An Estimate of Goodness of Cubatures for the Unit Circle in $\mathbb{R}^2$

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Abstract. The Sarma-Eberlein estimate $s_E$ is an estimate of goodness of cubature formulae for $n$-cubes defined as the integral of the square of the formula truncation error, over a function space provided with a measure. In this paper, cubature formulae for the unit circle in $\mathbb{R}^2$ are considered and an estimate of the above type is constructed with the desirable property of being compatible with the symmetry group of the circle.

1. Isometries and Two-dimensional Cubature Formulae. Let

$$S_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

be the unit circle in the two-dimensional Euclidean space $\mathbb{R}^2$ and let $\mathcal{U}(S_2)$ denote the symmetry group of $S_2$. This group consists of all linear bijective maps $u: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which preserve the Euclidean distance (that is, isometries of $\mathbb{R}^2$ leaving the origin invariant). Each element of $\mathcal{U}(S_2)$ can be identified with a $2 \times 2$ real orthogonal matrix and therefore

$$\mathcal{U}(S_2) = \{u_\alpha, u_\alpha \circ v ; \alpha \in [0, 2\pi)\},$$

where $u_\alpha$ denotes the rotation of $\alpha$ radians around the origin and $v$ is the reflection about any fixed straight line passing through the origin; thus

$$u_\alpha(x, y) = (x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha),$$
$$v(x, y) = (x, -y).$$

Let $w(x, y)$ be a normalized weight function compatible with $\mathcal{U}(S_2)$, that is, a real positive continuous function in the interior of $S_2$ such that

$$\iint_{S_2} w(x, y) \, dx \, dy = 1 \quad \text{and} \quad w \circ u = w \quad \text{for all } u \in \mathcal{U}(S_2).$$

A cubature formula for the $w$-weighted circle $S_2$ has the form

$$I(f) = Q_N(f) + E(f),$$
where
\[ I(f) = \iint_{S_2} w(x, y)f(x, y) \, dx \, dy, \]
(1.6)
\[ Q_N(f) = \sum_{i=1}^{N} A_i f(x_i, y_i), \quad (x_i, y_i) \in S_2, \]
and the constants \( A_i \) are independent of \( f \).

Let us consider a symmetry \( u \in \mathcal{U}(S_2) \) acting on (1.5). Since \( I(f \circ u) = I(f) \), it leads to another cubature formula
\[ I(f) = Q'_N(f) + E'(f), \]
(1.7)
where
\[ Q'_N(f) = Q_N(f \circ u) = \sum_{i=1}^{N} A_i f(u(x_i, y_i)), \]
(1.8)
\[ E'(f) = E(f \circ u). \]

**Definition 1.** For every \( u \in \mathcal{U}(S_2) \), the cubature formulae (1.5) and (1.7) are said to be \( \mathcal{U}(S_2) \)-equivalent or equivalent with respect to the symmetry group of \( S_2 \).

The integration of a function on the \( w \)-weighted circle \( S_2 \) is independent of the pair of orthogonal axis \( OX, OY \) whose origin \( O \) lies in the center of the circle. Therefore, all \( \mathcal{U}(S_2) \)-equivalent formulae have identical characteristics when they are considered as approximations of the operator \( I \).

Therefore, any estimate of goodness for cubature formulae (1.5) should be compatible with the \( \mathcal{U}(S_2) \)-equivalence relation, that is, all \( \mathcal{U}(S_2) \)-equivalent formulae should have the same estimate of goodness. For instance, the degree of precision of a cubature formula (1.5) is an estimate compatible with \( \mathcal{U}(S_2) \), because the space of polynomials of degree at most \( n \) is invariant under all the symmetries in (1.2).

The aim of this paper is to construct an \( \mathcal{U}(S_2) \)-compatible estimate of goodness of cubature formulae for \( S_2 \) similar to that defined by V. L. N. Sarma in [3] for cubatures for the square.

The next section is devoted to recalling briefly some characteristics of the Sarma-Eberlein estimate that are useful for our purpose. A detailed exposition of the construction of this estimate can be found in [3], [4] and [5] and an excellent summary of these results in [6, pp. 188–192].

