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L_∞-Boundedness of L_2-Projections on Splines for a Multiple Geometric Mesh*

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Abstract. This paper concerns the L_2-projectors from L_∞ to the normed linear space of polynomial splines. It is shown that for the multiple geometric meshes the L_∞ norms of the corresponding L_2-projectors are bounded independently of the mesh ratio.

0. Introduction. Let us begin with some notation. Let k be a positive integer, and $\mathbf{x} = (x_i)_{i \in \mathbf{Z}}$ a real nondecreasing knot sequence with $x_i < x_{i+k}$, all i. Set

$$x_{-\infty} := \lim_{i \to -\infty} x_i, \qquad x_\infty := \lim_{i \to \infty} x_i, \qquad I := (x_{-\infty}, x_\infty).$$

For $i \in \mathbf{Z}$ and $x \in I$, define

$$M_{i,k}(x) := k[x_i, \ldots, x_{i+k}](\cdot - x)_+^{k-1}, \qquad N_{i,k}(x) := (x_{i+k} - x_i) M_{i,k}(x)/k,$$

where $[\rho_0, \ldots, \rho_r]f$ denotes the rth divided difference of the function f at the points $\rho_0, \ldots, \rho_r$. Then $N_{i,k}$ is called the ith B-spline of order k on the knot sequence $\mathbf{x}$.

For $\mathbf{a} \in \mathbf{R}^{\mathbf{Z}}$, the rule

$$f(x) := \sum_i \mathbf{a}(i) N_{i,k}(x)$$

defines a function on I if we take the sum to be pointwise. Every such function is called a polynomial spline of order k with the knot sequence $\mathbf{x}$, and their collection is denoted by $\mathbb{S}_{k,\mathbf{x}}$. Further, let $\mathbb{S}$ denote the normed linear space of bounded polynomial splines of order k with the knot sequence $\mathbf{x}$ and norm

$$\|f\| := \sup_{x \in I} |f(x)|.$$

We shall be concerned with $P_{\mathbb{S}}$, the orthogonal projector onto $\mathbb{S}$ with respect to the ordinary inner product

$$(f, g) = \int_I f(x) g(x)\, dx,$$

but restricted to $L_\infty(I)$. We want to bound its norm

$$\|P_{\mathbb{S}}\| := \sup_{f \in L_\infty(I)} \|P_{\mathbb{S}} f\| / \|f\|_\infty.$$

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In 1973, de Boor [1] made the conjecture that

$$\sup_{\mathbf{x}} \|P_{\mathbb{S}}\| \leqslant \text{const}_k < \infty .$$

This conjecture has been verified only for $k = 2$ (Ciesielski [8]), $k = 3$ and 4 (de Boor [5]). Moreover, de Boor [4] showed that $P_{\mathbb{S}}$ is bounded in terms of the global mesh ratio. For geometric meshes, Höllig [12] proved that $P_{\mathbb{S}}$ is bounded independently of the mesh ratio. Later on, Feng and Kozak [9] reproved this result. For a tri-multiple geometric mesh, Mityagin [15] established the uniform boundedness of $P_{\mathbb{S}}$ for $k = 6$.

In this paper we shall be concerned with multiple geometric knot sequences $(x_i)_{i \in \mathbf{Z}}$; that is,

$$x_{li} = x_{li+1} = \cdots = x_{li+l-1} = q^i, \quad \text{all } i, 0 < q < \infty,$$

where $l \in \mathbf{N}$ is the multiplicity of the knots. Our main result is

$$\sup_{\mathbf{x}} \|P_{\mathbb{S}}\| \leqslant \text{const}_k < \infty$$

where $\mathbf{x}$ runs through all the multiple geometric knot sequences. This result extends the results of Höllig and Mityagin.

Let $A = A_q$ be the $\mathbf{Z} \times \mathbf{Z}$ matrix given by

$$A(i, j) := \int M_{i,k} N_{j,k} \quad \text{for } i, j \in \mathbf{Z}.$$

It is shown by de Boor ([1], also [2]), that

$$D_k^{-2} \|A^{-1}\|_\infty \leqslant \|P_{\mathbb{S}}\|_\infty \leqslant \|A^{-1}\|_\infty ,$$

where D_k is a constant depending only on k. Thus, bounding $P_{\mathbb{S}}$ is equivalent to bounding A^{-1}.

Now since the mesh $(x_i)_{i \in \mathbf{Z}}$ is an l-multiple geometric one, we have

$$\begin{aligned} A(i + l, j + l) &= \int M_{i+l}(x) N_{j+l}(x)\, dx \\ &= \int M_{i+l}(x) N_j(q^{-1}x)\, dx = \int q M_{i+l}(qx) N_j(x)\, dx \\ &= \int M_i(x) N_j(x)\, dx = A(i, j). \end{aligned}$$

This shows that A is an l-block Toeplitz matrix. Moreover, A is totally positive (see [13, Chapter 10], also [2]). This motivates us to investigate totally positive block Toeplitz matrices.

1. Totally Positive Block Toeplitz Matrices. Let A be a bi-infinite N-block Toeplitz matrix. Then there exists a sequence of $N \times N$ matrices A_j ($j \in \mathbf{Z}$) such that A has the following form:

$$A = \begin{bmatrix} \cdots & \cdot & \cdot & \cdot & \cdots \\ \cdots & A_0 & A_1 & A_2 & \cdots \\ \cdots & A_{-1} & A_0 & A_1 & \cdots \\ \cdots & A_{-2} & A_{-1} & A_0 & \cdots \\ \cdots & \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

