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Some Asymptotic Properties of Padé Approximants to e^{-x}

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Abstract. The method of matched asymptotic expansions is used to analyze the asymptotic behavior of the real zeros of, and error incurred by, Padé approximants to e^{-x}. These approximants are of interest because of their application in solving systems of ordinary differential equations arising from mathematical models of physical processes, for example, the heat equation.

1. Introduction. There is considerable interest in properties of Padé approximants to the exponential, not least because of their application in methods for solving those systems of ordinary differential equations which arise in mathematical models of physical processes. In this paper, some asymptotic properties of Padé approximants to e^{-x} are found by using the method of matched asymptotic expansions. First, we analyze the behavior of the real zero of odd-degree denominators along any straight line path through the Padé table; second, a result due to Saff, Varga and Ni [6] concerning the error in the uniform norm on $[0, \infty)$ incurred by these approximants is reconsidered.

2. The Real Poles. It is well known that the denominator of the (ν, n)th entry of the Padé table for e^{-x} is

$$\theta_n\left(\frac{x}{2}; \nu - n + 2\right),$$

where $\theta_n(x; a)$ is a generalized Bessel polynomial given by

$$\theta_n(x; a) = \frac{n!}{2^n\Gamma(n + a - 1)} \sum_{j=0}^{n} \frac{\Gamma(2n + a - j - 1)(2x)^j}{(n - j)!j!}.$$

Recently, Grosswald [4] and de Bruin, Saff and Varga [2] have considered the asymptotic behavior as $n \to \infty$, through odd values, of $\beta_n(a)$, the unique (negative) real zero of $\theta_n(x; a)$. They confined themselves to the case where the value of a is fixed, so that the equivalent path in the Padé table is a diagonal; in particular, if $a = 2$, it is the main diagonal, and the polynomials are ordinary Bessel polynomials. Here, we consider more generally the asymptotic behavior of $2\beta_n(\nu - n + 2)$, i.e., the real zero of the denominator of the (ν, n) entry, as ν and/or $n \to \infty$ along any straight line path in the Padé table. We start by representing the denominator as an integral.

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3. An Integral Expression for the Denominator. We rely on an extension of an integral representation of $\theta_n(x;2)$ due to Eweida [3]. Using the notation of [6], let the Padé denominator be

$$P_{\nu,n}(x),$$

and suppose the real zero is $\gamma_{\nu,n} := 2\beta_n(\nu - n + 2)$. $P_{\nu,n}(-x)$ is given by

$$P_{\nu,n}(-x) = \frac{(-1)^n x^{\nu+n+1} e^{-x/2}}{2^{\nu+2n+1}\Gamma(\nu+1)} \int_{-1}^{\infty} e^{-xs/2}(s+1)^{\nu}(s-1)^n\, ds, \quad (\operatorname{Re} x > 0),$$

so that if n is odd, $-x = \gamma_{\nu,n}$ when $I_1 = I_2$, where

$$I_i := \int_{\alpha_i}^{\beta_i} e^{-xs/2}(s+1)^{\nu}|1-s|^n\, ds, \qquad i = 1,2,$$

with $\alpha_1 := -1$, $\alpha_2 := 1$, $\beta_1 := 1$, $\beta_2 := \infty$.

We shall find the first few terms in the asymptotic expansion of $\gamma_{\nu,n}$ by matching the asymptotic expansions of I_1 and I_2 obtained by Laplace-type methods. We distinguish three cases:

(i) $\nu = fn + (a-2)$, f and a fixed, $0 < f < \infty$, a integral, $n \to \infty$. If $f = 1$, the direction of travel is diagonal, the case considered in [4] and [2].

(ii) ν fixed, $n \to \infty$ (a Padé column; if $\nu = 0$, the column contains inverses of partial sums of the Taylor series of e^x about $x = 0$).

(iii) n fixed, $n \geqslant 1$, $\nu \to \infty$ (a row of the table).

4. The Form of the Asymptotic Expansion of $\gamma_{\nu,n}$. From [2, pp. 4,6], we have

$$-n - \nu - \tfrac{4}{3} < \gamma_{\nu,n} < -\mu(n+\nu),$$

where $\mu \simeq 0.278\,465$ is the unique positive root of the equation $\mu e^{1+\mu} = 1$ and $n \geqslant 1$, $\nu \geqslant 0$. In cases (i) and (ii) we therefore assume that $\gamma_{\nu n} = -(\lambda_i n + \phi_i(n))$, where $\phi_i(n) = o(n)$ as $n \to \infty$, $i = 1,2$, and in case (iii) that $\gamma_{\nu n} = -(\lambda_3 \nu + \phi_3(\nu))$, where $\phi_3(\nu) = o(\nu)$ as $\nu \to \infty$. In each case, $\lambda_i > 0$. It is interesting that under these assumptions the Laplace-type arguments that establish λ and ϕ are different in all three cases.

