

Crosswind Smear and Pointwise Errors in Streamline Diffusion Finite Element Methods

By C. Johnson, A. H. Schatz, and L. B. Wahlbin*

Abstract. For a model convection-dominated singularly perturbed convection-diffusion problem, it is shown that crosswind smear in the numerical streamline diffusion finite element method is minimized by introducing a judicious amount of artificial crosswind diffusion. The ensuing method with piecewise linear elements converges with a pointwise accuracy of almost $h^{5/4}$ under local smoothness assumptions.

1. Introduction. The streamline diffusion method is a finite element method for convection-dominated convection-diffusion problems which combines formal high accuracy with decent stability properties. The method was introduced in the case of stationary problems by Hughes and Brooks [7], cf. Raithby and Torrance [14] and Wahlbin [15] for earlier thoughts in this direction. The mathematical analysis of the method was started in Johnson and Nävert [8] and continued with extensions to, e.g., time-dependent problems in Nävert [12], Johnson, Nävert and Pitkäranta [9] and Johnson and Saranen [10]. In these papers local error estimates in L_2 of order $O(h^{k+1/2})$, in regions of smoothness, with piecewise polynomial finite elements of degree k , were derived, together with estimates stating, as a typical example, that in the zero diffusion limit a sharp discontinuity in the exact solution across a streamline will be captured in a numerical crosswind layer of width $O(h^{1/2})$, essentially.

The purpose of the present paper is first to improve the result just mentioned on numerical crosswind smear to $O(h^{3/4})$. The improvement from $O(h^{1/2})$ to $O(h^{3/4})$ is obtained by adding a small amount, $O(h^{3/2})$, of artificial crosswind diffusion to the method. In the piecewise linear case ($k = 1$) this does not destroy the known $O(h^{3/2})$ accuracy in L_2 in smooth regions. Using our first result, we then obtain our second main result, localized pointwise error estimates of order $O(h^{5/4})$ in regions of smoothness. (The previously known best pointwise error estimate in the piecewise linear situation is $O(h^{1/2})$.) Another consequence is a global L_1 -estimate of order $O(h^{1/2})$ in the presence of typical crosswind and downwind singularities.

We shall consider the model problem of finding $u = u(x, y)$ such that

$$(1.1a) \quad -\delta u_{xx} - \epsilon u_{yy} + u_x + u = f \quad \text{in } \Omega,$$

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$$(1.1b) \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded convex domain in the plane and δ, ε are small parameters. Its numerical solution is sought in a family of finite element spaces $S_h \subseteq H_0^1(\Omega)$ which, for simplicity and concreteness, we take to be piecewise linear functions on a quasiuniform family of triangulations (of subdomains) of Ω . Exact hypotheses are given at the end of this introduction.

Our interest is in the singularly perturbed case and we assume throughout that

$$(1.2) \quad \delta, \varepsilon \leq h.$$

Hence, we do not seek to resolve boundary layers or other singularities, but rather aim for numerical methods in which singularities do not pollute into regions where the solution is smooth. The ordinary Galerkin method, i.e., finding $u^h \in S_h$ such that

$$(1.3) \quad \delta(u_x^h, \chi_x) + \varepsilon(u_y^h, \chi_y) + (u_x^h + u^h, \chi) = (f, \chi) \quad \text{for } \chi \in S_h,$$

is well known for its severe pollution. (Here (v, w) denotes the standard L_2 -inner product over Ω .)

The streamline diffusion method for (1.1) is, in essence, derived by replacing the test functions χ in (1.3) by test functions

$$(1.4) \quad \tilde{\chi} \equiv \chi + h\chi_x, \quad \chi \in S_h.$$

Ensuing terms of the form $\gamma h(Du^h, D\chi_x)$, $D = \partial/\partial x$ or $\partial/\partial y$, $\gamma = \delta$ or ε , would have to be given a suitable interpretation and, in our piecewise linear setting, we shall for simplicity of analysis discard them. (Their formal order is $O((\delta + \varepsilon)h) \leq O(h^2)$, which is below the asymptotic rate we can ever hope for.)

Changing also the crosswind diffusion ε artificially to ε_{mod} , the numerical method is thus to find $u^h \in S_h$ such that

$$(1.5) \quad B(u^h, \chi) = (f, \tilde{\chi}) \quad \text{for } \chi \in S_h,$$

where

$$(1.6) \quad \begin{aligned} B(u^h, \chi) &\equiv \delta(u_x^h, \chi_x) + \varepsilon_{\text{mod}}(u_y^h, \chi_y) + (u_x^h + u^h, \tilde{\chi}) \\ &\equiv (h + \delta)(u_x^h, \chi_x) + \varepsilon_{\text{mod}}(u_y^h, \chi_y) + (1 - h)(u_x^h, \chi) + (u^h, \chi). \end{aligned}$$

The ‘‘stabilizing’’ term $h(u_x^h, \chi_x)$ added in the streamline (wind) direction is an important feature of the method and, indeed, gives it its name.

The artificially modified crosswind diffusion ε_{mod} is given as follows: With $0 \leq \varepsilon_{\text{co}} \leq h$ a crossover point, typically dependent on h ,

$$(1.7) \quad \varepsilon_{\text{mod}} = \begin{cases} \varepsilon & \text{for } \varepsilon \geq \varepsilon_{\text{co}}, \\ \varepsilon_{\text{co}} & \text{for } \varepsilon < \varepsilon_{\text{co}}. \end{cases}$$

In the traditional streamline diffusion method, $\varepsilon_{\text{mod}} \equiv \varepsilon$.

We next briefly describe how (1.5) differs from the continuous problem (1.1). An easy calculation (differentiate (1.1) with respect to x , multiply by $-h$, add to (1.1) itself, and integrate some by parts) establishes that the solution u of (1.1) satisfies

$$(1.8) \quad B(u, \varphi) = (f, \tilde{\varphi}) + \text{Per}(u, \varphi) \quad \text{for } \varphi \in H_0^1(\Omega),$$

with the perturbation form given as

$$(1.9) \quad \begin{aligned} \text{Per}(u, \varphi) &= h(\delta u_{xx} + \epsilon u_{yy}, \varphi_x) + \epsilon_{\text{per}}(u_{yy}, \varphi) \\ &\equiv \text{Per}_1(u, \varphi) + \text{Per}_2(u, \varphi), \end{aligned}$$

where

$$(1.10) \quad \epsilon_{\text{per}} = \begin{cases} 0 & \text{for } \epsilon \geq \epsilon_{\text{co}}, \\ -(\epsilon_{\text{co}} - \epsilon) & \text{for } \epsilon < \epsilon_{\text{co}}. \end{cases}$$

Formally, these perturbation terms are of order $h\delta + h\epsilon + \epsilon_{\text{per}}$.