Now

$$\hat{A}(z) := \sum_{n=-\infty}^{\infty} A_n z^n, \qquad z \in \mathbf{C},$$

is a formal power series with matrix coefficients. It is called the symbol of A. If all the rows of A are in l_1, i.e.,

$$\sum_{j=-\infty}^{\infty} |a_{ij}| < \infty \quad \text{for all } i \in \mathbf{Z}, \tag{1.1}$$

then $\hat{A}(z)$ makes sense for $|z| \leqslant 1$. In this case, A determines a bounded linear operator from l_∞ to l_∞. This operator is also denoted by A. A basic question is when A is boundedly invertible. In the case where A is also totally positive, the following theorem gives an answer. In its statement and further on, we use the abbreviation

$$\mathbf{1}' := \left((-1)^{j-1}\right)_{j \in J}$$

for the vector on the index set J whose entries are alternately 1 and –1. Typically, $J = (1, \dots, N)$ or else $J = \mathbf{Z}$. We also use

$$\omega := (-1)^N.$$

THEOREM 1.1. *Let A be a bi-infinite totally positive N-block Toeplitz matrix with all the rows in l_1. Then A is boundedly invertible if and only if*

$$\det \hat{A}(\omega) \neq 0.$$

Moreover, $\det \hat{A}(\omega) \neq 0$ *if and only if there exists* $b \in \mathbf{R}^N$ *such that*

$$\hat{A}(\omega)\mathbf{b} = \mathbf{1}', \tag{1.2}$$

and, *for such* $\mathbf{b}$, $\|A^{-1}\| = \|\mathbf{b}\|_\infty$.

Remark. When A is a banded matrix, the first part of this theorem is already obtained in [7]. The above theorem removes the restriction of bandedness.

To prove Theorem 1.1, we need a lemma.

LEMMA 1.1. *Let φ be a real number. If $\varphi/(2\pi)$ is not an integer, then for any positive integer n_0 there exist integers k, l such that $l > k \geqslant n_0$ and*

$$|\cos k\varphi| \geqslant \tfrac{1}{2}, \quad |\cos l\varphi| \geqslant \tfrac{1}{2} \quad \textit{and} \quad \cos k\varphi \cos l\varphi < 0. \tag{1.3}$$

Furthermore, if φ/π is not an integer, then k and l can be chosen to be even numbers.

Proof. Suppose first that $\varphi/2\pi$ is not an integer. Without loss of generality we may assume $0 < \varphi < 2\pi$. We shall argue case by case.

1°. $0 < \varphi \leqslant 2\pi/3$. By $[x]$ we denote the integer part of x, i.e., the largest integer $\leqslant x$. Let

$$k := \left[(2n_0\pi + \pi/3)/\varphi\right], \qquad l := \left[((2n_0 + 1)\pi + \pi/3)/\varphi\right].$$

Then

$$k\varphi \leqslant 2n_0\pi + \pi/3 \leqslant (k+1)\varphi,$$
$$k\varphi \geqslant 2n_0\pi + \pi/3 - \varphi \geqslant 2n_0\pi - \pi/3;$$

hence,

$$k\varphi \in \left[2n_0\pi - \pi/3, 2n_0\pi + \pi/3\right].$$

Similarly,

$$l\varphi \in \left[(2n_0+1)\pi - \pi/3, (2n_0+1)\pi + \pi/3\right].$$

Thus (1.3) is fullfilled.

2°. $4\pi/3 \leqslant \varphi < 2\pi$. In this case we set

$$k := \left[2n_0\pi/(2\pi-\varphi)\right] \quad \text{and} \quad l := \left[(2n_0+1)\pi/(2\pi-\varphi)\right].$$

3°. $2\pi/3 < \varphi < \pi$. Since $4\pi/3 < 2\varphi < 2\pi$, this case reduces to 2°.

4°. $\varphi = \pi$. This case is trivial.

5°. $\pi < \varphi < 4\pi/3$. Since $2\pi < 2\varphi < 8\pi/3$, this case reduces to 1°.

Assume now that φ/π is not an integer. Then $2\varphi/2\pi$ is not an integer and we obtain (1.3) with φ replaced by 2φ; hence we obtain (1.3) itself with both k and l even. This ends the proof of Lemma 1.1. □

Proof of Theorem 1.1. It is known (see [11]) that A is boundedly invertible if and only if $\det \hat{A}(e^{i\theta}) \neq 0$ for all real θ. Thus, to prove the theorem, it is sufficient to show that the solvability of (1.2) for $\mathbf{b}$ implies that $\det \hat{A}(e^{i\theta}) \neq 0$ for all θ. We prove this implication by contradiction. Suppose that $\det \hat{A}(e^{i\theta}) = 0$ for some real θ. Then there exists a nonzero $\mathbf{y} \in \mathbf{C}^N$ such that $\hat{A}(e^{i\theta})\mathbf{y} = 0$. Without loss of generality we may assume $y_1 \neq 0$, and, after multiplying by a suitable complex number, we may assume further that y_1 is a positive real number. Let

$$\begin{aligned} y_{kN+j} &:= \left(e^{i\theta}\right)^k y_j, \qquad k \in \mathbf{Z};\ j = 1,2,\ldots,N, \\ \tilde{\mathbf{y}} &:= (\ldots, y_{-1}, y_0, y_1, y_2, \ldots). \end{aligned} \tag{1.4}$$

Then $A\tilde{\mathbf{y}} = 0$. Suppose that $\mathbf{b}$ is a solution of (1.2). Let

$$\begin{aligned} b_{kN+j} &:= \omega^k b_j, \qquad k \in \mathbf{Z};\ j = 1,2,\ldots,N, \\ \tilde{\mathbf{b}} &:= (\ldots, b_{-1}, b_0, b_1, b_2, \ldots). \end{aligned}$$

Then $A\tilde{\mathbf{b}} = \mathbf{1}'$.

We consider the two possibilities: $e^{i\theta} \neq \omega$ or $e^{i\theta} = \omega$ separately. First we consider the case $e^{i\theta} \neq \omega$. Fix $\varepsilon > 0$ such that $y_1/2 > \varepsilon|b_1|$ and let $\mathbf{u} := \operatorname{Re}\tilde{\mathbf{y}} + \varepsilon\tilde{\mathbf{b}}$. Then $\mathbf{u} \in l_\infty$ and $A\mathbf{u} = \varepsilon\mathbf{1}'$. By (1.1) there exists a positive integer m such that

$$\sum_{|k-j|>m} |a_{jk}|\,\|u\|_\infty < \varepsilon. \tag{1.5}$$

In the following we shall argue separately in terms of N even or odd.