5. Case (i): A Straight Line Across the Padé Table. In I_i, let $\nu = fn + (a-2)$, f and a fixed, $0 < f < \infty$, a integral and let $x = -\gamma_{\nu n} = \lambda_1 n + \phi_1(n)$, where $\lambda_1 > 0$ and $\phi_1(n) = o(n)$ as $n \to \infty$. Then

$$I_i = \int_{\alpha_i}^{\beta_i} e^{nh(s)} e^{-\phi_1(n)s/2}(s+1)^{a-2}\, ds,$$

where

$$h(s) = -\frac{\lambda_1 s}{2} + f\log(s+1) + \log|1-s|.$$

The stationary points of $h(s)$ occur at

$$s_2, s_1 = \frac{(1+f) \pm \sqrt{(1+f)^2 + \lambda_1(\lambda_1 + 2 - 2f)}}{\lambda_1},$$

and it is easy to show that $-1 < s_1 < 1$ and $s_2 > 1$. The application of Laplace's method (see, e.g., [1, pp. 60–65]) gives

$$I_i \sim \frac{e^{nh(s_i)} e^{-\phi_1(n)s_i/2}(s_i+1)^{a-2}\sqrt{2\pi}}{\left[-nh^{(2)}(s_i)\right]^{1/2}}$$

as $n \to \infty$, $i = 1, 2$. Since $s_1 \neq s_2$, I_1 matches I_2 only if $h(s_1) = h(s_2)$, and if $\phi_1(n)$ has the form $\mu_1 + o(1)$, so that

$$\frac{e^{-\mu_1 s_1/2}(s_1+1)^{a-2}}{\left[-h^{(2)}(s_1)\right]^{1/2}} = \frac{e^{-\mu_1 s_2/2}(s_2+1)^{a-2}}{\left[-h^{(2)}(s_2)\right]^{1/2}}.$$

Rearranging these equations, we obtain

THEOREM 5.1. *Suppose f and a are fixed, $0 < f < \infty$, a integral and $\nu = fn + (a-2)$, and that $\gamma_{\nu n} = -\lambda_1 n + o(n)$ as $n \to \infty$, where λ_1 is a constant.*

Then $\gamma_{\nu n} = -\lambda_1 n - \mu_1 + o(1)$ as $n \to \infty$, where μ_1 is a constant and λ_1 and μ_1 are given by the equations

$$\lambda_1 = \frac{2}{(s_2 - s_1)}\left\{ f \log\left(\frac{1+s_2}{1+s_1}\right) + \log\left(\frac{s_2-1}{1-s_1}\right)\right\}$$

and

$$\mu_1 = \frac{2}{(s_2 - s_1)}\left\{ (a-2)\log\left(\frac{1+s_2}{1+s_1}\right) + \log\left(\frac{s_2^2-1}{1-s_1^2}\right) + \frac{1}{2}\log\left(\frac{f(s_1-1)^2+(s_1+1)^2}{f(s_2-1)^2+(s_2+1)^2}\right)\right\},$$

where

$$s_2, s_1 = \frac{(1+f) \pm \sqrt{(1+f)^2 + \lambda_1(\lambda_1 + 2(1-f))}}{\lambda_1}.$$

COROLLARY. *Under the same conditions,*

$$\beta_n((f-1)n + a) = -\left(\frac{\lambda_1 n}{2} + \frac{\mu_1}{2}\right) + o(1) \quad \text{as } n \to \infty.$$

6. An Extension. In the special case $f = 1$, considered in [4] and [2], the equations of Theorem 5.1 reduce to the more tractable

$$\frac{\lambda_1}{2} e^{[1+\lambda_1^2/4]^{1/2}} = 1 + \left[1 + \frac{\lambda_1^2}{4}\right]^{1/2},$$

i.e., $\lambda_1 \simeq 1.325\,487$, and

$$\mu_1 = \frac{\lambda_1}{2} + (a-2)\frac{\lambda_1}{2}\left[1 + \frac{\log\left(\lambda_1/2 + \left[1+\lambda_1^2/4\right]^{1/2}\right)}{\left[1+\lambda_1^2/4\right]^{1/2}}\right],$$

i.e., $\mu_1 \simeq (a-2)\,1.006\,290 + 0.662\,744$. In this case (i.e., when $\nu = n + (a-2)$), we now find the next term in the asymptotic expansion of $\gamma_{\nu n}$.