The question naturally arises as to what crossover point ϵ_{co} to choose. Here we are guided by a careful analysis of the numerical crosswind spread in (1.5). By this we mean the following: How far in the crosswind direction do data f significantly influence the solution? In the continuous problem (1.1) the solution at a point (x_0, y_0) is appreciably influenced by f only from within an $\epsilon^{1/2} \ln(1/\epsilon)$ neighborhood in the crosswind direction and from within $\delta \ln(1/\delta)$ in the downwind direction, cf., e.g., Eckhaus [4], Eckhaus and De Jager [5] and Lions [11].

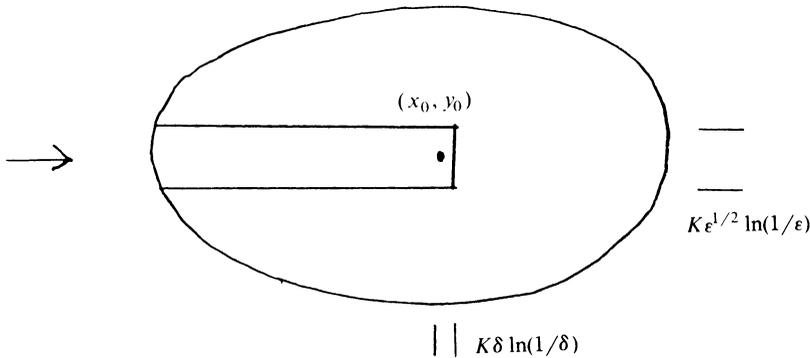


FIGURE 1.1
Influence of data in the continuous problem.

It is known, cf. [8], [9], that the numerical crosswind smear is limited to $h^{1/2} \ln(1/h)$ for any choice of $\epsilon_{\text{co}} \leq h$. We prove in Section 2 that it is actually restricted to

$$(1.11) \quad \begin{aligned} \epsilon_{\text{mod}}^{1/2} \ln(1/h) & \quad \text{for } h^{3/2} \leq \epsilon_{\text{mod}} \leq h, \\ h^{3/2} \epsilon_{\text{mod}}^{-1/2} \ln(1/h) & \quad \text{for } h^2 \leq \epsilon_{\text{mod}} < h^{3/2}, \\ h^{1/2} \ln(1/h) & \quad \text{for } \epsilon_{\text{mod}} < h^2, \end{aligned}$$

see Theorem 2.1 for a precise statement.

Since one can hardly expect less crosswind spread in the numerical scheme than in the continuous problem, and since excessive crosswind spread (in the precise sense of Theorem 2.1) seems in practice to imply excessive smearing of fronts following characteristics (in the limit of zero diffusion) of (1.1a), cf. Brooks and Hughes [2, Figure 3.7, p. 219], we choose the crossover point as

$$(1.12) \quad \epsilon_{\text{co}} = h^{3/2}.$$

This choice minimizes (1.11) when $\varepsilon < \varepsilon_{\text{co}}$. For $\varepsilon < h^{3/2}$, then, crosswind smear is limited to $h^{3/4} \ln(1/h) \ll h^{1/2} \ln(1/h)$, while for $\varepsilon > h^{3/2}$ the numerical method has essentially the same spread as the continuous problem.

Theorem 2.1 also shows that data f downwind only influence the numerical solution upwind within an $h \ln(1/h)$ distance; this is known from [8], [9], so that this part of our investigation is not new. (For completeness we include it in our proof.) In the continuous problem the corresponding distance is $\delta \ln(1/\delta)$, cf. Figure 1.1, but in the numerical scheme nothing can happen on a scale less than h .

In Section 3 we use the results above on crosswind and downwind smear in the numerical solution (1.5), with the choice $\varepsilon_{\text{co}} = h^{3/2}$, to show the following pointwise error estimate with local smoothness assumptions. At a point (x_0, y_0) we have

$$(1.13) \quad |(u - u^h)(x_0, y_0)| \leq Ch^{5/4} \ln^{3/2}(1/h)$$

provided $\delta u_{xx} + \varepsilon u_{yy} \in L_1(\Omega)$, $\nabla u \in L_1(\Omega)$, $f \in L_2(\Omega)$ and provided u is twice continuously differentiable on Ω_0 , the following region extending upstream from (x_0, y_0) , cf. Figure 1.2.

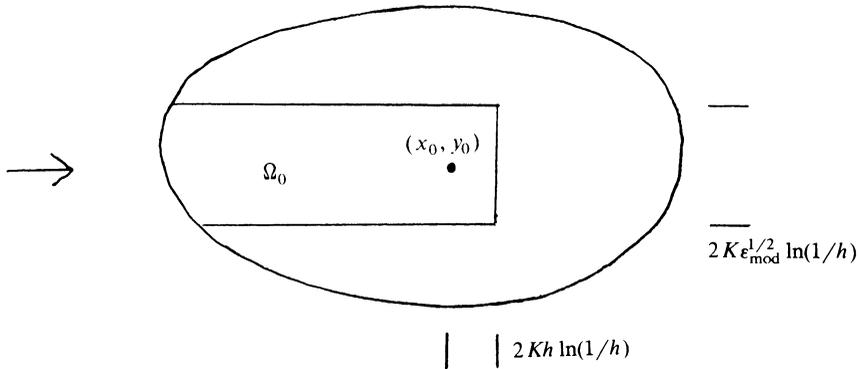


FIGURE 1.2

Influence on the error in the numerical solution.

Singularities in (1.1) can typically be expected of the form, with some a , b , “ $\exp((x - a)/\delta)$ ” in the downwind direction and “ $\exp(-|y - b|/\varepsilon^{1/2})$ ” in the crosswind direction. Hence our global assumptions for (1.13) are reasonable in practice.

The previously best known pointwise error estimate in smooth regions in our problem for general meshes is $O(h^{1/2} \ln^{1/2}(1/h))$, following from an L_2 -estimate of $O(h^{3/2})$, [8], by Sobolev’s inequality on S_h .