(i) N is even. In this case, $e^{i\theta} \neq \omega = 1$; hence $\theta/2\pi$ is not an integer. By Lemma 1.1, there exist positive integers k_i, l_i $(i = 0,\ldots,2m)$ such that

$$k_0 < l_0 < k_1 < l_1 < \cdots < k_{2m} < l_{2m}, \tag{1.6}$$

$$|\cos k_i\theta| \geqslant \tfrac{1}{2}, \quad |\cos l_i\theta| \geqslant \tfrac{1}{2}, \quad \text{and} \quad \cos k_i\theta \cos l_i\theta < 0. \tag{1.7}$$

Since $\varepsilon|b_1| < y_1/2$, we have

$$u_{k_iN+1}u_{l_iN+1} = (y_1\cos k_i\theta + \varepsilon b_1)(y_1 \cos l_i\theta + \varepsilon b_1) < 0.$$

But $(l_iN+1) - (k_iN+1)$ is an even number; therefore

$$S^-(u_{k_iN+1},\ldots,u_{l_iN+1}) \leqslant (l_iN - k_iN) - 1, \qquad i = 0,1,\ldots,2m, \tag{1.8}$$

where by S^- we denote the number of strong sign changes in a sequence (see [13, Chapter 5]). Now set

$$\mathbf{v} := \left(u_{-m+k_0N+1}, \ldots, u_{m+l_{2m}N+1}\right),$$

$$B := \left(a_{ij}\right)_{k_0N+1 \leqslant i \leqslant l_{2m}N+1, -m+k_0N+1 \leqslant j \leqslant m+l_{2m}N+1}.$$

Then B is a totally positive $((l_{2m} - k_0)N + 1) \times ((l_{2m} - k_0)N + 2m + 1)$-matrix. The fact that $A\mathbf{u} = \varepsilon\mathbf{1}'$, together with (1.5), implies that

$$S^-(B\mathbf{v}) = (l_{2m} - k_0)N.$$

On the other hand, (1.8) gives

$$S^-(\mathbf{v}) \leqslant (l_{2m} - k_0)N - (2m + 1) + 2m = (l_{2m} - k_0)N - 1.$$

Since B is totally positive, it is variation-diminishing, namely

$$S^-(B\mathbf{v}) \leqslant S^-(\mathbf{v})$$

(see [13, Theorem 5.1.4]). This is a contradiction.

(ii) N is odd. In this case, $e^{i\theta} \neq \omega = -1$. If $e^{i\theta} \neq 1$ also, then θ/π is not an integer. By Lemma 1.1 there exist even positive integers k_i, l_i $(i = 0, \ldots, 2m)$ such that (1.6) and (1.7) hold. Thus the argument for case (i) is also valid for this case.

Now, assume $e^{i\theta} = 1$. Let

$$k_i = i, \quad l_i = (i + 1), \qquad i = 0, 1, \ldots, 2m.$$

Then $(l_iN + 1) - (k_iN + 1)$ is an odd number, but $\cos k_i\theta \cos l_i\theta > 0$; hence (1.8) is still true. Following the argument for case (i), we obtain the desired result.

It remains to treat the case $e^{i\theta} = \omega$. In this case, (1.4) becomes

$$y_{kN+j} = \omega^k y_j.$$

Thus we may assume that $\tilde{\mathbf{y}}$ is a real sequence. For any $\lambda \in \mathbf{R}$, $\mathbf{c} := \tilde{\mathbf{b}} + \lambda\tilde{\mathbf{y}}$ is a real sequence and satisfies $A\mathbf{c} = \mathbf{1}'$. Thus we have

$$c_j c_{j+1} < 0 \quad \text{for all } j.$$

(This can be proved, as done before, by truncating A to a finite matrix, invoking the variation-diminishing property for it and using the periodicity of $\mathbf{c}$.) But that is impossible unless $\tilde{\mathbf{y}} = \mathbf{0}$.

Finally, since $A\tilde{\mathbf{b}} = \mathbf{1}'$, we have $\|A^{-1}\| = \|\tilde{\mathbf{b}}\|_\infty$ (see [6]). Therefore

$$\|A^{-1}\| = \|\mathbf{b}\|_\infty.$$

The proof of Theorem 1.1 is complete.

2. Exponential Splines.

Definition 2.1. $S \in \mathbb{S}_{2k}$ *is called an exponential spline, if for some constant* $\lambda \in \mathbf{R}$,

$$S(qx) = \lambda S(x) \quad \text{for all } x \in I.$$

"The exponential splines", said Schoenberg in his elegant monograph [17], "will be used as thoroughly as the American Indians utilized the buffalo, to the last bone." The exponential splines also play an important role in this paper. By the above definition, every exponential spline is uniquely determined by its polynomial component in $[1, q]$. For reasons that will become clear later, we are particularly interested in the exponential splines

$$\varphi_j, \psi_j \qquad (j = 0, 1, \ldots, l - 1)$$

given by

$$\varphi_0(x) := x^k,$$

$$\varphi_j(x) := (-1)^{k-j+1} q^k [0,1,\ldots,k-1,k+1,\ldots,2k-j]_z \frac{x^z}{q^z - q^k}$$

for $x \in [1,q]$ and $j = 1,\ldots,l-1$,

$$\psi_j(x) := (-1)^{k-j+1} q^k [0,1,\ldots,2k-j-1]_z \frac{x^z}{q^z + q^k}$$

for $x \in [1,q]$ and $j = 0,\ldots,l-1$

(cf. [14]). Extend the domain of φ_j and ψ_j to $(0,\infty)$ by the following rule:

$$\varphi_j(q^m x) = q^{mk}\varphi_j(x), \qquad m \in \mathbf{Z},\ x \in [1,q],$$

$$\psi_j(q^m x) = (-q^k)^m \psi_j(x), \qquad m \in \mathbf{Z},\ x \in [1,q].$$

We have to verify that φ_j and ψ_j are in $\mathbb{S}_{2k}$. Obviously, φ_j and ψ_j are polynomials on each interval $[q^m, q^{m+1}]$. Thus it suffices to show that

$$q^p \varphi_j^{(p)}(q) = q^k \varphi_j^{(p)}(1), \qquad p = 0,1,\ldots,2k-2, \tag{2.1}$$

$$q^p \psi_j^{(p)}(q) = -q^k \psi_j^{(p)}(1), \qquad p = 0,1,\ldots,2k-2. \tag{2.2}$$