2. **The Sarma-Eberlein Estimate of Goodness** \( s_E \). Let us consider the square
\[ C_2 = \{ (x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1 \} \]
and cubature formulae
\[ I(f) = Q_N(f) + E(f), \]
(2.1)
where
\[ I(f) = \frac{1}{4} \iint_{C_2} f(x, y) \, dx \, dy, \]
(2.2)
\[ Q_N(f) = \sum_{i=1}^{N} A_i f(x_i, y_i), \quad (x_i, y_i) \in C_2. \]
Sarma in [3], [4] defines the estimate of goodness of the cubature formula (2.1) as

$$s^2_E = \int_{F_{S_\infty}} E(f)^2 \, df,$$

where the integral is defined over the unit sphere of a normed space of functions provided with a measure defined as follows:

Let $l_1$ be the space of real sequences

$$f = \{f_{nk}; n = 0, 1, \ldots; k = 0, 1, \ldots, n\}$$

such that

$$\|f\|_1 = \sum_{n,k} |f_{nk}| < \infty; \quad n = 0, 1, \ldots; k = 0, 1, \ldots, n.$$

The unit sphere $S_\infty = \{f \in l_1: \|f\|_1 \leq 1\}$ is compact in the weak*-topology of $l_1$, and an elementary integral defined for the weak*-continuous real functions on $S_\infty$ can be extended by the Daniell process inducing a countably additive measure on $S_\infty$.

Among the properties of this measure, let us recall that

$$\int_{S_\infty} f_{nk} f_{ml} df = 0 \quad \text{if} \quad (n, k) \neq (m, l),$$

$$\int_{S_\infty} f_{nk}^2 df = \frac{2^{n+2}}{(n + 2)!(n + 3)!} = q_n^2.$$

Real two-dimensional power series

$$f(x, y) = \sum_{n,k} f_{nk} x^{n-k} y^k; \quad n = 0, 1, \ldots; k = 0, 1, \ldots, n,$$

whose coefficients satisfy the condition

$$\|f\|_1 = \sum_{n,k} |f_{nk}| < \infty$$

converge uniformly and absolutely for all points $(x, y) \in C_2$.

The space $F_{l_1}$ of all functions defined by (2.8) and (2.9) can be identified with the sequence space $l_1$ and is dense in the space $\mathcal{C}(C_2)$ of all real continuous functions on $C_2$ with the uniform norm. This identification allows us to consider the above integral as an integral over the unit sphere $F_{S_\infty}$ of the function space $F_{l_1}$.

The truncation error $E(f)$ of the cubature formula (2.1) is a continuous linear form over $\mathcal{C}(C_2)$ with the uniform norm and therefore also over $F_{l_1}$ with the $\| \cdot \|_1$-norm. Using (2.6) and (2.7), it follows that the estimate $s_E$ defined by (2.3) can be written as

$$s^2_E = \sum_{n=0}^{\infty} q_n^2 e_n^2,$$

where $q_n$ is defined in (2.7) and

$$e_n^2 = \sum_{k=0}^{n} E(x^{n-k}y^k)^2.$$
section. In effect, \( \mathcal{U}(C_2) \) consists of the eight symmetries
\[
(x, y) \to (\pm x, \pm y); \quad (x, y) \to (\pm y, \pm x)
\]
and the equalities
\[
e_2^2 = \sum_{k=0}^{n} E(x^{n-k}y^k)^2 = \sum_{k=0}^{n} E((\pm x)^{n-k}(\pm y)^k)^2
\]
\[
e_2^2 = \sum_{k=0}^{n} E((\pm y)^{n-k}(\pm x)^k)^2
\]

imply that \( \mathcal{U}(C_2) \)-equivalent cubature formulae have the same estimate of goodness \( s_E \). Unfortunately, this estimate of goodness is not useful for cubature formulae (1.5), (1.6) for the unit circle \( S_2 \), because it is not compatible with \( \mathcal{U}(S_2) \), as can be computationally checked. For instance, taking \( w(x, y) = 1/\pi \), the cubature formula (degree 3, 4 points) given by
\[
Q_4(f) = \frac{1}{4} \left[ f(\sqrt{2}/2, 0) + f(-\sqrt{2}/2, 0) + f(0, \sqrt{2}/2) + f(0, -\sqrt{2}/2) \right]
\]
has an estimate of goodness \( s_E = (-4)^{1.75032} \), whereas the \( \mathcal{U}(S_2) \)-equivalent formula (use a rotation of \( \pi/4 \) radians) given by
\[
Q_4(f) = \frac{1}{4} \left[ f(1/2, 1/2) + f(-1/2, 1/2)
\]
\[
+ f(1/2, -1/2) + f(-1/2, -1/2) \right]
\]
has an estimate of goodness \( s_E = (-4)^{3.81547} \).