The perturbations off (1.1) in the numerical scheme (1.5) are described in (1.8)–(1.10). The term Per_1 is of order $h(\delta + \varepsilon) \leq h^2$, while for $\varepsilon < h^{3/2}$ the crosswind perturbation Per_2 is of order $h^{3/2}$. (This motivates our calling the crosswind diffusion ε_{mod} artificial.) Methods for including these perturbation terms in the numerical method, with higher-order spaces S_h , have been considered in [8], [12] for Per_1 , and for Per_2 in Axelsson and Layton [1].

We next describe two corollaries of our main results. Whereas the continuous problem (1.1) has a maximum principle, so that $\|u\|_{L_\infty} \leq C\|f\|_{L_\infty}$, for the numerical problem we can only prove a weaker result, $\|u^h\|_{L_\infty} \leq Ch^{-1/4} \ln^{3/2}(1/h)\|f\|_{L_\infty}$, in

Corollary 3.5. For a typical singularity following a characteristic and typical boundary layers we find in Corollary 3.6 that for the global error in L_1 , with (1.12),

$$\|u - u^h\|_{L_1(\Omega)} \leq Ch^{-1/4} \varepsilon_{\text{mod}}^{1/2} \ln^{5/2}(1/h),$$

which for $\varepsilon < h^{3/2}$ translates to an $O(h^{1/2} \ln^{5/2}(1/h))$ error estimate.

Our results are probably not sharp. Numerical experiments by Pitkäranta [13] and by ourselves suggest that the pure streamline diffusion method, i.e., $\varepsilon_{\text{co}} = 0$ so that $\varepsilon_{\text{mod}} \equiv \varepsilon$, has better crosswind spread than $h^{1/2} \ln(1/h)$ for low ε , and one may guess that it is $h^{3/4} \ln(1/h)$ also for $\varepsilon_{\text{co}} < h^{3/2}$ in (1.11) (or, $h^{2/3} \ln(1/h)$; the numerical experiments vacillate somewhat, in particular when the characteristics are sharply curved). Similarly, one may guess that (1.13) should be replaced by an $O(h^{3/2})$ -estimate, or even $O(h^2)$ if one is daring. Also, our L_1 -estimate above is curious in that it gets worse as ε increases above $h^{3/2}$, while the typical singularities in (1.1) then attenuate. For the possible root of this possible lack of sharpness, see Remark 3.4 below.

We conclude this introduction by describing our hypotheses for the piecewise linear spaces $S_h \subseteq H_0^1(\Omega)$ and listing some well-known results for them that will be used in the sequel. Let $\mathcal{T}_h = \{\tau_i^h\}_{i=1}^N$ be a family of edge-to-edge triangulations of $\Omega_h \subseteq \Omega$ with the parameter $h \equiv N^{-1/2}$ uniformly comparable to $\max_i(\text{diam}(\tau_i^h))$ and

$$S_h = \left\{ \chi \in \mathcal{C}^0(\Omega_h), \chi = 0 \text{ on } \partial\Omega_h, \chi|_{\tau_i^h} \text{ linear in } x \text{ and } y \right\}.$$

When necessary, such functions are extended by zero to Ω . Here, with M a constant,

$$(1.14) \quad \max_{x \in \partial\Omega} (\text{dist}(\partial\Omega_h), x) \leq Mh^2,$$

as any family of triangulations used in practice would satisfy. The family is assumed to be quasiuniform, so that (here and below we continue to use M for various quantities associated with the family S_h),

$$(1.15) \quad \text{diam}(\tau_i^h) \leq Mh \leq M^2 \rho(\tau_i^h), \quad i = 1, \dots, N,$$

where $\rho(\tau)$ denotes the diameter of the largest inscribed disc of τ .

With (1.14) and (1.15) we have the following: For $\text{Int}(v)$, the interpolant of v , there holds for any triangle τ ,

$$(1.16) \quad h \|\nabla(v - \text{Int}(v))\|_{L_p(\tau)} + \|v - \text{Int}(v)\|_{L_p(\tau)} \leq Mh^2 \sum_{|\gamma|=2} \|D^\gamma v\|_{L_p(\tau)}.$$

We shall also need an inverse estimate for $\chi \in S_h$:

$$(1.17) \quad \|\chi\|_{W_p^l(\tau_i^h)} \leq Mh^{-l-2(1/q-1/p)} \|\chi\|_{L_q(\tau_i^h)},$$

for $l = 0, 1$ and $1 \leq q \leq p \leq \infty$. Finally, with P_0 the $L_2(\Omega)$ -projection into S_h , we have for any domain $D \subseteq \Omega_h$,

$$(1.18) \quad h \|\nabla(v - P_0 v)\|_{L_\infty(D)} + \|v - P_0 v\|_{L_\infty(D)} \leq Mh^2 \|v\|_{\mathcal{C}^2(D^+)} + Mh^2 \|v\|_{L_2(\Omega)},$$

where $D^+ = (D + Mh) \cap \Omega$. This is easily derived with the techniques of Douglas, Dupont, and Wahlbin [3].

2. Crosswind (and Downwind) Smear in the Numerical Method. For a domain D we set

$$(2.1) \quad [[v]]_D = h^{1/2} \|v_x\|_D + \varepsilon_{\text{mod}}^{1/2} \|v_y\|_D + \|v\|_D,$$

where $\|v\|_D = \|v\|_{L_2(D)}$. Recall that M denotes various constants related to the family S_h , and recall also the notation (1.4) and (1.6). The following is the result of this section.

THEOREM 2.1. *For any $s > 0$ there exists a constant $K = K(M, s)$ such that the following holds: Let $u^h \in S_h$ satisfy $B(u^h, \chi) = (f, \tilde{\chi})$ for $\chi \in S_h$ and let*

$$(2.2) \quad \Omega_0 = \{x \leq A, B_1 \leq y \leq B_2\} \cap \Omega,$$

$$(2.3) \quad \Omega_0^+ = \{x \leq A + \rho \ln(1/h), B_1 - \sigma \ln(1/h) \leq y \leq B_2 + \sigma \ln(1/h)\} \cap \Omega,$$

where the downwind spread is $\rho = Kh$ and the crosswind spread σ is given by

$$(2.4) \quad \sigma = \begin{cases} K\varepsilon_{\text{mod}}^{1/2} & \text{for } h^{3/2} \leq \varepsilon_{\text{mod}} \leq h, \\ Kh^{3/2}\varepsilon_{\text{mod}}^{-1/2} & \text{for } h^2 \leq \varepsilon_{\text{mod}} < h^{3/2}, \\ Kh^{1/2} & \text{for } \varepsilon_{\text{mod}} < h^2. \end{cases}$$

Then,

$$(2.5) \quad [[u^h]]_{\Omega_0} \leq K \|f\|_{\Omega_0^+} + h^s \|f\|_{\Omega}.$$

The rest of the present section is devoted to the proof of this. For typographical reasons we write U for u^h . The proof will be executed in detail for $h^2 \leq \varepsilon_{\text{mod}} \leq h$; the case $\varepsilon_{\text{mod}} < h^2$ is contained in [8], [12], cf. Remark 2.3 below.

