MATHEMATICS OF COMPUTATION
VOLUME 49, NUMBER 179
JULY 1987, PAGES 17-23

Quasi-Optimal Estimates for Finite Element Approximations Using Orlicz Norms*

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Abstract. We consider the approximation by linear finite elements of the solution of the Dirichlet problem $-\Delta u = f$. We obtain a relation between the error in the infinite norm and the error in some Orlicz spaces. As a consequence, we get quasi-optimal uniform estimates when u has second derivatives in the Orlicz space associated with the exponential function. This estimate contains, in particular, the case where f belongs to L^∞ and the boundary of the domain is regular. We also show that optimal order estimates are valid for the error in this Orlicz space provided that u be regular enough.

1. Introduction. Consider the problem of finding u such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain contained in R^n and f is a given function.

We shall use standard notation for the Sobolev spaces $W_p^k(\Omega)$ and $H^k(\Omega) = W_2^k$ with the norms

$$\|f\|_{k,p,\Omega} = \sum_{j \leqslant k} |f|_{j,p,\Omega},$$

where

$$|f|_{j,p,\Omega} = \sum_{|\alpha| = j} \|D^\alpha f\|_{L^p(\Omega)}.$$

We shall write $\|f\|_{k,p} = \|f\|_{k,p,\Omega}$ and $|f|_{k,p} = |f|_{k,p,\Omega}$ when there is no confusion.

The letter C will denote a constant, not necessarily the same at each occurrence.

For simplicity we will consider Ω to be a convex polyhedral domain, but the results are valid in more general domains as in [9].

Let $\{\mathscr{T}_h\}$ be a quasi-regular family of triangulations of Ω and denote by u_h the H_0^1-projection of u into the space of piecewise linear functions $M_h \subset H_0^1$, that is,

$$\int_\Omega \nabla u_h \nabla v_h \, dx = \int_\Omega f v \, dx, \qquad v_h \in M_h.$$

It is well known (see [1]) that

$$|u - u_h|_{0,2} \leqslant Ch^2 |u|_{2,2} \quad \text{and} \quad |u - u_h|_{1,2} \leqslant Ch |u|_{2,2}.$$

Received September 16, 1985; revised May 15, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 65N30.
*Supported by Consejo Nacional de Investigaciones Científicas y Técnicas de Argentina.

Many authors have studied estimates for $u - u_h$ in W_p^1-norms and L^p-norms. In [8] the following optimal estimate for the gradient of the error in L^p is obtained,

$$|u - u_h|_{1,p} \leqslant Ch\|u\|_{2,p} \quad \text{for } 1 < p \leqslant \infty.$$

Then by the usual duality argument (see [1]) they get

$$|u - u_h|_{0,p} \leqslant Ch^2\|u\|_{2,p} \quad \text{for } 2 \leqslant p < \infty,$$

provided that Ω is a convex polygonal domain or $\partial\Omega$ is smooth.

As is known, this duality argument cannot be applied for $p = \infty$.

A quasi-optimal estimate for the error in L^∞ was obtained in [9], where it is proved that

$$|u - u_h|_{0,\infty} \leqslant Ch^2 \log\frac{1}{h}\|u\|_{2,\infty}.$$

Moreover, in [4] an example is given that shows that the logarithm in this estimate cannot be removed.

We will work here with Orlicz spaces defined in the following way. Given a convex function ϕ: $R_+ \to R_+$, $\phi(0) = 0$, let

$$L^\phi(\Omega) = \left\{ f \,|\, \exists b > 0 \,|\, \int_\Omega \phi\left(\frac{|f|}{b}\right) dx < \infty \right\}.$$

L^ϕ is a Banach space with the norm

$$\|f\|_{L^\phi} = \inf\left\{ b > 0 \,|\, \int_\Omega \phi\left(\frac{|f(x)|}{b}\right) dx \leqslant 1 \right\}.$$

We will call W_ϕ^k the space of functions in L^ϕ with derivatives up to the order k in L^ϕ, and we will use analogous notation as in the L^p case for the norms and seminorms.

When the boundary of Ω is regular and $1 < p < \infty$ [3],

$$\|u\|_{2,p} \leqslant C|f|_{0,p},$$

and consequently,

$$|u - u_h|_{0,p} \leqslant Ch^2|f|_{0,p}.$$

As is well known, the regularity result mentioned above is not true for $p = \infty$, but if $f \in L^\infty$ the solution $u \in W_{\phi_1}^2$, where $\phi_1(t) = e^t - t - 1$. Moreover, the second derivatives of u are in the space of functions with bounded mean oscillation BMO (same proof as in the L^p case [3], using the result of [6]) and this space is contained in L^{ϕ_1} when the domain is bounded, [5]. Then it is natural to seek an estimate for $|u - u_h|_{0,\infty}$ when u has second derivatives in L^{ϕ_1}.