Put $-\gamma_{\nu n} = \lambda_1 n + \mu_1 + \psi_1(n)$, where λ_1 and μ_1 are given above and $\psi_1(n) = o(1)$ as $n \to \infty$. Then

$$I_i = \int_{\alpha_i}^{\beta_i} e^{nh(s)} e^{-[\mu_1 + \psi_1(n)]s/2}(s+1)^{a-2}\,ds,$$

with $h(s) = -\lambda_1 s/2 + \log|1 - s^2|$. $h(s)$ has its stationary points at

$$s_1 = \frac{2 - \sqrt{4 + \lambda_1^2}}{\lambda_1} \in (-1, 1) \quad \text{and} \quad s_2 = \frac{2 + \sqrt{4 + \lambda_1^2}}{\lambda_1} \in (1, \infty).$$

Let $s - s_i = \sigma$. Then

$$\begin{aligned} &e^{-[\mu_1 + \psi_1(n)]s/2}(s+1)^{a-2} \\ &\quad = e^{-\mu s_i/2}(s_i+1)^{a-2}\left[1 - \psi_1(n)\frac{s_i}{2} + \left\{\frac{a-2}{s_i+1} - \frac{\mu_1}{2}\right\}\sigma \right. \\ &\qquad\qquad \left. + \left\{\frac{\mu_1^2}{8} - \frac{\mu_1(a-2)}{2(s_i+1)} + \frac{(a-2)(a-3)}{2(s_i+1)^2}\right\}\sigma^2 + \cdots\right], \end{aligned}$$

and

$$\begin{aligned} e^{nh(s)} &= e^{nh(s_i)}e^{\frac{1}{2}nh^{(2)}(s_i)\sigma^2} \\ &\quad \cdot\left[1 + \frac{1}{6}h^{(3)}(s_i)\tau + \frac{1}{24}h^{(4)}(s_i)\tau\sigma + \frac{1}{72}\left[h^{(3)}(s_i)\right]^2\tau^2 + \cdots\right], \end{aligned}$$

where $\tau = n\sigma^3$. So the integrand, keeping only those terms which will survive under the Laplace process, is

$$e^{-\mu s_i/2}(s_i+1)^{a-2}e^{nh(s_i)}e^{\frac{1}{2}nh^{(2)}(s_i)\sigma^2}S_i,$$

where

$$\begin{aligned} S_i &= 1 - \psi_1(n)\frac{s_i}{2} + \left[\frac{\mu_1^2}{8} - \frac{\mu_1(a-2)}{2(s_i+1)} + \frac{(a-2)(a-3)}{2(s_i+1)^2}\right]\sigma^2 \\ &\quad + \left[\frac{1}{6}h^{(3)}(s_i)\left[\frac{a-2}{s_i+1} - \frac{\mu_1}{2}\right] + \frac{1}{24}h^{(4)}(s_i)\right]\sigma\tau + \frac{1}{72}\left[h^{(3)}(s_i)\right]^2\tau^2 + \cdots, \end{aligned}$$

and therefore, as $n \to \infty$,

$$I_i \sim \frac{e^{-\mu s_i/2}(s_i+1)^{a-2}e^{nh(s_i)}\Gamma(\frac{1}{2})T_i}{n^{1/2}\left[-\frac{1}{2}h^{(2)}(s_i)\right]^{1/2}},$$

where

$$\begin{aligned} T_i &= 1 - \tfrac{1}{2}\psi_1(n)s_i \\ &\quad + \frac{1}{n}\left\{\frac{\left[\mu_1^2/8 - \mu_1(a-2)/2(s_i+1) + (a-2)(a-3)/2(s_i+1)^2\right]}{2\left[-\frac{1}{2}h^{(2)}(s_i)\right]}\right. \\ &\qquad + \frac{\left[\frac{1}{2}h^{(3)}(s_i)((a-2)/(s_i+1) - \mu_1/2) + \frac{1}{8}h^{(4)}(s_i)\right]}{4\left[-\frac{1}{2}h^{(2)}(s_i)\right]^2} \\ &\qquad\qquad \left. + \frac{5}{192}\frac{\left[h^{(3)}(s_i)\right]^2}{\left[-\frac{1}{2}h^{(2)}(s_i)\right]^3}\right\} + \cdots. \end{aligned}$$