3. An Estimate of Goodness of Cubatures for the Unit Circle. In the previous section, the sequence space \( l_1 \) was identified with the space of functions \( F_l \) by using the family of monomials \( \{x^n, y^n; n = 0, 1, \ldots; k = 0, 1, \ldots, n\} \), but we can also identify \( l_1 \) with other subspaces of \( \mathcal{U}(C_2) \) or \( \mathcal{U}(S_2) \) by using other families of polynomials. For each \( n \), let us denote
\[
M_n = \{a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n; a_i \in \mathbb{R}\}
\]
and let
\[
\psi_n = \{\psi_{n0}, \ldots, \psi_{nn}\} \subset M_n
\]
be a basis of \( M_n \), i.e., \( M_n = \text{span}\, \psi_n \).

If the polynomials \( \psi_{nk} \) satisfy
\[
\|\psi_{nk}\|_\infty = \max_{(x, y) \in S_2} |\psi_{nk}(x, y)| \leq c; \quad n = 0, 1, \ldots; k = 0, 1, \ldots, n,
\]
then the series
\[
f(x, y) = \sum_{n,k} f_{nk}\psi_{nk}(x, y)
\]
whose coefficients satisfy (2.9) converge uniformly and absolutely for all points \((x, y) \in S_2\). If we denote \( \Phi = \{\varphi_1, \varphi_2, \ldots\} \), the space \( F_l(\Phi) \) of all functions defined by (3.4) and (2.9) can be identified with the sequence space \( l_1 \). Let us note that \( F_l(\Phi) \) contains all real polynomials in two variables and therefore is dense in \( \mathcal{U}(S_2) \) with the uniform norm.

This identification allows us to define, in a natural way, an estimate of goodness for cubatures (1.5) by
\[
s_E^2(\Phi) = \int_{FS_c(\Phi)} E(f)^2 \, df,
\]
where

\[(3.6) \quad FS_{\infty}(\Phi) = \{ f \in FL_1(\Phi) : \sum_{n,k} |f_{nk}| \leq 1 \} .\]

It is straightforward to deduce that this estimate can be expressed by

\[(3.7) \quad s^2_E(\Phi) = \sum_{n=0}^{\infty} q^2_n e^2_n(\Phi_n),\]

where \(q^2_n\) is given in (2.7) and

\[(3.8) \quad e^2_n(\Phi_n) = \sum_{k=0}^{n} E(\varphi_{nk})^2.\]

Our problem at this stage is to choose suitable families \(\Phi_n\) satisfying (3.3), such that the estimate \(s^2_E(\Phi)\) is compatible with the symmetry group \(\mathcal{G}(S_2)\) in the sense described in Section 1.

As the matrix

\[(3.9) \quad \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}\]

associated with the rotation \(u_\alpha \in \mathcal{G}(S_2)\) has eigenvalues \(e^{i\alpha}, e^{-i\alpha}\) and eigenvectors \((1, i)T, (1, -i)T\), the use of complex arithmetic will simplify the calculations. Let us denote

\[(3.10) \quad M_n^* = \{ a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n ; a_i \in \mathbb{C} \},\]

and let

\[(3.11) \quad \Phi_n^* = \{ \varphi_{n0}^*, \ldots, \varphi_{nn}^* \}\]

be a basis of \(M_n^*\), i.e., \(M_n^* = \text{span}^*(\Phi_n^*)\).