Following [8], [12] we start by introducing a suitable cutoff function. Let $g(s) \in \mathcal{C}^2(-\infty, \infty)$ with $g(s) = |s|$ for $|s| \geq 1$ and set

$$(2.6) \quad \psi(t) = \int_t^\infty \exp(-g(s)) ds.$$

Then, as is easily checked, there exist positive constants c and C such that

$$(2.7) \quad c \leq \psi(t) \leq C \quad \text{for } t \leq 1,$$

$$(2.8) \quad \begin{aligned} \psi(t) &= e^{-t} && \text{for } t > 1, \\ \psi'(t) &\leq 0 && \text{all } t, \\ |\psi'(t)| + |\psi''(t)| &\leq C|\psi(t)| && \text{all } t, \\ |\psi''(t)| &\leq -C\psi'(t) && \text{all } t. \end{aligned}$$

Further, with the relative oscillation on a domain D defined as

$$\text{RO}(D, v) = \max_{x \in D} |v(x)| / \min_{x \in D} |v(x)|,$$

on any interval I of length 1,

$$\text{RO}(I, \psi) + \text{RO}(I, \psi') \leq C.$$

Define now

$$\omega(x, y) = \psi\left(\frac{x-A}{\rho}\right) \psi\left(\frac{B_1-y}{\sigma}\right) \psi\left(\frac{y-B_2}{\sigma}\right).$$

From the properties above it follows that $\omega_x \leq 0$ and that

$$(2.9) \quad |D_x^\alpha D_y^\beta \omega| \leq C \rho^{-\alpha} \sigma^{-\beta} \omega \quad \text{for } \alpha + \beta \leq 2,$$

$$(2.10) \quad |D_x^\alpha D_y^\beta \omega| \leq -C \rho^{-\alpha+1} \sigma^{-\beta} \omega_x \quad \text{for } \sigma \geq 1, \alpha + \beta \leq 2,$$

$$(2.11) \quad \text{RO}(\tau, \omega) \text{ and } \text{RO}(\tau, \omega_x) \text{ are bounded independently of } h \\ \text{on any element } \tau.$$

From (2.7) and (2.8) it is clear that Theorem 2.1 would obtain from the following result: Let

$$(2.12) \quad L \equiv \max\left((h/\rho)^{1/2}, h^{3/2}/(\sigma \epsilon_{\text{mod}}^{1/2}), \epsilon_{\text{mod}}/\sigma^2\right)$$

and

$$(2.13) \quad Q(U) \equiv \left((h + \delta) \|\omega U_x\|^2 + \epsilon_{\text{mod}} \|\omega U_y\|^2 \right. \\ \left. + \|\omega U\|^2 + (1 - h) \left\| (\omega |\omega_x|)^{1/2} U \right\|^2 \right)^{1/2}.$$

Then for L sufficiently small,

$$(2.14) \quad Q(U) \leq C \|\omega f\|.$$

The reader may be interested in following the proof without prior knowledge of the choices of ρ and σ . If so, we remark that we assume $h \leq \rho \leq \sigma$ throughout the proof.

We shall need the following ‘‘superapproximation’’ result which follows from (1.16) and (1.17). Let

$$E = \omega^2 U - \text{Int}(\omega^2 U).$$

LEMMA 2.2. *There exists a constant $C = C(M)$ such that for $U \in S_h$,*

$$(2.15) \quad h \|\omega^{-1} \nabla E\| + \|\omega^{-1} E\| \leq Ch^{1/2} L Q(U).$$

The proof of this is postponed until the end of this section, and we proceed to prove (2.14). Note first that

$$0 = ((\omega U)_x, \omega U) = (\omega_x U, \omega U) + (U_x, \omega^2 U).$$

Hence, using (1.5) for $\chi = \text{Int}(\omega^2 U) \in S_h$,

$$(2.16) \quad Q^2(U) \equiv (h + \delta)(\omega U_x, \omega U_x) + \epsilon_{\text{mod}}(\omega U_y, \omega U_y) \\ + (\omega U, \omega U) - (1 - h)(\omega_x U, \omega U) \\ = (h + \delta)(U_x, (\omega^2 U)_x - 2\omega \omega_x U) + \epsilon_{\text{mod}}(U_y, (\omega^2 U)_y - 2\omega \omega_y U) \\ + (U, \omega^2 U) + (1 - h)(U_x, \omega^2 U) \\ = B(U, \omega^2 U) - 2(h + \delta)(\omega U_x, \omega_x U) - 2\epsilon_{\text{mod}}(\omega U_y, \omega_y U) \\ = B(U, E) - (f, \tilde{E}) + (f, (\omega^2 U)^\sim) \\ - 2(h + \delta)(\omega U_x, \omega_x U) - 2\epsilon_{\text{mod}}(\omega U_y, \omega_y U) \\ \equiv I_1 + \cdots + I_5.$$

Here, by Lemma 2.2 and since $\varepsilon_{\text{mod}} \leq h$,

$$\begin{aligned} I_1 &= (h + \delta)(\omega U_x, \omega^{-1} E_x) + \varepsilon_{\text{mod}}(\omega U_y, \omega^{-1} E_y) \\ &\quad + (1 - h)(\omega U_x, \omega^{-1} E) + (\omega U, \omega^{-1} E) \\ &\leq \left(\|\omega U_x\| + \frac{\varepsilon_{\text{mod}}}{h} \|\omega U_y\| + \|\omega U\| \right) (h \|\omega^{-1} \nabla E\| + \|\omega^{-1} E\|) \\ &\leq \left(h^{1/2} \|\omega U_x\| + \varepsilon_{\text{mod}}^{1/2} \|\omega U_y\| (\varepsilon_{\text{mod}}/h)^{1/2} + h^{1/2} \|\omega U\| \right) CLQ(U) \\ &\leq CLQ^2(U). \end{aligned}$$