Indeed, for $j = 0$, (2.1) is trivial. For $1 \leqslant j \leqslant l-1$, we have

$$\begin{aligned} q^p \varphi_j^{(p)}(q) &= q^p (-1)^{k-j+1} q^k [0,1,\ldots,k-1,k+1,\ldots,2k-j]_z \\ &\quad \cdot \frac{z(z-1)\cdots(z-p+1)q^{z-p}}{q^z - q^k} \\ &= (-1)^{k-j+1} q^k [0,1,\ldots,k-1,k+1,\ldots,2k-j]_z \\ &\quad \left\{ z(z-1)\cdots(z-p+1)\left(1 + q^k \frac{1}{q^z - q^k}\right)\right\} \\ &= q^k \varphi_j^{(p)}(1). \end{aligned}$$

One can prove (2.2) in the same fashion.

We want to investigate some properties of these exponential splines. In this section, we mean by i the imaginary unit: $i = \sqrt{-1}$. Also we make the abbreviation $t := \log q$. We notice that the case $q \leqslant 1$ is symmetric to the case $q \geqslant 1$, so we need to treat the case $q \geqslant 1$ only. We assume also that $k \geqslant l+1$.

In the following, we use the abbreviations

$$h_{m,j}(z) := \prod_{\nu=1}^{k-m} 1/(\nu + iz) \prod_{\nu=1}^{k-j} 1/(\nu - iz), \qquad z_r := r\pi/t.$$

THEOREM 2.1. (i) *For* $x \in [1,q]$, $m = 0,1,\ldots,l-1$, *and* $j = 1,\ldots,l-1$,

$$\varphi_j^{(m)}(x) = \sum_{n\in\mathbf{Z}} \frac{1}{t} e^{(k-m+2n\pi i/t)\log x} h_{m,j}(z_{2n}).$$

(ii) *For* $x \in [1,q]$ *and* $m, j = 0,1,\ldots,l-1$,

$$\psi_j^{(m)}(x) = \sum_{n\in\mathbf{Z}} \frac{1}{itz_{2n+1}} e^{(k-m+(2n+1)\pi i/t)\log x} h_{m,j+1}(z_{2n+1}).$$

Proof. Suppose that Ω is a simply connected domain in the complex plane, and that C is a rectifiable Jordan closed curve in Ω. If f is analytic in the domain Ω, and if x is inside the curve C, then Cauchy's formula holds:

$$[x]f = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-x}\,dz. \tag{2.3}$$

More generally, we have the following well-known formula for divided differences (see [10, Chapter 1]):

$$[x_0, x_1, \ldots, x_n]f = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-x_0)\cdots(z-x_n)}\,dz, \tag{2.4}$$

as long as $x_0, \ldots, x_n$ are all inside C. This result may be generalized to an arbitrary domain (not necessarily simply connected). Let Ω be a domain, C a cycle in Ω (see [16, Chapter 10]). Thus C is a sum of closed paths,

$$C = \gamma_1 + \cdots + \gamma_n.$$

Each path is understood to be a continuous map from the unit circle to Ω. The image of a path γ is denoted by γ^*. Correspondingly,

$$C^* := \gamma_1^* + \cdots + \gamma_n^*.$$

For any $x \notin C^*$, the index of x with respect to C is defined by

$$\operatorname{Ind}_C(x) := \frac{1}{2\pi i}\int_C \frac{dz}{z-x}.$$

Suppose that $\operatorname{Ind}_C(\zeta) = 0$ for any ζ not in Ω. If f is analytic in Ω, and if $\operatorname{Ind}_C(x) = 1$, then Cauchy's formula (2.3) is still valid. More generally, if

$$\operatorname{Ind}_C(x_j) = 1, \qquad j = 0, \ldots, n,$$

then (2.4) is still true.

Now we want to use (2.4) to prove Theorem 2.1. Fix $x \in [1, q]$. Consider the function f defined by

$$f(z) := x^z/(q^z - q^k) = e^{z\log x}/(e^{zt} - e^{kt}).$$

Then f is an analytic function of z in $\mathbf{C} \setminus \{k + 2n\pi i/t;\ n \in \mathbf{Z}\}$. Let γ_n be the path defined by

$$\gamma_n\colon e^{i\theta} \mapsto k + 2n\pi i/t + e^{i\theta}/t, \qquad 0 \leqslant \theta \leqslant 2\pi\ (n \in \mathbf{Z}),$$

and δ_N the path given by

$$\delta_N\colon e^{i\theta} \mapsto k + ((2N+1)\pi/t)e^{i\theta}, \qquad 0 \leqslant \theta \leqslant 2\pi.$$

Let

$$C_N := \delta_N - \sum_{n=-N}^{N} \gamma_n.$$

Then by what has been proved, we have

$$[0, 1, \ldots, k-1, k+1, \ldots, 2k-j]f = \frac{1}{2\pi i}\int_{C_N} g(z)\,dz, \tag{2.5}$$

where

$$g(z) = f(z) \prod_{\substack{0 \leqslant \nu \leqslant 2k-j \\ \nu \neq k}} \frac{1}{z-\nu}.$$

We claim

$$\lim_{N\to\infty}\int_{\delta_N} g(z)\,dz = 0. \tag{2.6}$$

First, we observe that

$$\left|\prod_{\substack{0\leqslant\nu\leqslant 2k-j\\ \nu\neq k}}\frac{1}{z-\nu}\right| \leqslant \left(\frac{1}{N-2k}\right)^k \quad \text{for } z\in\delta_N^*.$$

Next, f is bounded on δ_N^*. This will be proved case by case. Let

$$z = k + e^{i\theta}(2N+1)\pi/t.$$

Then

$$(z-k)t = e^{i\theta}(2N+1)\pi = (2N+1)\pi(\cos\theta + i\sin\theta).$$

Case 1. $\cos\theta \geqslant 1/(4N+2)$. In this case, since

$$\left|e^{(z-k)t}\right| = e^{(2N+1)\pi\cos\theta} \geqslant 2,$$

one has

$$\left|e^{(z-k)t}-1\right| \geqslant e^{(2N+1)\pi\cos\theta}/2.$$

It follows that

$$|f(z)| \leqslant e^{k\log x+(2N+1)\pi\cos\theta(\log x/t)}\Big/\left(e^{kt+(2N+1)\pi\cos\theta}/2\right) \leqslant 2,$$

noting that $\log x/t \leqslant 1$.