Considering the natural complexification of linear operators

\[(3.12) \quad E(f + ig) = E(f) + iE(g)\]

with the standard complex notation

\[(3.13) \quad |E(f + ig)|^2 = \overline{E(f + ig)} E(f + ig) = E(f)^2 + E(g)^2,\]

we can define

\[(3.14) \quad e^2_n(\Phi_n^*) = \sum_{k=0}^{n} |E(\varphi_{nk}^*)|^2.\]

**Theorem 1.** For every \(n\), let \(\Phi_n^* = \{ \varphi_{n0}^*, \ldots, \varphi_{nn}^* \}\) and \(\Phi_n = \{ \varphi_{n0}, \ldots, \varphi_{nn} \}\) be bases of \(M_n^*\) and \(M_n\), respectively, satisfying

(i) \((\varphi_{n0}, \ldots, \varphi_{nn})^T = A_n(\varphi_{n0}^*, \ldots, \varphi_{nn}^*)^T\) where \(A_n\) is an \(n \times n\) complex unitary matrix, i.e., \(A^H = A^{-1}\);

(ii) \(\sum_{k=0}^{n}|E(\varphi_{nk}^*)|^2 = \sum_{k=0}^{n} |E(\varphi_{nk}^* u_\alpha)|^2 = \sum_{k=0}^{n} |E(\varphi_{nk}^* u_\alpha v)|^2\) for all \(\alpha \in [0, 2\pi]\);

(iii) there exists a \(c \in \mathbb{R}\) such that \(||\varphi_{nk}||_\infty \leq c\) for all \(n, k\).

Then, the estimate \(s^2_E(\Phi)\) associated with the family \(\Phi = \{ \Phi_0, \Phi_1, \ldots \}\) is compatible with the symmetry group \(\mathcal{G}(S_2)\).
Proof. Let us remark that the operators
\[ f^* \in M_n^* \rightarrow E(f^*) \in \mathbb{C} , \]
\[ f^* \in M_n^* \rightarrow f^* \circ u_a \in M_n^* , \]
\[ f^* \in M_n^* \rightarrow f^* \circ u_a \circ v \in M_n^* \]
are linear and therefore "pass through the matrix \( A_n^* \)."
Moreover, \( E(\varphi_{nk}) \) and \( E(\varphi_{nk} \circ u) \) are real and therefore
\[
\sum_{k=0}^{n} E(\varphi_{nk} \circ u_a)^2 = \left( E(\varphi_{n0} \circ u_a), \ldots, E(\varphi_{nn} \circ u_a) \right) \left( E(\varphi_{n0} \circ u_a), \ldots, E(\varphi_{nn} \circ u_a) \right)^T
\]
\[
= (E(\varphi_{n0} \circ u_a), \ldots, E(\varphi_{nn} \circ u_a)) A_n^H A_n (E(\varphi_{n0} \circ u_a), \ldots, E(\varphi_{nn} \circ u_a))^T
\]
\[
= \sum_{k=0}^{n} |E(\varphi_{nk}^* \circ u_a)|^2 = \sum_{k=0}^{n} |E(\varphi_{nk}^*)|^2
\]
\[
= (E(\varphi_{n0}^*), \ldots, E(\varphi_{nn}^*)) A_n A_n^H (E(\varphi_{n0}), \ldots, E(\varphi_{nn}))^T = \sum_{k=0}^{n} E(\varphi_{nk}^*)^2,
\]
given that \( A_n \) is unitary. Similarly, it can be shown that
\[
\sum_{k=0}^{n} E(\varphi_{nk} \circ u_a \circ v)^2 = \sum_{k=0}^{n} E(\varphi_{nk})^2 ,
\]
and therefore it follows in a straightforward way that \( s_k(\Phi) \) is compatible with \( \mathcal{U}(S_2) \). \( \Box \)

Now let us consider the complex polynomials
\[
\varphi_{nk}^* = (x + iy)^{n-k}(x - iy)^k \in M_n^* \tag{3.15}
\]
obtained from the monomials \( x^{n-k}y^k \) by a linear transformation with Jacobian
\[
J = \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} = -2i ,
\]
so that \( \varphi_{n0}^*, \ldots, \varphi_{nn}^* \) are linearly independent in \( M_n^* \).