Since $s_1 \neq s_2$, T_1 and T_2 match only if $\psi_1(n) = \nu_1/n + o(1/n)$, and then, after some algebra and using the relations

$$1 - s_i^2 = -4s_i/\lambda_1 \quad \text{and} \quad s_1 s_2 = -1,$$

we obtain

$$\nu_1 = \frac{\lambda_1}{2(1+\lambda_1^2/4)}\left\{\left(\frac{\mu_1}{\lambda_1}\right)^2 - \left(\frac{\mu_1}{\lambda_1}\right) + \frac{5}{12(1+\lambda_1^2/4)} + (a-2)\left[\left(\frac{\mu_1}{\lambda_1}\right)\left(\frac{\lambda_1}{2}\right)\left(\frac{\lambda_1}{2}+1\right) + \frac{(1-\lambda_1)}{2} - \frac{(a-3)}{2}\right]\right\}.$$

Hence we have proved

THEOREM 6.1. *Suppose a is a fixed integer, and $\nu = n + (a-2)$, and suppose that $\gamma_{\nu n} = -\lambda_1 n + o(n)$ as $n \to \infty$, where λ_1 is a constant. Then $\gamma_{\nu n} = -\lambda_1 n - \mu_1 - \nu_1/n + o(1/n)$ as $n \to \infty$, where μ_1 and ν_1 are constants, and λ_1, μ_1 and ν_1 are given by the equations*

$$\frac{\lambda_1}{2} e^{[1+\lambda_1^2/4]^{1/2}} = 1 + \left[1 + \frac{\lambda_1^2}{4}\right]^{1/2},$$

i.e., $\lambda_1 \simeq 1.325\,487$,

$$\mu_1 = \frac{\lambda_1}{2} + (a-2)\frac{\lambda_1}{2}\left[1 + \frac{\log\left(\lambda_1/2 + \left[1+\lambda_1^2/4\right]^{1/2}\right)}{\left[1+\lambda_1^2/4\right]^{1/2}}\right],$$

i.e., $\mu_1 \simeq (a-2)\,1.006\,290 + 0.662\,744$, *and*

$$\nu_1 = \frac{\lambda_1}{2(1+\lambda_1^2/4)}\left\{\left(\frac{\mu_1}{\lambda_1}\right)^2 - \left(\frac{\mu_1}{\lambda_1}\right) + \frac{5}{12(1+\lambda_1^2/4)} + (a-2)\left[\left(\frac{\mu_1}{\lambda_1}\right)\frac{\lambda_1}{2}\left(\frac{\lambda_1}{2}+1\right) + \frac{(1-\lambda_1)}{2} - \frac{(a-3)}{2}\right]\right\}.$$

Of special interest is the case $a = 2$, where the polynomials are related to ordinary Bessel polynomials and the path is the main diagonal of the Padé table. In this case

$$\lambda_1 \simeq 1.325\,487, \qquad \mu_1 = \frac{\lambda_1}{2} \simeq 0.662\,744,$$

and

$$\nu_1 = \frac{\lambda_1}{2(1+\lambda_1^2/4)}\left\{-\frac{1}{4} + \frac{5}{12(1+\lambda_1^2/4)}\right\} \simeq 0.018\,192.$$

Let

$$-\tilde{\gamma}_{nn} = \lambda_1\left(n + \tfrac{1}{2}\right) + \frac{\lambda_1\left\{-\frac{1}{4} + 5/12(1+\lambda_1^2/4)\right\}}{2n(1+\lambda_1^2/4)}.$$

In Table I, the entries in the first two columns are taken from [2]; the last column shows the effect of the inclusion of the extra term.