Case 2. $\cos\theta \leqslant -1/(4N+2)$. In this case,

$$\left|e^{(z-k)t}-1\right| \geqslant 1 - e^{-\pi/2},$$

and

$$\left|e^{z\log x}\right| \leqslant e^{kt}.$$

Case 3. $|\cos\theta| \leqslant 1/(4N+2)$. In this case,

$$|\sin\theta| \geqslant 1-|\cos\theta| \geqslant 1 - 1/(4N+2),$$

hence

$$\operatorname{Im}((z-k)t) \in [2N\pi+\pi/2, 2N\pi+\pi] \text{ or } [-2N\pi-\pi, -2N\pi-\pi/2].$$

It follows that

$$\left|e^{(z-k)t}-1\right| \geqslant 1.$$

Moreover,

$$\left|e^{z\log x}\right| \leqslant e^{(k+\pi/2)t}.$$

This finishes the proof of (2.6).

Now let $N\to\infty$ in (2.5), and take account of (2.6). We obtain

$$[0,1,\ldots,k-1,k+1,\ldots,2k-j]f = -\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{\gamma_n} g(z)\,dz.$$

Each integral $\int_{\gamma_n} g(z)\,dz$ can be computed by residue calculus. The function g has only one pole $z = k + 2n\pi i/t$ inside γ_n. The residue of g at this point is

$$\frac{1}{q^k t}e^{(k+2n\pi i/t)\log x}\prod_{\substack{0\leqslant\nu\leqslant 2k-j\\ \nu\neq k}}\frac{1}{k+2n\pi i/t-\nu}.$$

Finally, we obtain

$$\varphi_j(x) = (-1)^{k-j+1} q^k [0,1,\ldots,k-1,k+1,\ldots,2k-j]_z \frac{x^z}{q^z - q^k}$$

$$= (-1)^{k-j} \sum_{n \in \mathbf{Z}} \frac{1}{t} e^{(k+2n\pi i/t)\log x} \prod_{\nu=0}^{k-1} \frac{1}{k-\nu+2n\pi i/t} \prod_{\nu=k+1}^{2k-j} \frac{1}{k-\nu+2n\pi i/t}$$

$$= \sum_{n \in \mathbf{Z}} \frac{1}{t} e^{(k+2n\pi i/t)\log x} \prod_{\nu=1}^{k} \frac{1}{\nu+2n\pi i/t} \prod_{\nu=1}^{k-j} \frac{1}{\nu-2n\pi i/t}.$$

Now (i) comes from this formula by differentiating m times. The proof of (ii) is similar. □

As a consequence of Theorem 2.1, the following relations hold:

$$\varphi_j^{(m)}(1) = \sum_{n \in \mathbf{Z}} \frac{1}{t} h_{m,j}(z_{2n}),$$

$$\psi_j^{(m)}(1) = \sum_{n \in \mathbf{Z}} \frac{1}{itz_{2n+1}} h_{m,j+1}(z_{2n+1}).$$

It follows from these two formulae that

$$\text{(2.7)} \qquad \begin{aligned} \lim_{q\to\infty} \varphi_j^{(m)}(1) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} h_{m,j}(z)\, dz, \\ \lim_{q\to\infty} \psi_j^{(m)}(1) &= \frac{1}{2\pi} \text{pr.v.} \int_{-\infty}^{\infty} h_{m,j+1}(z)/(iz)\, dz. \end{aligned}$$

(Here, pr.v. means the Cauchy principal value.) Indeed, since $(1/2\pi)\int_{-\infty}^{\infty} |h_{m,j}(z)|\, dz < \infty$, by the very definition of integration we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} h_{m,j}(z)\, dz = \lim_{t\to\infty} \frac{1}{2\pi} \sum_{n \in \mathbf{Z}} h_{m,j}(z_{2n})(z_{2n} - z_{2n-2})$$

$$= \lim_{t\to\infty} \sum_{n \in \mathbf{Z}} h_{m,j}(z_{2n})/t = \lim_{q\to\infty} \varphi_j^{(m)}(1).$$

The proof of the second relation in (2.7) is similar.

Now set

$$\Phi(q) := \left(\varphi_j^{(m)}(1)\right)_{m,j=0}^{l-1} \quad \text{and} \quad \Psi(q) := \left(\psi_j^{(m)}(1)\right)_{m,j=0}^{l-1}.$$

Theorem 2.2. *The matrices $\Phi(q)$ and $\Psi(q)$ are invertible for $1 < q < \infty$. Moreover,*

$$\Phi(\infty) := \lim_{q\to\infty} \Phi(q), \qquad \Psi(\infty) := \lim_{q\to\infty} \Psi(q)$$

both exist and are invertible.

Proof. We observe that

$$\varphi_0^{(m+1)}(1)/\varphi_0^{(m)}(1) = k - m, \qquad h_{m+1,j}(z)/h_{m,j}(z) = k - m + zi/t.$$

Thus we subtract $(k-m)$ times row m from row $m+1$ in $\Phi(q)$ $(m = l-2, \ldots, 0)$. We obtain

$$\det \Phi = \begin{vmatrix} 1 & \varphi_1 & \cdots & \varphi_{l-1} \\ 0 & \varphi_1' - k\varphi_1 & \cdots & \varphi_{l-1}' - k\varphi_{l-1} \\ \vdots & & & \vdots \\ 0 & \varphi_1^{(l-1)} - (k-l+2)\varphi_1^{(l-2)} & \cdots & \varphi_{l-1}^{(l-1)} - (k-l+2)\varphi_{l-1}^{(l-2)} \end{vmatrix},$$

all functions evaluated at 1. We have

$$\varphi_j^{(m+1)}(1) - (k-m)\varphi_j^{(m)}(1) = \sum_{n\in\mathbf{Z}} \left[h_{m+1,j}(z_{2n}) - h_{m,j}(z_{2n})\right]/t$$
$$= \sum_{n\in\mathbf{Z}} iz_{2n}h_{m,j}(z_{2n})/t.$$