Again using Lemma 2.2, cf. (1.4),

$$I_2 \leq \|\omega f\| (\|\omega^{-1} E\| + h \|\omega^{-1} \nabla E\|) \leq \|\omega f\|^2 + ChL^2Q^2(U).$$

Further, by (2.9), since $\rho \geq h$,

$$\begin{aligned} I_3 &\leq \|\omega f\| (\|\omega U\| + 2h \|\omega_x U\| + h \|\omega U_x\|) \leq \|\omega f\| (C \|\omega U\| + h \|\omega U_x\|) \\ &\leq \frac{1}{8} Q^2(U) + C \|\omega f\|^2. \end{aligned}$$

By (2.10),

$$\begin{aligned} I_4 &\leq 2(h + \delta) \|\omega U_x\| \|\omega_x U\| \leq \frac{h}{8} \|\omega U_x\|^2 + Ch \|\omega_x U\|^2 \\ &\leq \frac{h}{8} \|\omega U_x\|^2 + C \left(\frac{h}{\rho} \right)^{1/2} \|(\omega|\omega_x|)^{1/2} U\|^2 \leq \left(\frac{1}{8} + CL \right) Q^2(U). \end{aligned}$$

Finally, from (2.9),

$$I_5 \leq \frac{C\varepsilon_{\text{mod}}}{\sigma} \|\omega U_y\| \|\omega U\| \leq \frac{\varepsilon_{\text{mod}}}{8} \|\omega U_y\|^2 + \frac{C\varepsilon_{\text{mod}}}{\sigma^2} \|\omega U\|^2 \leq \left(\frac{1}{8} + CL \right) Q^2(U).$$

Using these estimates in (2.16), it is clear that (2.14) follows if L is small enough. As we have already noted, this would prove Theorem 2.1 in the case $h^2 \leq \varepsilon_{\text{mod}} \leq h$.

It remains to verify Lemma 2.2. On any triangle τ we have by (1.16),

$$(2.17) \quad h \|\nabla E\|_\tau + \|E\|_\tau \leq Mh^2 \sum_{|\gamma|=2} \|D^\gamma(\omega^2 U)\|_\tau.$$

Since $D^2 U = 0$, it follows by use of (2.9) that

$$|D_y^2(\omega^2 U)| \leq |(\omega^2)_{,yy} U| + 2|(\omega^2)_{,y} U_y| \leq C\sigma^{-2} |\omega^2 U| + C\sigma^{-1} |\omega^2 U_y|.$$

For the second mixed derivative,

$$D_x D_y(\omega^2 U) = 2\omega_x \omega_y U + 2\omega \omega_{xy} U + 2\omega \omega_x U_y + 2\omega \omega_y U_x,$$

we use (2.9) and (2.10) to arrive at

$$|D_x D_y(\omega^2 U)| \leq C\sigma^{-1} |\omega \omega_x U| + C |\omega \omega_x U_y| + C\sigma^{-1} |\omega^2 U_x|.$$

Similarly,

$$|D_x^2(\omega^2 U)| \leq C\rho^{-1} |\omega \omega_x U| + C |\omega \omega_x U_x|.$$

Inserting the above in (2.17) and employing (2.11),

$$(2.18) \quad \begin{aligned} & h \|\omega^{-1} \nabla E\|_{\tau} + \|\omega^{-1} E\|_{\tau} \\ & \leq C \left[h^2 \sigma^{-2} \|\omega U\|_{\tau} + h^2 \sigma^{-1} \|\omega U_y\|_{\tau} + h^2 \sigma^{-1} \|\omega_x U\|_{\tau} + h^2 \|\omega_x U_y\|_{\tau} \right. \\ & \quad \left. + h^2 \sigma^{-1} \|\omega U_x\|_{\tau} + h^2 \rho^{-1} \|\omega_x U\|_{\tau} + h^2 \|\omega_x U_x\|_{\tau} \right]. \end{aligned}$$

We proceed to operate further on the last five terms. By (2.9) and since $\sigma \geq h$,

$$\begin{aligned} h^2 \sigma^{-1} \|\omega_x U\|_{\tau} & \leq Ch^2 \sigma^{-1} \rho^{-1/2} \|(\omega|\omega_x|)^{1/2} U\|_{\tau} \\ & \leq Ch \rho^{-1/2} \|(\omega|\omega_x|)^{1/2} U\|_{\tau}. \end{aligned}$$

By the inverse property (1.17), and as above,

$$h^2 \|\omega_x U_y\|_{\tau} \leq Ch \|\omega_x U\|_{\tau} \leq Ch \rho^{-1/2} \|(\omega|\omega_x|)^{1/2} U\|_{\tau}$$

and, since $\sigma \geq \rho$,

$$h^2 \sigma^{-1} \|\omega U_x\|_{\tau} \leq Ch^2 \rho^{-1} \|\omega^2 U_x\|_{\tau}.$$

Finally, as for the first term treated above,

$$h^2 \rho^{-1} \|\omega_x U\|_{\tau} \leq Ch \rho^{-1/2} \|(\omega|\omega_x|)^{1/2} U\|_{\tau},$$

and by (2.9),

$$h^2 \|\omega_x U_x\|_{\tau} \leq Ch^2 \rho^{-1} \|\omega U_x\|_{\tau}.$$

Inserting these estimates into (2.18) and rearranging,

$$(2.19) \quad \begin{aligned} & h \|\omega^{-1} \nabla E\|_{\tau} + \|\omega^{-1} E\|_{\tau} \\ & \leq C \left[h^2 \rho^{-1} \|\omega U_x\|_{\tau} + h^2 \sigma^{-1} \|\omega U_y\|_{\tau} + h^2 \sigma^{-2} \|\omega U\|_{\tau} \right. \\ & \quad \left. + h \rho^{-1/2} \|(\omega|\omega_x|)^{1/2} U\|_{\tau} \right] \\ & = Ch^{1/2} \left[(h/\rho) h^{1/2} \|\omega U_x\|_{\tau} + (h^{3/2}/\sigma \epsilon_{\text{mod}}^{1/2}) \epsilon_{\text{mod}}^{1/2} \|\omega U_y\|_{\tau} \right. \\ & \quad \left. + (h^{3/2}/\sigma^2) \|\omega U\|_{\tau} + (h/\rho)^{1/2} \|(\omega|\omega_x|)^{1/2} U\|_{\tau} \right]. \end{aligned}$$

Since $L \geq h^{3/2} \sigma^{-2}$ we obtain (2.15) upon squaring and summing over all elements. This completes the proof of Lemma 2.2.