In this paper we obtain a relation between the error in L^∞ and the error in some Orlicz spaces that implies in particular the following quasi-optimal estimate,

$$|u - u_h|_{0,\infty} < Ch^2\left(\log\frac{1}{h}\right)^2\|u\|_{2,\phi_1}.$$

This estimate contains as a particular case the following one proved in [9],

$$|u - u_h|_{0,\infty} < Ch^2\left(\log\frac{1}{h}\right)^2|f|_{0,\infty}.$$

A similar estimate was obtained also in [7] but with a higher power of the logarithm and with the BMO norm of the second derivatives in the right-hand side.

Our result is more general because BMO is strictly contained in L^{ϕ_1} (for example, in $\Omega = (-1,1)$ the function

$$f(x) = \begin{cases} \log x, & x > 0, \\ 0, & x < 0, \end{cases}$$

is in L^{ϕ_1} but not in BMO).

Error estimates for problems where u has other kinds of singularities can be obtained by our theorem. As examples, consider $\Omega = \{x \in R^2 \,|\, |x| < 1/e\}$ and

$$u(x) = |x|^2 \left(\log \frac{1}{|x|}\right)^{1/n} - 1/e^2, \qquad n \in N.$$

In this case, $D^\alpha u \in L^\phi(\Omega)$ for $|\alpha| = 2$, where $\phi(t) = e^{t^n} - t^n - 1$, and then we will get the following estimate,

$$|u - u_h|_{0,\infty} \leqslant Ch^2 \left(\log \frac{1}{h}\right)^{1+1/n} \|u\|_{2,\phi}.$$

Finally, we show in the two-dimensional case that

$$|u - u_h|_{0,\phi_1} \leqslant Ch^2 \|u\|_{2,\infty},$$

provided that $\partial\Omega$ is smooth or Ω is a Lipschitz convex domain. In this way we show that the logarithm factor can be removed if we replace the L^∞-norm on the left by a slightly weaker Orlicz norm.

2. Error Estimates.

LEMMA 1. *If $v \in M_h$ the following inverse inequality holds,*

$$(1) \qquad |v|_{0,\infty} \leqslant C\phi^{-1}(1/h^n)|v|_{0,\phi}.$$

Proof. Let $T \in \mathcal{T}_h$ such that $|v|_{0,\infty,T} = |v|_{0,\infty}$. By usual scaling arguments one can see that

$$|v|_{0,\infty,T} \leqslant C(1/h^n) \int_T |v(x)|\, dx.$$

Let ψ be the complementary function of ϕ; then we can apply the Hölder inequality for Orlicz spaces, and we have

$$(2) \qquad |v|_{0,\infty,T} \leqslant C(1/h^n)|v|_{0,\phi}|\chi|_{0,\psi},$$

where χ is the characteristic function of T. But $|\chi|_{0,\psi} = b$, where b satisfies

$$\int_T \psi(1/b)\, dx = 1,$$

so $b = 1/\psi^{-1}(1/|T|)$ and then, using the inequality $t \leqslant \phi^{-1}(t)\psi^{-1}(t)$, we get

$$(3) \qquad b \leqslant |T|\phi^{-1}(1/|T|) \leqslant Ch^n\phi^{-1}(1/h^n),$$

and (2) and (3) imply (1). □

The result of the following lemma is proved in [2] but we give here a more direct proof.

LEMMA 2. *Let g be a continuous function such that $\partial g/\partial x_j \in L^\Phi(Q)$, where $Q \subset R^n$ is an open set with Lipschitz boundary. Assume that*

$$\mu(t) = \int_0^t \phi^{-1}(1/s^n)\, ds$$

is finite. Then,

$$|g(x+y) - g(x)| \leqslant C|g|_{1,\phi,Q}\,\mu(|y|). \tag{4}$$

Proof. Taking an extension, we can assume that g is in $W^1_\phi(R^n)$. Let $\eta \in C_0^\infty$ such that $\int \eta = 1$ and $0 \leqslant \eta(x) \leqslant 1$, $\eta_t(x) = t^{-n}\eta(x/t)$ and $v(x,t) = g * \eta_t(x)$; then

$$(\partial v/\partial x_j)(x,t) = \int (\partial g/\partial x_j)(y)\eta_t(x-y)\, dy,$$

and applying the Hölder inequality, we have

$$\left|(\partial v/\partial x_j)(x,t)\right| \leqslant 2|\partial g/\partial x_j|_{0,\phi}|\eta_t|_{0,\psi}. \tag{5}$$