TABLE I

n	$-\gamma_{nn}$	$\lambda_1(n+\frac{1}{2})$	$-\tilde{\gamma}_{nn}$
1	2.000 000	1.988 230	2.006 422
3	4.644 370	4.639 204	4.645 268
5	7.293 478	7.291 018	7.293 816
7	9.943 574	9.941 152	9.943 650
9	12.594 038	12.592 124	12.594 146
11	15.244 680	15.243 098	15.244 750
13	17.895 420	17.894 072	17.895 472
15	20.546 220	20.545 046	20.546 258

7. Case (ii): Padé Columns. Let ν be fixed and $x = \lambda_2 n + \phi_2(n)$, where $\phi_2(n) = o(n)$ as $n \to \infty$. Then

$$I_i = \int_{\alpha_i}^{\beta_i} e^{nh(s)} e^{-\phi_2(n)s/2} (s+1)^{\nu}\, ds,$$

where $h(s) = -\lambda_2 s/2 + \log|1-s|$. The stationary point of $h(s)$ occurs at $s_2 = 1 + 2/\lambda_2 > 1$ and $h'(s) < 0$ if $s < 1$. Hence, if s_1 is the maximum value of $h(s)$ in $[-1, 1]$, then $s_1 = -1$ and

$$I_1 \sim \frac{e^{n[\lambda_2/2 + \log 2]} e^{\phi_2(n)} \nu! 2^{\nu+1}}{(\lambda_2+1)^{\nu+1} n^{\nu+1}},$$

while

$$I_2 \sim \frac{e^{-n[\lambda_2/2 + 1 + \log(\lambda_2/2)]} e^{-\phi_2(n)(1+2/\lambda_2)/2} 2^{\nu+1} [1 + 1/\lambda_2]^{\nu} \sqrt{2\pi}}{\lambda_2 \sqrt{n}}$$

as $n \to \infty$.

In this case it is clear that if $\phi_2(n) = O(1)$, we cannot match I_1 and I_2. Let $\phi_2(n) = \psi(n) + \nu_2 + o(1)$, ν_2 being a constant. Then the powers of n in I_1 and I_2 agree only if

$$\psi(n) = \frac{[\nu + \frac{1}{2}] \log n}{1 + 1/\lambda_2} + O(1).$$

In addition, matching the exponentials, we have

$$\lambda_2 e^{1+\lambda_2} = 1,$$

and, matching the constants,

$$\nu_2 = \frac{1}{[1 + 1/\lambda_2]} \log\left[\frac{[1 + 1/\lambda_2]^{2\nu+1} \sqrt{2\pi}\, \lambda_2^{\nu}}{\nu!}\right],$$

resulting in

THEOREM 7.1. *Suppose that ν is fixed and that $\gamma_{\nu n} = -\lambda_2 n + o(n)$ as $n \to \infty$, where λ_2 is a constant. Then $\gamma_{\nu n} = -\lambda_2 n - \mu_2 \log n - \nu_2 + o(1)$ as $n \to \infty$, where μ_2 and ν_2 are constants, and λ_2, μ_2, ν_2 are given by*

$$\lambda_2 e^{1+\lambda_2} = 1, \qquad \mu_2 = [\nu + \tfrac{1}{2}]/[1 + 1/\lambda_2],$$

and

$$\nu_2 = \frac{1}{[1 + 1/\lambda_2]} \log\left[\frac{[1 + 1/\lambda_2]^{2\nu+1}\sqrt{2\pi}\,\lambda_2^{\nu}}{\nu!}\right].$$

COROLLARY 1. *Under the same conditions,*

$$\beta_n(\nu - n + 2) = -\left[\frac{\lambda_2 n}{2} + \frac{\mu_2}{2}\log n + \frac{\nu_2}{2}\right] + o(1) \quad \text{as } n \to \infty.$$

COROLLARY 2. *Suppose*

$$E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

(*the denominators in column number* 0 *of the Padé table*), *and, if* n *is odd, let* δ_n *be the unique negative zero of* $E_n(x)$. *Then as* $n \to \infty$,

$$\delta_n = -(\lambda_2 n + \mu_2 \log n + \nu_2) + o(1),$$

where $\lambda_2 e^{1+\lambda_2} = 1$, *i.e.*, $\lambda_2 \simeq 0.278\,465$,

$$\mu_2 = \frac{1}{2}\Big/\left[1 + \frac{1}{\lambda_2}\right] \simeq 0.108\,906$$

and

$$\nu_2 = \frac{1}{[1 + 1/\lambda_2]} \log\left[\sqrt{2\pi}\left[1 + \frac{1}{\lambda_2}\right]\right] \simeq 0.523\,128.$$

Table II shows the actual values of $-\delta_n$ and of $-\tilde{\delta}_n = 0.278\,465n + 0.108\,906 \log n + 0.532\,128$.