We notice that Φ is a real matrix. If Φ is singular, then there exist *real* numbers $\beta_1, \dots, \beta_{l-1}$, not all zero, such that

$$\sum_{j=1}^{l-1} \beta_j \sum_{n\in\mathbf{Z}} iz_{2n}h_{m,j}(z_{2n})/t = 0, \qquad m = 1, \dots, l-1;$$

or

$$(2.8) \qquad \sum_{n\in\mathbf{Z}} iz_{2n}h_{0,1}(z_{2n})\left[\sum_{j=1}^{l-1} \beta_j \prod_{\nu=k-j+1}^{k-1} (\nu - iz_{2n})\right] \prod_{\nu=k-m+2}^{k} (\nu + iz_{2n}) = 0.$$

Let

$$(2.9) \qquad \begin{aligned} p(u) &:= \sum_{j=1}^{l-1} \beta_j \prod_{\nu=k-j+1}^{k-1} (\nu + u), \\ g_m(u) &:= \prod_{\nu=k-m+2}^{k} (\nu + u), \qquad m = 1, \dots, l-1. \end{aligned}$$

Then p and g_m are all real polynomials, and (2.8) becomes

$$(2.10) \qquad \sum_{n\in\mathbf{Z}} iz_{2n}h_{0,1}(z_{2n})p(-iz_{2n})g_m(iz_{2n}) = 0.$$

We denote by $\mathbf{P}_n$ the linear space of all polynomials of degree less than n. Since g_m has the leading term u^{m-1} $(m = 1, \dots, l)$, $\{g_1, \dots, g_{l-1}\}$ forms a basis for $\mathbf{P}_{l-1}$. Since $p \in \mathbf{P}_{l-1}$, we can find *real* numbers $\alpha_1, \dots, \alpha_{l-1}$ such that

$$(2.11) \qquad p = \sum_{m=1}^{l-1} \alpha_m g_m.$$

It follows from (2.10) that

$$\sum_{m=1}^{l-1} \alpha_m \sum_{n\in\mathbf{Z}} iz_{2n}h_{0,1}(z_{2n})p(-iz_{2n})g_m(iz_{2n}) = 0,$$

or, by interchanging the order of summation,

$$(2.12) \qquad \sum_{n\in\mathbf{Z}} iz_{2n}h_{0,1}(z_{2n})p(-iz_{2n}) \sum_{m=1}^{l-1} \alpha_m g_m(iz_{2n}) = 0.$$

Now (2.11) tells us that

$$p(-iz_{2n}) \sum_{m=1}^{l-1} \alpha_m g_m(iz_{2n}) = p(-iz_{2n})p(iz_{2n}) = |p(iz_{2n})|^2.$$

Moreover,

$$iz_{2n}h_{0,1}(z_{2n}) = iz_{2n}(k - iz_{2n})h_{0,0}(z_{2n}),$$

and

$$\mathrm{Re}(iz_{2n}(k - iz_{2n})) = |z_{2n}|^2,$$

while

$$h_{0,0}(z_{2n}) = \prod_{\nu=1}^{k} 1/(\nu^2 + z_{2n}^2) > 0.$$

Thus (2.12) yields

$$\sum_{n\in\mathbf{Z}} |p(iz_{2n})|^2 |z_{2n}|^2 h_{0,0}(z_{2n}) = 0,$$

and, therefore,

$$p(iz_{2n}) = 0 \quad \text{for all } n \in \mathbf{Z}.$$

Since p is a polynomial, this implies that $p = 0$. Recalling (2.9), we conclude that

$$\beta_j = 0 \quad \text{for all } j = 1,\dots,l-1.$$

This shows that Φ is invertible. We can prove in the same fashion, and even more easily, that Ψ is also invertible.

As to $\Phi(\infty)$ and $\Psi(\infty)$, the series appearing in the definition of φ_j and ψ_j are replaced by integrals, but the above proof is readily translated to this case. The proof of Theorem 2.2 is complete.

3. Divided Differences and Derivatives. For simplicity, we write

$$\varphi_j := \psi_{j-l}, \qquad j = l,\dots,2l-1.$$

Let S be the span of $\varphi_0,\dots,\varphi_{2l-1}$. Then, by (2.1) and (2.2), we have

$$f(q^2x) = q^{2k}f(x) \quad \text{all } f \in S.$$

Recall that $(x_j)_{j\in\mathbf{Z}}$ is an l-multiple geometric mesh. Hence, $x_{j+2l} = q^2x_j$.

We use the following abbreviation,

$$\lambda_{j,m} := [x_j,\dots,x_{j+m}],$$

for the mth divided difference linear functional. According to the standard convention,

$$\lambda_{j,m} = [x_j]D^m \quad \text{in case } x_j = \cdots = x_{j+m},$$

where $D = d/dx$ is differentiation. By standard properties of divided differences,

$$\lambda_{j,m+1} = (\lambda_{j+1,m} - \lambda_{j,m})/(x_{j+1+m} - x_j) \quad \text{if } x_{j+1+m} > x_j, \tag{3.1}$$

$$\lambda_{j+1,m} = \lambda_{j,m} \quad \text{if } x_{j+1+m} = x_j. \tag{3.2}$$

Now we restrict $\lambda_{j,m}$ to the space S. Then one verifies that

$$\lambda_{2l,m} = q^{2k-2m}\lambda_{0,m}. \tag{3.3}$$

Let Λ_m be the cone generated by the $\lambda_{j,m}$, i.e.,

$$\Lambda_m := \left\{ \sum_{j=0}^{2l-1} a_j\lambda_{j,m};\ a_j \geqslant 0 \right\}.$$

We claim that

$$\Lambda_m \subset \Lambda_{m+1}. \tag{3.4}$$

Indeed, (3.1) and (3.2) tell us that

$$\lambda_{j+1,m} - \lambda_{j,m} \in \Lambda_{m+1}, \qquad j = 0, \ldots, 2l-1.$$