Set $b = t^{-n}/\psi^{-1}(t^{-n})$; then, since $\eta(x/t) \leqslant 1$ and ψ is convex, we have

$$\int \psi(t^{-n}\eta(x/t)/b)\, dx = \int \psi(\psi^{-1}(t^{-n})\eta(x/t))\, dx \leqslant \int \eta(x/t)t^{-n}\, dx = 1.$$

Consequently,

$$|\eta_t|_{0,\psi} \leqslant t^{-n}/\psi^{-1}(t^{-n}) \leqslant \phi^{-1}(t^{-n}),$$

and by (5),

$$\left|(\partial v/\partial x_j)(x,t)\right| \leqslant 2|\partial g/\partial x_j|_{0,\phi}\phi^{-1}(t^{-n}).$$

A similar estimate for $\partial v/\partial t$ can be obtained in the following way. First observe that

$$\partial \eta_t/\partial t = -\sum_{i=1}^n \partial(x_i\eta)_t/\partial x_i;$$

then,

$$(\partial v/\partial t)(x,t) = (g * \partial\eta_t/\partial t)(x) = -\sum_{i=1}^n (g * \partial(x_i\eta)_t/\partial x_i)$$
$$= -\sum_{i=1}^n \partial g/\partial x_i * (x_i\eta)_t,$$

and now we are in the same situation as before, with η replaced by $x_i\eta$. In the same way we can prove that

$$|(x_i\eta)_t|_{0,\psi} \leqslant \phi^{-1}(t^{-n})\max\{\|x_i\eta\|_{L^1}, \|x_i\eta\|_{L^\infty}\}$$

and then,

$$|(\partial v/\partial t)(x,t)| \leqslant C|g|_{1,\phi}\phi^{-1}(t^{-n}),$$

where C depends on η.

Now (4) follows easily, writing

$$g(x+y)-g(x)=\left[g(x+y)-v(x+y,|y|)\right]+\left[v(x+y,|y|)-v(x,|y|)\right]$$
$$+\left[v(x,|y|)-g(x)\right]$$

and estimating each summand separately. □

Now we restrict ourselves to functions of the form $\phi(t)=\sum_{j=2}^{\infty}a_jt^j$ with $a_j\geqslant 0$, because our main example is of this form. For this class of functions it is easy to prove results about the error for Lagrange interpolation in the ϕ-norm. In fact, using the known estimates for L^p-norms and the series expansion of ϕ, we get the following result,

$$|u-I_hu|_{j,\phi}\leqslant Ch^{2-j}\|u\|_{2,\phi},\qquad j=0,1,$$

where I_hu is the Lagrange interpolation of u. Then we can state the following corollary of Lemma 2.

COROLLARY 1. *Let* $\phi(t)=\sum_{j=2}^{\infty}a_jt^j$, $a_j\geqslant 0$, *be an Orlicz function; then*

$$|u-I_hu|_{0,\infty}\leqslant Ch\mu(h)\|u\|_{2,\phi}.$$

We can now give a theorem which compares the error in L^∞- and L^ϕ-norms.

THEOREM 1. *If* ϕ *satisfies the condition of Corollary* 1 *and* μ *is the function associated with* ϕ *in Lemma* 2, *then there exists a constant* C *such that*

$$|u-u_h|_{0,\infty}\leqslant Ch\mu(h)\left[\|u\|_{2,\phi}+\frac{|u-u_h|_{0,\phi}}{h^2}\right].$$

Proof. By Lemma 1 and Corollary 1 we have

$$\begin{aligned}|u-u_h|_{0,\infty}&\leqslant|u-I_hu|_{0,\infty}+|I_hu-u_h|_{0,\infty}\\&\leqslant C\left[h\mu(h)\|u\|_{2,\phi}+\phi^{-1}(1/h^n)|I_hu-u_h|_{0,\phi}\right].\end{aligned}$$