TABLE II

n	$-\delta_n$	$-\tilde{\delta}_n$
1	1.000 000	0.810 592
3	1.596 072	1.487 167
5	2.180 607	2.099 727
7	2.759 003	2.693 300
9	3.333 551	3.277 599
11	3.905 452	3.856 382
13	4.475 412	4.431 504

8. Case (iii): Padé Rows. Let n be fixed and $x = \lambda_3\nu + \phi_3(\nu)$, where $\phi_3(\nu) = o(\nu)$ as $\nu \to \infty$. Then

$$I_i = \int_{\alpha_i}^{\beta_i} e^{\nu h(s) - \phi_3(\nu)s/2}|1 - s|^n\, ds, \qquad i = 1, 2,$$

where $h(s) = -\lambda_3 s/2 + \log(1 + s)$, with one maximum at $s_0 = 2/\lambda_3 - 1$. Suppose $\lambda_3 \neq 1$. If $\lambda_3 > 1$, the maximum occurs in $(-1, 1)$ and $h'(s) < 0$ in $[1, \infty)$. If $0 < \lambda_3 < 1$, the maximum occurs in $(1, \infty)$ and $h'(s) > 0$ in $(-1, 1]$. In both cases,

matching the dominant term in I_1 and I_2 gives $h(s_0) = h(1)$, i.e., $\lambda_3 - 1 - \log\lambda_3 = 0$, i.e., $\lambda_3 = 1$, a contradiction. Hence, $\lambda_3 = 1$. Let

$$I = \int_{-1}^{\infty} e^{\nu h(s)} e^{-\phi_3(\nu)s/2} |1 - s|^n \, ds \quad \text{with } \lambda_3 = 1.$$

$h'(s) = 0$ when $s = 1$, so that

$$e^{\nu h(s)} = e^{\nu h(1)} e^{\nu h^{(2)}(1)\sigma^2/2} \left[1 + h^{(3)}(1)\nu\sigma^3/6 + \cdots\right]$$

and

$$e^{-\phi_3(\nu)s/2} = e^{-\phi_3(\nu)/2}\left[1 - \phi_3(\nu)\sigma/2 + \cdots\right],$$

where $\sigma = s - 1$. Keeping only those terms which will survive the Laplace process, we have

$$I \sim e^{\nu h(1)} e^{-\phi_3(\nu)/2} \int e^{-\nu\sigma^2/8} \left[\frac{\nu}{24}\sigma^{n+3} - \frac{\phi_3(\nu)\sigma^{n+1}}{2} + \cdots\right] d\sigma,$$

and the leading term can only be zero if

$$\phi_3(\nu) = \tfrac{2}{3}(n/2 + 1) + o(1) \quad \text{as } \nu \to \infty.$$

Hence $x = \nu + \frac{2}{3}(n/2 + 1) + o(1)$ as $\nu \to \infty$. The result is

THEOREM 8.1. *Suppose n is fixed and $\gamma_{\nu n} = -\lambda_3\nu + o(\nu)$ as $\nu \to \infty$, where λ_3 is a constant. Then $\gamma_{\nu n} = -\lambda_3\nu - \mu_3 + o(1)$ as $\nu \to \infty$, where μ_3 is a constant and $\lambda_3 = 1$, $\mu_3 = \frac{2}{3}(n/2 + 1)$.*

COROLLARY. *Under the same conditions,*

$$\beta_n(\nu - n + 2) = -\left[\frac{\nu}{2} + \frac{1}{3}\left[\frac{n}{2} + 1\right]\right] + o(1) \quad \text{as } \nu \to \infty.$$

Note. It is interesting to confirm the result of Theorem 5.3 in the cases $n = 1$ and $n = 3$.

$$P_{\nu,1}(x) = x + \nu + 1, \quad \text{so in fact} \quad \gamma_{\nu 1} = -(\nu + 1),$$

$$P_{\nu,3}(x) = x^3 + 3(\nu + 1)x^2 + 3(\nu + 2)(\nu + 1)x + (\nu + 1)(\nu + 2)(\nu + 3).$$

Solving the equation $P_{\nu,3} = 0$ for real solutions x, we obtain $x = -y - (\nu + 1)$, where $y = 2(\nu + 1)^{1/2}\sinh\theta$ and $\sinh 3\theta = (\nu + 1)^{-1/2}$. This can be rewritten as

$$y = \frac{(\nu + 1)^{1/3}\left[\left(1 + (\nu + 2)^{1/2}\right)^{2/3} - (\nu + 1)^{1/3}\right]}{\left[1 + (\nu + 2)^{1/2}\right]^{1/3}},$$

and it is easy to show that $y \sim 2/3$ as $\nu \to \infty$. Hence $x \sim -[\nu + 5/3]$ as $\nu \to \infty$.