Also,
\[
(\varphi_{nk}^* \circ u_a)(x, y) = \varphi_{nk}^*(x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha)
\]
\[
= e^{i(n-k)\alpha}(x + iy)^{n-k}e^{-ik\alpha}(x - iy)^k = e^{i(n-2k)\alpha}\varphi_{nk}^*(x, y),
\]
thus
\[
\sum_{k=0}^{n} |E(\varphi_{nk}^* \circ u_a)|^2 = \sum_{k=0}^{n} |E(\varphi_{nk}^*)|^2 . \tag{3.16}
\]

Similarly,
\[
(\varphi_{nk}^* \circ u_a \circ v)(x, y) = (\varphi_{nk}^* \circ u_a)(x, -y) = e^{i(n-2k)\alpha}(x - iy)^{n-k}(x + iy)^k
\]
\[
= e^{i(n-2k)\alpha}\varphi_{n-n-k}^*(x, y) ,
\]
and then

\[ (3.17) \quad \sum_{k=0}^{n} |E(q_n^* \circ u_{o \circ v})|^2 = \sum_{k=0}^{n} |E(q_n^*)|^2. \]

Therefore, for each \( n \), the family \( \Phi_n^* = \{ q_{n0}^*, \ldots, q_{nn}^* \} \) is a basis of \( M_n^* \) which satisfies the hypothesis (ii) of Theorem 1.

For \( k < n/2 \) let us define

\[ (3.18) \quad q_{nk} = \frac{1}{\sqrt{2}} (q_{nk}^* + q_{n,n-k}^*) \]

\[ = \frac{1}{\sqrt{2}} (x^2 + y^2)^k [(x + iy)^{n-2k} + (x - iy)^{n-2k}], \]

\[ (3.19) \quad q_{n,n-k} = \frac{1}{\sqrt{2}i} (q_{nk}^* - q_{n,n-k}^*) \]

\[ = \frac{1}{\sqrt{2}i} (x^2 + y^2)^k [(x + iy)^{n-2k} - (x - iy)^{n-2k}], \]

and if \( n \) is even,

\[ (3.20) \quad q_{n,n/2}^* = q_{n,n/2} = (x^2 + y^2)^{n/2}. \]

Then, \( \Phi_n = \{ q_{n0}, \ldots, q_{nn} \} \) is formed by polynomials with real coefficients and is a basis of \( M_n \). Also the matrix \( A_n \) of Theorem 1 that relates the elements of \( \Phi_n \) and \( \Phi_n^* \) is a unitary matrix, because the matrices

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}, & -\frac{1}{\sqrt{2}}i
\end{pmatrix}
\]

that relate the pairs \( (q_{nk}, q_{n,n-k}) \) and \( (q_{nk}^*, q_{n,n-k}^*) \) are unitary. Moreover, it can easily be shown that

\[ \|q_{nk}\|_\infty = \|q_{n,n-k}\|_\infty = \sqrt{2}, \quad k < n/2, \]

and \( \|q_{n,n/2}\|_\infty = 1 \) for \( n \) even.

Using the results above, and applying Theorem 1, we deduce the following

**Theorem 2.** Let \( \Phi = \{ \Phi_0, \Phi_1, \ldots \} \) where, for each \( n \), \( \Phi_n = \{ q_{n0}, \ldots, q_{nn} \} \) is the basis of \( M_n \) defined by (3.18), (3.19) and (3.20). Then the estimate \( s_\mathcal{E}(\Phi) \) defined by (3.5) is an estimate of goodness of cubature formulae for the unit circle that is compatible with the symmetry group \( \mathfrak{H}(S_2) \).

Following the proof of Theorem 1, we can also deduce that

\[ (3.21) \quad s_\mathcal{E}^2(\Phi) = \sum_{n=0}^{\infty} q_n^2 \sum_{k=0}^{n} E(q_{nk})^2 = \sum_{n=0}^{\infty} q_n^2 \sum_{k=0}^{n} |E(q_{nk}^*)|^2, \]

and therefore the estimate \( s_\mathcal{E}(\Phi) \) can be calculated using any of these two expressions.
Table 1 shows the values of $s_E(\Phi)$ for some cubature formulae (1.5) for the unit circle with $w(x, y) = 1/\pi$. The nomenclature of these formulae corresponds to the one in [6, pp. 277–289]. N stands for the number of nodes and D for the degree of precision.

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