But $|I_hu-u|_{0,\phi}\leqslant Ch^2\|u\|_{2,\phi}$ and then,

$$|u-u_h|_{0,\infty}\leqslant C\left[h\mu(h)\|u\|_{2,\phi}+h^2\phi^{-1}(h^{-n})\|u\|_{2,\phi}+\phi^{-1}(h^{-n})|u-u_h|_{0,\phi}\right].$$

Noting that $h\phi^{-1}(h^{-n})\leqslant\mu(h)$, we obtain the result. □

COROLLARY 2. *There exists a constant* C *such that*

$$|u-u_h|_{0,\infty}\leqslant Ch(\log h^{-1})\mu(h)\|u\|_{2,\phi}\tag{6}$$

and, in particular,

$$|u-u_h|_{0,\infty}<Ch^2(\log h^{-1})^2\|u\|_{2,\phi_1}.\tag{7}$$

Proof. By the known estimates [9], [1]

$$|u-u_h|_{0,\infty}\leqslant Ch^2\log h^{-1}\|u\|_{2,\infty}\quad\text{and}\quad|u-u_h|_{0,2}\leqslant Ch^2\|u\|_{2,2}$$

we get by interpolation

$$|u-u_h|_{0,p}\leqslant Ch^2\log h^{-1}\|u\|_{2,p}\quad\text{for } 2\leqslant p<\infty,$$

with C independent of p. Using the expansion in power series of ϕ, we get

$$|u-u_h|_{0,\phi}<Ch^2\log h^{-1}\|u\|_{2,\phi},$$

hence, by Theorem 1, we get (6).

When $\phi = \phi_1$ it is easily shown that $\mu_1(h) \leqslant Ch \log h^{-1}$ for small h and this proves (7). □

We will show in the following theorem that as a consequence of the estimates for $|u - u_h|_{1,\infty}$ [8] we have optimal-order estimates in the ϕ_1-norm if $u \in W_\infty^2$.

THEOREM 2. *Let $\Omega \subset R^2$ be such that $\partial\Omega$ is smooth or Ω is convex with Lipschitz boundary. Then there exists a constant C such that*

$$|u - u_h|_{0,\phi_1} \leqslant Ch^2 \|u\|_{2,\infty}.$$

Proof. In [8] it is proved that

$$|u - u_h|_{1,p} \leqslant Ch\|u\|_{2,p}, \qquad 2 \leqslant p \leqslant \infty.$$

On the other hand, if $v \in H_0^1(\Omega)$ and $-\Delta v = g$,

$$\|v\|_{2,q} \leqslant \frac{C}{q-1}\|g\|_q \quad \text{for } 1 < q \leqslant 2. \tag{8}$$

In fact, if Ω has a smooth boundary (for instance $C^{1,1}$), (8) can be shown by the classical proof [3], examining carefully the constants involved. In the case of a Lipschitz convex domain this result was proven recently by T. Wolff in unpublished work. Indeed, he has proven a weak type inequality for L^1 that, together with the known result for $q = 2$, implies (8) by usual interpolation methods.

By the known duality argument of Aubin-Nitsche [1], and using (8), we get

$$|u - u_h|_{0,p} \leqslant Cph^2\|u\|_{2,p}, \qquad 2 \leqslant p < \infty,$$

with C independent of p.

But, in general, if we have two functions g_1 and g_2 such that

$$|g_1|_{0,p} \leqslant C_1 p |g_2|_{0,p}, \qquad 2 \leqslant p < \infty,$$

then,

$$|g_1|_{0,\phi_1} \leqslant C_1 C_2 |g_2|_{0,\infty},$$

where C_2 depends only on Ω. In fact,

$$\begin{aligned}
\int_\Omega \phi_1\left(\frac{|g_1(x)|}{K|g_2|_{0,\infty}}\right) dx &= \int_\Omega \sum_{j=2}^{\infty} \frac{|g_1(x)|^j}{K^j |g_2|_{0,\infty}^j} \frac{1}{j!}\, dx \\
&= \sum_{j=2}^{\infty} \frac{1}{j!K^j|g_2|_{0,\infty}^j} \int_\Omega |g_1(x)|^j dx \leqslant \sum_{j=2}^{\infty} \frac{C_1^j j^j |g_2|_{0,j}^j}{j!K^j|g_2|_{0,\infty}^j} \\
&\leqslant \sum_{j=2}^{\infty} \left(\frac{C_1}{K}\right)^j \frac{j^j}{j!} |\Omega|,
\end{aligned}$$

and the last series is convergent and less than 1 if we choose $K = C_1 C_2$ with C_2 sufficiently large, depending only on Ω. □

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