9. The Padé Error on $[0, \infty)$. Following [6], let

$$\varepsilon_{\nu n}(x) := Q_{\nu n}(x)/P_{\nu n}(x) - e^{-x},$$

where $Q_{\nu n}(x)/P_{\nu n}(x)$ is the (ν, n) Padé approximant to e^{-x}. Saff, Varga and Ni, [6], showed that if $\{\nu(n)\}$ is a sequence so that

$$\lim_{n\to\infty} \frac{\nu(n)}{n} = f, \qquad 0 < f < 1,$$

then

$$\lim_{n\to\infty}\left\{\|\varepsilon_{\nu(n),n}\|_{\mathscr{L}_\infty[0,\infty)}\right\}^{1/n}=\frac{f^f(1-f)^{1-f}}{2^{1-f}},$$

and that this limit is minimized when $f = 1/3$, in which case the limit is also $1/3$.

In what follows, we show that in fact

$$\|\varepsilon_{fn,n}\|_{\mathscr{L}_\infty[0,\infty)}\sim\sqrt{f}\left\{\frac{f^f(1-f)^{1-f}}{2^{1-f}}\right\}^n$$

and hence that

$$\|\varepsilon_{n/3,n}\|_{\mathscr{L}_\infty[0,\infty)}\sim\left(\frac{1}{3}\right)^{n+1/2}\quad\text{as } n\to\infty.$$

Using the Perron representation of the error [5, p. 436], and the Eweida form used earlier, it is easy to show that

$$(-1)^n\varepsilon_{\nu n}(x)=e^{-x}I_1/I_2,\qquad x>0,$$

where

$$I_i=\int_{\alpha_i}^{\beta_i}e^{-xs/2}(1+s)^n|1-s|^\nu\,ds,\qquad i=1,2.$$

$\varepsilon_{\nu n}(x)$ has one sign on $[0,\infty)$ and its modulus has a single maximum for $x>0$, occurring at $x=m$, given by $\varepsilon'_{\nu n}(m)=0$. m is of course a function of both n and ν. So,

$$I_1'/I_1-I_2'/I_2=1, \tag{1}$$

where

$$I_1'=-\tfrac{1}{2}\int_{\alpha_i}^{\beta_i}se^{-ms/2}(1+s)^n|1-s|^\nu\,ds,\qquad i=1,2.$$

Then

$$\|\varepsilon_{\nu,n}\|_{\mathscr{L}_\infty[0,\infty)}=|\varepsilon_{\nu,n}(m)|.$$

Let $\nu=fn+\psi(n)$, where $\psi(n)=o(n)$ as $n\to\infty$. Then

$$I_i=\int_{\alpha_i}^{\beta_i}e^{h^*(s)}|1-s|^{\psi(n)}\,ds,$$

where $h^*(s)=-ms/2+n\log(1+s)+nf\log|1-s|$.

$$h^{*\prime}(s)=0\quad\text{when } s_2,s_1=\frac{(1+f)\pm\sqrt{(1+f)^2-2m(1-f-m/2n)/n}}{m/n},$$

and in this more general situation it is still possible to show that $-1<s_1<1$, $s_2>1$. Then,

$$\frac{I_1'}{I_1s_1}\to-\frac{1}{2}\quad\text{and}\quad\frac{I_2'}{I_2s_2}\to-\frac{1}{2}\quad\text{as } n\to\infty,$$

and in (1) we have $\lim_{n\to\infty}(s_2-s_1)=2$. Hence we obtain

$$\lim_{n\to\infty}\frac{m}{n}=\frac{(1+f)^2}{2(1-f)},$$

i.e.,

$$m = \frac{(1+f)^2 n}{2(1-f)} + o(n) \quad \text{as } n \to \infty;$$

this should be compared with Eq. 3.10 of [6].