Remark 2.3. In the case $\epsilon_{\text{mod}} \leq h^2$, take

$$L = \max\left((h/\rho)^{1/2}, h^{1/2} \sigma^{-1}\right).$$

It is then easily seen that Lemma 2.2 still holds. The only change in the proof occurs in (2.19), where the term $h^2 \sigma^{-1} \|\omega U_y\|_{\tau}$ is now bounded by $Ch \sigma^{-1} \|\omega U\|_{\tau}$ from the inverse property (1.17). The rest of the proof of Theorem 2.1 is as before.

Remark 2.4. By a more complicated argument one may show that (2.5) holds in a completely local fashion, viz.,

$$[[u^h]]_{\Omega_0} \leq K \|f\|_{\Omega_0^*} + h^s \|u^h\|_{\Omega_0^*}.$$

This may be of value in analyzing very rough flows.

3. A Pointwise Error Estimate with Local Smoothness Assumptions. Let (x_0, y_0) be any point in Ω and let K be as in Theorem 2.1 (for $s = 6$). Set in this section

$$(3.1) \quad \Omega_0 = \{x \leq x_0 + 2Kh \ln(1/h), |y - y_0| \leq 2K\epsilon_{\text{mod}}^{1/2} \ln(1/h)\} \cap \Omega,$$

cf. Figure 1.2.

THEOREM 3.1. *Assume that $\epsilon_{\text{co}} = h^{3/2}$ and that*

$$\|u\|_{\mathcal{H}^2(\bar{\Omega}_0)} + \|\delta u_{xx} + \epsilon u_{yy}\|_{L_1(\Omega)} + \|\nabla u\|_{L_1(\Omega)} + \|f\|_{L_2(\Omega)} \leq Q.$$

There exists a constant $C = C(Q, M)$ such that

$$|(u - u^h)(x_0, y_0)| \leq Ch^{5/4} \ln^{3/2}(1/h).$$

Proof. Let $G = G_h^{(x_0, y_0)} \in S_h$ be the discrete Green's function,

$$(3.2) \quad B(\chi, G) = \chi(x_0, y_0) \quad \text{for } \chi \in S_h.$$

With $P_0 u$ the L_2 -projection into S_h , we have

$$(3.3) \quad \begin{aligned} (u^h - P_0 u)(x_0, y_0) &= B(u^h - P_0 u, G) = (f, \tilde{G}) - B(P_0 u, G) \\ &= (f, \tilde{G}) + \text{Per}(u, G) - B(P_0 u, G) - \text{Per}(u, G) \\ &= B(u - P_0 u, G) - \text{Per}(u, G). \end{aligned}$$

Let $\Omega'_0 \subseteq \Omega_0$ be as in (3.1), with $2K$ replaced by K . We claim that

$$(3.4) \quad \|G\|_{W_\infty^1(\Omega \setminus \Omega'_0)} \leq Ch^3.$$

To see this, let δ_h be a linear function on the element τ containing (x_0, y_0) such that $(\delta_h, \chi) = \chi(x_0, y_0)$ for χ linear on τ , with δ_h vanishing outside τ . Then (3.2) is equivalent to $B(\chi, G) = (\delta_h, \chi)$ for $\chi \in S_h$, so that by the counterpart of Theorem 2.1 for the adjoint problem, which has the wind direction reversed,

$$\|G\|_{L_2(\Omega \setminus \Omega'_0)} \leq Ch^6 \|\delta_h\|_{L_2}.$$

Since the dimensions involved in (3.1) are much greater than h , and since G vanishes outside Ω_h , we may assume that $\Omega \setminus \Omega'_0$ is a mesh domain. Then (3.4) follows from the inverse property (1.17), since clearly $\|\delta_h\|_{L_2} \leq Ch^{-1}$.

Now let $B_D(v, \varphi)$ denote that the integrations in (1.6) are extended only over the domain D . Then

$$B_{\Omega \setminus \Omega'_0}(u - P_0 u, G) \leq (\|u\|_{W_1^1} + \|P_0 u\|_{W_1^1}) \|G\|_{W_\infty^2(\Omega \setminus \Omega'_0)}.$$

Since by assumption $\|u\|_{W_1^1} \leq C$, and since by the inverse assumption (1.17),

$$\|P_0 u\|_{W_1^1} \leq C \|P_0 u\|_{W_2^1} \leq Ch^{-1} \|P_0 u\| \leq Ch^{-1} \|u\| \leq Ch^{-1},$$

we get from (3.4),

$$(3.5) \quad B_{\Omega \setminus \Omega'_0}(u - P_0 u, G) \leq Ch^2.$$

For the remaining part of B we have

$$\begin{aligned} B_{\Omega'_0}(u - P_0 u, G) &= \int_{\Omega'_0 \cap \Omega_h} [(h + \delta)(u - P_0 u)_x G_x + \epsilon_{\text{mod}}(u - P_0 u)_y G_y \\ &\quad - (1 - h)(u - P_0 u) G_x + (u - P_0 u) G] \\ &\leq C [h \|\nabla(u - P_0 u)\|_{L_\infty(\Omega'_0 \cap \Omega_h)} + \|u - P_0 u\|_{L_\infty(\Omega'_0 \cap \Omega_h)}] I \end{aligned}$$

where

$$I = \|G_x\|_{L_1(\Omega'_0)} + \varepsilon_{\text{mod}} h^{-1} \|G_y\|_{L_1(\Omega'_0)} + \|G\|_{L_1(\Omega'_0)}.$$

Since $\text{meas}(\Omega'_0) \leq C\varepsilon_{\text{mod}}^{1/2} \ln(1/h)$, we obtain by Cauchy-Schwarz' inequality,

$$I \leq C\varepsilon_{\text{mod}}^{1/4} \ln^{1/2}(1/h) \mathcal{J}, \quad \text{where } \mathcal{J} = \|G_x\| + \varepsilon_{\text{mod}} h^{-1} \|G_y\| + \|G\|,$$

so that using also (1.18),

$$(3.6) \quad B_{\Omega'_0}(u - P_0 u, G) \leq Ch^2 \varepsilon_{\text{mod}}^{1/4} \ln^{1/2}(1/h) \mathcal{J}.$$