Therefore,

$$\lambda_{2l,m} - \lambda_{0,m} = \sum_{j=0}^{2l-1} \left(\lambda_{j+1,m} - \lambda_{j,m}\right) \in \Lambda_{m+1}.$$

From (3.3), we have $\lambda_{2l,m} = q^{2k-2m}\lambda_{0,m}$, hence $(q^{2k-2m} - 1)\lambda_{0,m} \in \Lambda_{m+1}$. This shows that $\lambda_{0,m} \in \Lambda_{m+1}$, and therefore

$$\lambda_{j,m} = \lambda_{0,m} + \sum_{i=0}^{j-1} \left(\lambda_{i+1,m} - \lambda_{i,m}\right) \in \Lambda_{m+1}.$$

Thus (3.4) is proved. Set

$$\begin{aligned} &\mu_i := \lambda_{0,i} \\ &\mu_{i+l} := \lambda_{l,i}/q^{k-i} \end{aligned} \quad \text{for } 0 \leqslant i < l. \tag{3.5}$$

Then, as a consequence of (3.4), we have

$$\mu_i \in \Lambda_k \quad \text{for } i = 0, \ldots, 2l-1.$$

We have already proved the first part of the following theorem.

THEOREM 3.1. *There exists a nonnegative matrix*

$$C := \left(c_{ij}\right)_{i,j=0}^{2l-1}$$

such that

$$\mu_i = \sum_{j=0}^{2l-1} c_{ij}\lambda_{j,k}.$$

Moreover, C is invertible, and C is bounded independently of q,

$$|c_{ij}| \leqslant \text{const}_k \quad \textit{for all } i, j.$$

Proof. To prove the second part of this theorem, we let μ_i act on the function $\varphi_0 \in S$. Then

$$\mu_i\varphi_0 = \sum_j c_{ij}\left(\lambda_{j,k}\varphi_0\right).$$

We observe that

$$\lambda_{j,k}\varphi_0 = 1 \quad \text{for all } j$$

while

$$\mu_i\varphi_0 = \mu_{i+l}\varphi_0 = k(k-1)\cdots(k-l+1) \quad \text{for } i = 0, \ldots, l-1.$$

Therefore,

$$\sum_j c_{ij} \leqslant \text{const}_k.$$

Since each c_{ij} is nonnegative, this implies that

$$c_{ij} \leqslant \text{const}_k \quad \text{for all } i, j.$$

We observe in the next section (see (4.7)) that $(\mu_i \varphi_j)$ is invertible. This shows that (μ_i) is linearly independent over S, hence C is invertible. With this, the proof of Theorem 3.1 is complete.

4. L-Boundedness of L_2-Projections on Splines for a Multiple Geometric Mesh. We are now in a position to prove the main result of this paper.

THEOREM 4.1. *Let* $\mathbf{x} = (x_i)_{-\infty}^{\infty}$ *be the mesh given by*

$$x_{li} = x_{li+1} = \cdots = x_{li+l-1} = q^i \quad \text{all } i, 0 < q < \infty,$$

where $l \in \mathbf{N}$ *is the multiplicity of the mesh. Let*

$$A := \left(\int M_{i,k} N_{j,k} \right)_{i,j \in \mathbf{Z}}.$$

Then

$$\sup_{0<q<\infty} \|A^{-1}\|_\infty \leqslant \text{const}_k.$$

Proof. When $l = k$, $\mathbb{S}_k$ becomes the space of all piecewise polynomials with breakpoints x_i, $i \in \mathbf{Z}$, so this case is trivial. Thus we may assume $l < k$.

We view A as a $(2l \times 2l)$-block Toeplitz matrix. Set

$$v_{p,j} := \sum_{m \in \mathbf{Z}} A(p, j + 2ml), \qquad p, j = 0, 1, \ldots, 2l - 1,$$

and

$$V := (v_{p,j})_{p,j=0}^{2l-1}.$$

Then V is just $\hat{A}(1)$. If $\mathbf{b} \in R^{2l-1}$ is a solution to the equation

$$V\mathbf{b} = -\mathbf{1}', \tag{4.1}$$

then, by Theorem 1.1,

$$\|A^{-1}\|_\infty = \|\mathbf{b}\|_\infty. \tag{4.2}$$

Here is our scheme for the proof of this theorem: As mentioned before, we need only to consider the case $1 \leqslant q < \infty$. First, we show that (4.1) has a solution for any $q \in (1, \infty)$. Next, we investigate the uniform boundedness of $\mathbf{b}$ as $q \to \infty$. Finally, we consider the behavior of $\mathbf{b}$ near $q = 1$.

We need the following lemma.

LEMMA 4.1. *If* $f \in S$ *satisfies*

$$[x_i, \ldots, x_{i+k}]f = (-1)^i/k! \quad \textit{for } i = 0, \ldots, 2l - 1, \tag{4.3}$$

then (4.1) *has a solution* $\mathbf{b}$ *satisfying*

$$\|\mathbf{b}\|_\infty \leqslant \text{const}_k \|f^{(k)}\|_\infty.$$

(Recall that S is the span of $\varphi_0, \ldots, \varphi_{2l-1}$.)