Let $m = (1+f)^2 n/2(1-f) + \phi(n)$, where $\phi(n) = o(n)$ as $n \to \infty$. With

$$h(s) = -\frac{(1+f)^2 s}{4(1-f)} + \log(1+s) + f\log|1-s|,$$

we obtain

$$I_i \sim \frac{e^{nh(s_i)} e^{-\phi(n)s_i/2} |1-s_i|^{\psi(n)} \sqrt{2\pi}}{\sqrt{n}\left[-h^{(2)}(s_i)\right]^{1/2}} \quad \text{as } n \to \infty.$$

Substituting in the equation $|\varepsilon_{\nu n}(m)| = e^{-m}|I_1/I_2|$ and simplifying, we then obtain

$$\|\varepsilon_{\nu,n}\|_{\mathscr{L}_\infty[0,\infty)} \sim \sqrt{f}\left(\frac{(1-f)^{1-f} f^f}{2^{1-f}}\right)^n \left(\frac{2f}{1-f}\right)^{\psi(n)} \quad \text{as } n \to \infty,$$

with $\nu = fn + \psi(n)$, $\psi(n) = o(n)$. This proves

THEOREM 9.1. *If in* $[0,\infty)$ *the maximum of* $|\varepsilon_{\nu n}(x)|$ *with* $\nu = fn + \psi(n)$, $\psi(n) = o(n)$ *as* $n \to \infty$, $0 < f < 1$, *occurs at* m, *then*

$$m = \frac{(1+f)^2 n}{2(1-f)} + o(n),$$

and

$$\|\varepsilon_{\nu n}\|_{\mathscr{L}_\infty[0,\infty)} \sim \sqrt{f}\left\{\frac{(1-f)^{1-f} f^f}{2^{1-f}}\right\}^n \left(\frac{2f}{1-f}\right)^{\psi(n)} \quad \text{as } n \to \infty.$$

Putting $\psi(n) = 0$ results in

COROLLARY 1. *If the maximum of* $|\varepsilon_{fn,n}(x)|$ *in* $[0,\infty)$ *occurs at* m, *then*

$$m = \frac{(1+f)^2 n}{2(1-f)} + o(n),$$

and

$$\|\varepsilon_{fn,n}\|_{\mathscr{L}_\infty[0,\infty)} \sim \sqrt{f}\left\{\frac{(1-f)^{1-f} f^f}{2^{1-f}}\right\}^n \quad \text{as } n \to \infty.$$

COROLLARY 2. *In the minimizing direction, i.e.,* $f = 1/3$, *we have*

$$\|\varepsilon_{n/3,n}\|_{\mathscr{L}_\infty[0,\infty)} \sim \left(\frac{1}{3}\right)^{n+1/2} \quad \text{as } n \to \infty.$$

We can also consider the case $\nu = 0$ when the approximants $1/E_n(x)$ are drawn from column number zero. Then at $x = m$,

$$I_i = \int_{\alpha_i}^{\beta_i} e^{h^*(s)}\, ds,$$

where $h^*(s) = -ms/2 + n\log(1+s)$. If s_1, s_2 are the points where h^* takes its maxima in the two intervals, then as in Theorem 9.1, $\lim_{n\to\infty}(s_2 - s_1) = 2$. It is then easy to show that $m = n/2 + \phi(n)$, where $\phi(n) = o(n)$ as $n \to \infty$. Put

$$I_i = \int_{\alpha_i}^{\beta_i} e^{nh(s)} e^{-\phi(n)s/2}\, ds,$$

where $h(s) = -s/4 + \log(1+s)$. $h'(s) = -1/4 + 1/(1+s)$, so that $h'(s) > 0$ when $s \in [-1, 1]$. Hence $s_1 = 1$; $h'(s) = 0$ when $s = s_2 = 3$. We obtain

$$I_1 \sim \frac{2^{n+2} e^{-n/4} e^{-\phi(n)/2}}{n}, \qquad I_2 \sim \frac{2^{2n+2} e^{-3n/4} e^{-3\phi(n)/2}\sqrt{2\pi}}{\sqrt{n}},$$

and $e^{-m} \sim e^{-n/2} e^{-\phi(n)}$ as $n \to \infty$. The result is

THEOREM 9.2. *If the maximum of* $|\varepsilon_{0,n}(x)| = |e^{-x} - 1/E_n(x)|$ *on* $[0, \infty)$ *occurs at* m, *then* $m = n/2 + o(n)$ *and*

$$\|\varepsilon_{0,n}\|_{\mathscr{L}_\infty[0,\infty)} \sim 2^{-(n+1/2)}(\pi n)^{-1/2} \quad \text{as } n \to \infty.$$

Added in Proof. It has come to the author's attention that the assumption of Theorems 5.1 and 6.1 that $\gamma_{\nu n} = -\lambda_1 n + o(n)$ as $n \to \infty$ where λ_1 is a constant is justified by Theorem 2.2 of [7].

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