We next estimate the perturbation term $\text{Per}(u, G) \equiv \text{Per}_1(u, G) + \text{Per}_2(u, G)$ in (3.3), cf. (1.9). For the first part we have by (3.4), using again Cauchy-Schwarz' inequality,

$$(3.7) \quad \begin{aligned} \text{Per}_1(u, G) &= h(\delta u_{xx} + \varepsilon u_{yy}, G_x) \\ &\leq Ch^2 \|u\|_{\mathcal{H}^2(\Omega_0)} \|G_x\|_{L_1(\Omega_0)} + Ch \|\delta u_{xx} + \varepsilon u_{yy}\|_{L_1(\Omega)} h^3 \\ &\leq Ch^2 \varepsilon_{\text{mod}}^{1/4} \ln^{1/2}(1/h) \|G_x\| + Ch^4, \end{aligned}$$

and for the second, after integration by parts over $\Omega \setminus \Omega_0$.

$$(3.8) \quad \begin{aligned} \text{Per}_2(u, G) &= \varepsilon_{\text{per}}(u_{yy}, G) = \varepsilon_{\text{per}} \left[(u_{yy}, G)_{\Omega_0} - \int_{\Omega \setminus \Omega_0} u_y G_y + \int_{\partial\Omega_0} u_y G \right] \\ &\leq C\varepsilon_{\text{per}} \varepsilon_{\text{mod}}^{1/4} \ln^{1/2}(1/h) \|G\| + C\varepsilon_{\text{per}} h^3, \end{aligned}$$

where Cauchy-Schwarz' inequality and (3.4) were used together with our assumptions on u .

Collecting (3.5)–(3.8) into (3.3) and using again (1.18) and the triangle inequality,

$$(3.9) \quad \begin{aligned} |(u - u^h)(x_0, y_0)| &\leq Ch^2 \varepsilon_{\text{mod}}^{1/4} \ln^{1/2}(1/h) [\|G_x\| + \varepsilon_{\text{mod}} h^{-1} \|G_y\| + \|G\|] \\ &\quad + C\varepsilon_{\text{per}} \varepsilon_{\text{mod}}^{1/4} \ln^{1/2}(1/h) \|G\| + Ch^2. \end{aligned}$$

We now use the following lemma whose proof will be postponed.

LEMMA 3.2. *We have*

$$\begin{aligned} \|G_x\| &\leq Ch^{-3/4} \varepsilon_{\text{mod}}^{-1/4} \ln(1/h), \\ \|G_y\| &\leq Ch^{-1/4} \varepsilon_{\text{mod}}^{-3/4} \ln(1/h), \\ \|G\| &\leq Ch^{-1/4} \varepsilon_{\text{mod}}^{-1/4} \ln(1/h). \end{aligned}$$

Admitting this lemma, we have from (3.9) that

$$\begin{aligned} |(u - u^h)(x_0, y_0)| &\leq Ch^2 \varepsilon_{\text{mod}}^{1/4} \ln^{3/2}(1/h) [h^{-3/4} \varepsilon_{\text{mod}}^{-1/4} + h^{-5/4} \varepsilon_{\text{mod}}^{1/4} + h^{-1/4} \varepsilon_{\text{mod}}^{-1/4}] \\ &\quad + C\varepsilon_{\text{per}} \varepsilon_{\text{mod}}^{1/4} \ln^{3/2}(1/h) h^{-1/4} \varepsilon_{\text{mod}}^{-1/4} \\ &\leq Ch^{5/4} \ln^{3/2}(1/h), \end{aligned}$$

since $h^{-5/4} \varepsilon_{\text{mod}}^{1/4} \leq h^{-3/4} \varepsilon_{\text{mod}}^{-1/4}$ and since $\varepsilon_{\text{per}} \leq h^{3/2}$ in the present case $\varepsilon_{\text{co}} = h^{3/2}$, cf. (1.10). This would conclude the proof of Theorem 3.1.

It remains to show Lemma 3.2, and for this we shall need the following variant of Sobolev's inequality.

PROPOSITION 3.3. For $v \in H_0^1(\Omega)$ and $p > 2$,

$$\|v\|_{L_p} \leq \|v_x\|_{L_2}^{1/2} \|v_y\|_{L_2}^{1/2} \frac{p}{4} \text{meas}(\Omega)^{1/p}.$$

Proof. Our proof is a minor modification of a standard proof of Sobolev's inequality. Let $w \in \mathcal{C}_0^\infty(\Omega)$. Then

$$|w(x, y)| \leq \frac{1}{2} \int |w_{x'}(x', y)| dx'$$

or

$$|w(x, y)| \leq \frac{1}{2} \int |w_{y'}(x, y')| dy'.$$

Thus,

$$|w(x, y)|^2 \leq \frac{1}{4} \int |w_{x'}(x', y)| dx' \int |w_{y'}(x, y')| dy'.$$

Integrating (and removing the primes),

$$\iint |w(x, y)|^2 dx dy \leq \frac{1}{4} \left(\iint |w_x(x, y)| dx dy \right) \left(\iint |w_y(x, y)| dx dy \right).$$

After a density argument we may apply this to $w = |v|^p$, so that $Dw = p|v|^{p-1} \text{sgn}(Dv)$. Using Cauchy-Schwarz' inequality,

$$\begin{aligned} \iint |v|^{2p} &\leq \frac{p^2}{4} \left(\iint |v|^{p-1} |v_x| \right) \left(\iint |v|^{p-1} |v_y| \right) \\ &\leq \frac{p^2}{4} \|v_x\| \|v_y\| \iint |v|^{2(p-1)}. \end{aligned}$$

By Hölder's inequality,

$$\iint |v|^{2(p-1)} \leq \|v\|_{L_{2p}}^{2(p-1)} \text{meas}(\Omega)^{1/p},$$

and hence

$$\|v\|_{L_{2p}}^{2p} \leq \|v_x\| \|v_y\| \|v\|_{L_{2p}}^{2(p-1)} \frac{p^2}{4} \text{meas}(\Omega)^{1/p}$$

or

$$\|v\|_{L_{2p}}^2 \leq \|v_x\| \|v_y\| \frac{p^2}{4} \text{meas}(\Omega)^{1/p}.$$

Changing $2p$ to p completes the proof.