Proof. By Peano's Theorem,

$$\int M_i f^{(k)} = k![x_i, \ldots, x_{i+k}]f \tag{4.4}$$

(see [2]). Since $f^{(k)} \in \mathbb{S}_k$, it can be uniquely expanded in a B-spline series:

$$f^{(k)} = \sum_{j \in \mathbf{Z}} b_j N_{j,k}$$

with coefficient vector $\tilde{\mathbf{b}} := (b_j)_{j \in \mathbf{Z}}$. By (4.3) and (4.4),

$$A\tilde{\mathbf{b}} = -\mathbf{1}'. \tag{4.5}$$

We claim that

$$b_{j+2l} = b_j \quad \text{for any } j \in \mathbf{Z}.$$

Since $f \in S$, $f(q^2x) = q^{2k}f(x)$, and therefore $f^{(k)}(q^2x) = f^{(k)}(x)$. This fact gives

$$\sum b_j N_j(q^2x) = \sum b_j N_j.$$

But $N_j(q^2x) = N_{j-2l}(x)$, hence

$$\sum b_j N_{j-2l} = \sum b_j N_j \quad \text{or} \quad \sum b_{j+2l} N_j = \sum b_j N_j.$$

This implies that

$$b_{j+2l} = b_j \quad \text{all } j,$$

by the linear independence of $\{N_j;\ j \in \mathbf{Z}\}$ (see [2]). Therefore, with (4.5), the vector $\mathbf{b} := (b_j)_{j=0}^{2l-1}$ solves (4.1). Moreover, $\|\mathbf{b}\|_\infty \leqslant \text{const}_k \|f^{(k)}\|_\infty$ (see [2]). The proof of Lemma 4.1 is complete. □

Back to the proof of Theorem 4.1. We want to find $f \in S$ such that (4.3) holds. Since S is the span of $\varphi_0, \ldots, \varphi_{2l-1}$, there exists $\mathbf{a} = (a_0, \ldots, a_{2l-1})$ such that $f = \sum a_j \varphi_j$. Hence,

$$\mu_i f = \sum (\mu_i \varphi_j) a_j.$$

On the other hand, Theorem 3.1 tells us that

$$\mu_i f = \sum_j c_{ij} \lambda_{j,k} f.$$

Therefore,

$$\sum_j (\mu_i \varphi_j) a_j = \sum_j c_{ij} (\lambda_{j,k} f).$$

Since $C = (c_{ij})$ is an invertible matrix, (4.3) is equivalent to

$$\sum_j (\mu_i \varphi_j) a_j = \sum_j c_{ij} (-1)^j / k!. \tag{4.6}$$

We have to take a closer look at the matrix $(\mu_i \varphi_j)$. By (3.5), we have

$$\mu_i \varphi_j = \begin{cases} \varphi_j^{(i)}(1) & \text{for } 0 \leqslant i, j < l, \\ \psi_j^{(i)}(1) & \text{for } 0 \leqslant i < l \leqslant j < 2l, \\ \varphi_j^{(i-l)}(1) & \text{for } l \leqslant i < 2l;\ 0 \leqslant j < l, \\ -\psi_j^{(i-l)}(1) & \text{for } l \leqslant i, j < 2l. \end{cases}$$

Therefore, recalling the definition of Φ and Ψ, we obtain

$$(\mu_i \varphi_j) = \begin{bmatrix} \Phi & \Psi \\ \Phi & -\Psi \end{bmatrix}. \tag{4.7}$$

Since Φ and Ψ are invertible, so is the matrix

$$\begin{bmatrix} \Phi & \Psi \\ \Phi & -\Psi \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} \Phi & \Psi \\ 0 & -2\Psi \end{bmatrix}.$$

(Here, I is the $l \times l$ identity matrix.) Moreover, since $\Phi(\infty)$ and $\Psi(\infty)$ are invertible,

$$\limsup_{q \to \infty} \left\| (\mu_i \varphi_j)^{-1} \right\|_\infty \leqslant \text{const}_k. \tag{4.8}$$

Since the matrix $(\mu_i \varphi_j)$ is invertible, we can find $\mathbf{a}$ so that (4.6) holds. Moreover, using Theorem 3.1,

$$\|\mathbf{a}\|_\infty \leqslant \text{const}_k \left\| (\mu_i \varphi_j)^{-1} \right\|_\infty. \tag{4.9}$$

For such $\mathbf{a}$, the function $f := \sum_j a_j \varphi_j$ satisfies (4.3). Thus, by Lemma 4.1, we have already proved that (4.1) has a solution $\mathbf{b}$ for any $q \in (1, \infty)$.

Now we want to prove that

$$\limsup_{q \to \infty} \| f^{(k)} \|_\infty \leqslant \text{const}_k. \tag{4.10}$$

By (4.8) and (4.9),

$$\limsup_{q \to \infty} \|\mathbf{a}\|_\infty \leqslant \text{const}_k.$$

Hence (4.10) is true, once we show that

$$\limsup_{q \to \infty} \left\| \varphi_j^{(k)} \right\|_\infty \leqslant \text{const}_k. \tag{4.11}$$

Appealing to Theorem 2.1, we have

$$\left\| \varphi_j^{(k)} \right\|_\infty \leqslant \sum_{n \in \mathbf{Z}} \left[\prod_{\nu=1}^{k-j} \left(\nu^2 + 4n^2\pi^2/t^2\right)^{-1/2} \right] \Big/ t \quad \text{for } 1 \leqslant j \leqslant l-1.$$

Recall the abbreviation $t = \log q$. Since $t \to \infty$ when $q \to \infty$, we have, by the very definition of the Riemann integral, that

$$\lim_{q \to \infty} \sum_{n \in \mathbf{Z}} \left[\prod_{\nu=1}^{k-j} \left(\nu^2 + 4n^2\pi^2/t^2\right)^{-1/2} \right] \Big/ t$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{\nu=1}^{k-j} \left(\nu^2 + z^2\right)^{-1/2} dz \leqslant \text{const}_k.$$

This proves (4.11) for $j = 1, \ldots, l-1$. For $l \leqslant j \leqslant 2l-1$, the proof is similar. The case $j = 0$ is trivial. With this, it follows from (4.2), (4.4), and (4.10) that

$$\limsup_{q \to \infty} \| A^{-1} \|_\infty \leqslant \text{const}_k. \tag{4.12}$$

Finally, we consider the behavior of $\mathbf{b}$ near $q = 1$. It is known (see [1]) that, for $q = 1$, A^{-1} is bounded. Therefore, V is invertible for $q = 1$. Since each entry of V is a continuous function of q, $\mathbf{b}$ is continuous in q near $q = 1$. This shows that

$$\lim_{q \to 1} \|\mathbf{b}\|_\infty \leqslant \text{const}_k.$$

Invoking (4.2) once again, we obtain

$$\lim_{q \to 1} \| A^{-1} \|_\infty \leqslant \text{const}_k.$$

The proof of Theorem 4.1 is now complete.

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