right| < \frac{3.4 \times 10^{-7}}{q^{2.795}}. \tag{40}$$

Now if a_i is the ith partial quotient, we have the following well-known identities (the first follows from Theorem 9.6 of [4]; for the second see Theorem 182 of [3]):

$$\frac{1}{(a_{i+2} + 2)q_{i+1}^2} < \left|2^{1/3} - \frac{p_{i+1}}{q_{i+1}}\right|$$

and

$$\left|q_{i+1}2^{1/3} - p_{i+1}\right| < \left|q2^{1/3} - p\right|.$$

These, together with (40), imply

$$\frac{1}{(a_{i+1} + 2)q_{i+1}} < \left|q2^{1/3} - p\right| < \frac{3.4 \times 10^{-7}}{q^{1.795}} < \frac{3.4 \times 10^{-7}}{q_i^{1.795}}.$$

Hence,

$$\frac{2.9 \times 10^{6}}{(a_{i+2} + 2)q_{i+1}} q_i^{1.795} < 1. \tag{41}$$

Employing the identity $q_{i+1} = a_{i+1}q_i + q_{i-1}$, we have

$$\frac{q_i}{q_{i+1}} = \frac{q_i}{a_{i+1}q_i + q_{i-1}} > \frac{1}{(a_{i+1} + 1)},$$

and hence from (41),

$$\frac{2.9 \times 10^{6} q_i^{.795}}{(a_{i+2} + 2)(a_{i+1} + 1)} < 1.$$

This, together with the observations that $q_i \geqslant \prod_{j=0}^{i} a_j$ and $\prod_{j=0}^{2000} a_j > 10^{478}$, enables us to readily verify that for all integers p, q

$$\left|2^{1/3} - p/q\right| > \frac{3.4 \times 10^{-7}}{q^{2.795}} \quad \text{when } 0 < q \leqslant 10^{478}. \tag{42}$$

Hence, from (39) and (42), we have Corollary 1 for $\alpha = 2^{1/3}$.

The rest of the Corollary is proved in a similar manner. We conclude by listing, for each value of α, the values for a and b with which we obtain the result. We also list the values obtained for q_0.

α	a	b	q_0
$6^{1/3}$	467^3	$6 \cdot 257^3$	10^{1976}
$10^{1/3}$	$5 \cdot 13^3$	$2^2 \cdot 14^3$	10^{846}
$15^{1/3}$	5^2	$3 \cdot 2^3$	10^{408}
$17^{1/3}$	18^3	$17 \cdot 7^3$	10^{1117}
$19^{1/3}$	$19 \cdot 3^3$	8^3	10^{802}
$20^{1/3}$	$20 \cdot 7^3$	19^3	10^{1141}
$22^{1/3}$	$11 \cdot 5^3$	$2^2 \cdot 7^3$	10^{789}
$26^{1/3}$	3^3	26	10^{417}
$28^{1/3}$	28	3^3	10^{422}
$37^{1/3}$	10^3	$37 \cdot 3^3$	10^{890}
$39^{1/3}$	$39^2 \cdot 2^3$	23^3	10^{1216}
$42^{1/3}$	7^2	$6 \cdot 2^3$	10^{498}
$43^{1/3}$	$43 \cdot 2^3$	7^3	10^{751}

The continued fraction expansions for the above values of α are available from the author upon request.

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