We proceed now to prove Lemma 3.2. We have

$$(3.10) \quad (h + \delta) \|G_x\|^2 + \varepsilon_{\text{mod}} \|G_y\|^2 + \|G\|^2 = B(G, G) = G(x_0, y_0).$$

By the inverse estimate (1.17) and by Proposition 3.3,

$$G(x_0, y_0) \leq Ch^{-2/p} \|G\|_{L_p} \leq Ch^{-2/p} p \|G_x\|^{1/2} \|G_y\|^{1/2}.$$

Choosing $p = \ln(1/h)$ and $A = (h/\varepsilon_{\text{mod}})^{1/4}$ below,

$$\begin{aligned} G(x_0, y_0) &\leq C \ln(1/h) \|G_x\|^{1/2} \|G_y\|^{1/2} \\ &\leq C \ln(1/h) (A \|G_x\| + A^{-1} \|G_y\|) \\ &= C \ln(1/h) (Ah^{-1/2} h^{1/2} \|G_x\| + A^{-1} \varepsilon_{\text{mod}}^{-1/2} \varepsilon_{\text{mod}}^{1/2} \|G_y\|) \\ &\leq \frac{1}{2} h \|G_x\|^2 + \frac{1}{2} \varepsilon_{\text{mod}} \|G_y\|^2 + C \ln^2(1/h) (A^2 h^{-1} + \varepsilon_{\text{mod}}^{-1} A^{-2}), \end{aligned}$$

so that from (3.10),

$$h\|G_x\|^2 + \varepsilon_{\text{mod}}\|G_y\|^2 + \|G\|^2 \leq C \ln^2(1/h)(h\varepsilon_{\text{mod}})^{-1/2},$$

which proves Lemma 3.2.

Remark 3.4. In the continuous problem it is easily seen (using the maximum principle and Fourier analysis) that $\|G\| \leq C\varepsilon^{-1/4}$ for the corresponding Green's function. Regrettably, we are a factor of $h^{-1/4} \ln(1/h)$ off this estimate in the discrete case. (An early and quite interesting example of the Green's function in this problem is given in Gore [6, pp. 574–575]. The exponential decay properties are clearly seen from the measurements of ashfall levels.)

We next give two corollaries of our main results. The first is analogous to Theorem 2.1 in a pointwise setting.

COROLLARY 3.5. *Let $\varepsilon_{\text{co}} = h^{3/2}$ and let Ω_0 be as in (3.1). Given $s > 0$, there exists a constant $C = C(M, s)$ such that*

$$|u^h(x_0, y_0)| \leq Ch^{-1/4} \ln^{3/2}(1/h) \|f\|_{L_\infty(\Omega_0)} + h^s \|f\|_{L_1(\Omega)}.$$

In particular,

$$\|u^h\|_{L_\infty(\Omega)} \leq Ch^{-1/4} \ln^{3/2}(1/h) \|f\|_{L_\infty(\Omega)}.$$

Proof. With G the discrete Green's function, (3.2), we have using Cauchy-Schwarz' inequality, the inverse estimate (1.17), the obvious higher-order analogue of (3.4), and Lemma 3.2,

$$\begin{aligned} |u^h(x_0, y_0)| &= |B(u^h, G)| = |(f, \tilde{G})| \\ &\leq |(f, \tilde{G})_{\Omega_0}| + |(f, \tilde{G})_{\Omega \setminus \Omega_0}| \\ &\leq C \|f\|_{L_\infty(\Omega_0)} \ln^{1/2}(1/h) \varepsilon_{\text{mod}}^{1/4} \|G\| + Ch^s \|f\|_{L_1(\Omega)} \\ &\leq C \|f\|_{L_\infty(\Omega_0)} \ln^{3/2}(1/h) h^{-1/4} + Ch^s \|f\|_{L_1(\Omega)}. \end{aligned}$$

This proves the corollary.

Finally we give a global L_1 -error estimate. For this assume that $\Omega_s \subseteq \Omega$ is a domain where u is smooth and that otherwise u has typical singularities of exponential type. More precisely, assume

$$(3.11) \quad \|u\|_{\mathcal{C}^2(\Omega_s)} + \|\delta u_{xx} + \varepsilon u_{yy}\|_{L_1(\Omega)} + \|\nabla u\|_{L_1(\Omega)} + \|f\|_{L_\infty} \leq Q.$$

Further assume that the domain where u may be rough, i.e., $\Omega \setminus \Omega_s$, is small. Typically, if u has a few singularities of exponential type, then $\text{meas}(\Omega \setminus \Omega_s) \leq Q \max(\delta \ln(1/\delta), \varepsilon^{1/2} \ln(1/\varepsilon))$. Our formal assumption is that

$$(3.12) \quad \text{meas}(\Omega \setminus \Omega_s) \leq Q \varepsilon_{\text{mod}}^{1/2} \ln(1/h).$$

COROLLARY 3.6. *Assume (3.11), (3.12) and that $\varepsilon_{\text{co}} = h^{3/2}$. There exists a constant $C = C(Q, M)$ such that*

$$\|u - u^h\|_{L_1(\Omega)} \leq Ch^{-1/4} \varepsilon_{\text{mod}}^{1/2} \ln^{5/2}(1/h).$$

In particular, for $\varepsilon < h^{3/2}$,

$$\|u - u^h\|_{L_1(\Omega)} \leq Ch^{1/2} \ln^{5/2}(1/h).$$

Proof. Clearly, Theorem 3.1 gives the required estimates on a subdomain Ω'_s of Ω_s with

$$(3.13) \quad \text{meas}(\Omega \setminus \Omega'_s) \leq C \epsilon_{\text{mod}}^{1/2} \ln(1/h).$$

Since f is bounded, so is u by the maximum principle, and hence on $\Omega \setminus \Omega'_s$, by Corollary 3.5 and (3.13),

$$\begin{aligned} \|u - u^h\|_{L_1(\Omega \setminus \Omega'_s)} &\leq (\|u\|_{L_\infty(\Omega)} + \|u^h\|_{L_\infty(\Omega)}) \text{meas}(\Omega \setminus \Omega'_s) \\ &\leq Ch^{-1/4} \ln^{5/2}(1/h) \epsilon_{\text{mod}}^{1/2}, \end{aligned}$$

which proves the desired result.

Department of Mathematics
Chalmers University of Technology
S-412 96 Göteborg, Sweden

Department of Mathematics
Cornell University
Ithaca, New York 14853

Department of Mathematics
Cornell University
Ithaca, New York 14853

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