On Weight Functions Admitting Chebyshev Quadrature

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Abstract. In this paper we prove the existence of Chebyshev quadrature for three new weight functions which are quite different from the two known examples given by Ullman [15] and Byrd and Stalla [2]. In particular, we indicate a simple method to construct weight functions for which there exist infinitely many Chebyshev quadrature rules.

1. Introduction. By a weight function $w$ we mean a real-valued nonnegative function on $[-1,1]$ for which the proper or improper Riemann integral exists and has positive value. We shall consider quadrature rules $Q_n$ of the type

$$Q_n[f] := \sum_{\nu=1}^{n} a_{\nu,n} f(x_{\nu,n}),$$

having real nodes $x_{\nu,n}$ and real weights $a_{\nu,n}$.

A quadrature rule (1) is called a Chebyshev quadrature rule (in the strict sense) if the following holds:

(2) $a_{1,n} = a_{2,n} = \cdots = a_{n,n},$

(3) $-1 < x_{1,n} < x_{2,n} < \cdots < x_{n,n} < 1,$

(4) $R_n[f] = 0$ for all $f \in P_n.$

($P_n$ denotes the class of polynomials of degree $\leq n$.) We say that a weight function $w$ admits Chebyshev quadrature if there exist Chebyshev quadrature rules $Q_n$ for all positive integers $n$.

The study of Chebyshev quadrature rules began in 1874 with the classical paper of Chebyshev [3]. Since then, there have been further investigations in the mathematical literature. For a review of recent advances in this field we refer to the paper of Gautschi [7].

Until 1966, the only known weight function admitting Chebyshev quadrature was the Chebyshev weight function

$$w_1(x) = (1 - x^2)^{-1/2}.$$
In 1966 Ullman [15] proved that the weight function
\[ w_2(x) = w_1(x) \frac{1 + ax}{1 + a^2 + 2ax}, \quad |a| \leq \frac{1}{2} \]
also admits Chebyshev quadrature. Recently, Byrd and Stalla [2] have shown that this result also holds for the weight function
\[ w_3(x) = w_1(x) \frac{1}{2a + 1 + x}, \quad a \geq 1. \]
There appears to be no other concrete example in the literature of weight functions admitting Chebyshev quadrature.

We now consider a weight function \( w \) as a product
\[ w(x) = w_1(x) v(x). \]
Kahaner [12] has shown that for \( w \in C(-1, 1) \) to admit Chebyshev quadrature, a necessary condition on \( v \) is
\[ v(x) \geq \frac{1}{2} c \quad \text{for all } x \in (-1, 1), \]
where
\[ c = \int_{-1}^{1} w(x) \, dx / \int_{-1}^{1} w_1(x) \, dx > 0. \]
Comparing the above weight functions \( w_1, w_2, \) and \( w_3 \), we see that in these cases \( v \) is a rational function continuous on \([-1, 1]\).

In this paper we shall prove that the weight functions
\[ w_4(x) = w_1(x) |x|^{-1/2} (1 + |x|)^{1/2}, \]
\[ w_5(x) = w_1(x) (1 + x)^{-1/4} (\sqrt{2} + (1 + x)^{1/2})^{1/2}, \]
\[ w_6(x) = w_1(x) (1 - x^2)^{-1/4} (1 + (1 - x^2)^{1/2})^{1/2} \]
also admit Chebyshev quadrature. Our method is quite different from those of Ullman [15] and Byrd and Stalla [2]. With regard to the still open problem of characterizing all weight functions admitting Chebyshev quadrature, it may be of interest that in these three cases the corresponding functions \( v \) in (8) have singularities either at an interior point or at one or both of the end points of the interval \([-1, 1]\).

After establishing the existence of the Chebyshev quadrature rules for the weight functions (10), (11) and (12), and obtaining their nodes, we shall indicate a simple method for constructing weight functions for which there exist infinitely many Chebyshev quadrature rules. Such examples may also help path the way toward a solution of the above-mentioned problem.

2. Construction of the Chebyshev Rules. We first consider the Chebyshev weight function \( w_1 \) given in (5). The corresponding Chebyshev rules \( Q_n^1 \) have Gaussian degree of precision \( 2n - 1 \), i.e., \( R_n^1[f] = 0 \) for all \( f \in \mathcal{P}_{2n-1} \) (see, e.g., Ghizzetti and Ossicini [11, p. 99 ff]). Transforming \( w_1 \) and \( Q_n^1 \) to the interval \([-1, 0]\) as well as to the interval \([0, 1]\) and compounding the two resulting weight functions and rules to the interval \([-1, 1]\) gives the weight function \( w_1 \) together with an equally weighted quadrature rule \( \tilde{Q}_{2n} \) having \( 2n \) nodes and degree of precision \( 2n - 1 \). The nodes of
the Chebyshev rule $Q_n^1$ are the zeros of the polynomial $T_n$, where $T_n$ denotes the Chebyshev polynomial of the first kind of degree $n$. Hence the nodes of $Q_{2n}^4$, whose nodes are the zeros of $p_{2n}$ given by

$$
\hat{p}_{2n}(x) = T_n(2x + 1)T_n(2x - 1).
$$

We shall now show that the interpolatory quadrature rule (for definition see, e.g., [1, p. 16]) $Q_{2n}^4$, whose nodes are the zeros of

$$
p_{2n}(x) = \hat{p}_{2n}(x) - \frac{1}{2},
$$

is a Chebyshev quadrature rule with $2n$ nodes for the weight function $w_4$. If $\hat{p}_{2n} - \alpha$ ($\alpha \in \mathbb{R}$) has only real zeros, then the interpolatory quadrature formula, whose nodes are the zeros of $\hat{p}_{2n} - \alpha$, is also equally weighted (see, e.g., [7, p. 103], [9], [4], [5]).

So, the proof is completed if it is shown that

(i) $R_{2n}^4[q_{2n}] = 0$, $q_{2n}(x) := x^{2n}$,

(ii) all zeros of $p_{2n}$ are real, pairwise distinct and contained in the open interval $(-1, 1)$.

Because $Q_{2n}^4$ is an interpolatory quadrature rule, we have $R_{2n}^4[f] = 0$ for all $f \in \mathcal{P}_{2n-1}$ and therefore

$$
2^{n-2}R_{2n}^4[q_{2n}] = R_{2n}^4[p_{2n}]
$$

$$
= \int_{-1}^{1} w_4(x)[T_n(2x + 1)T_n(2x - 1) - \frac{1}{2}] \, dx
$$

$$
= -\pi + 2 \int_{0}^{1} (x - x^2)^{-1/2} T_n(2x + 1)T_n(2x + 1) \, dx
$$

$$
= -\pi + 2 \int_{-1}^{1} w_1(x)T_n(x)T_n(x + 2) \, dx
$$

$$
= -\pi + 2 \int_{-1}^{1} w_1(x)T_n(x)2^{n-1}x^n \, dx
$$

$$
= -\pi + 2 \int_{-1}^{1} w_1(x)\{T_n(x)\}^2 \, dx = 0,
$$

using the known properties of $T_n$ (cf. here and in the following, e.g., Tricomi [14, p. 187 ff] or Paszkowski [13]) as well as the symmetry of $w_4$ and $p_{2n}$. This proves (i).

To prove (ii), we need consider only the interval $[0, 1]$ since $p_{2n}$ is symmetric. $T_n(2x - 1)$ has in $(0, 1)$ $n$ pairwise distinct real zeros. All $n - 1$ relative maxima of $|T_n(2x - 1)|$ as well as $T_n(1)$ have the value 1. Since $T_n(2x + 1) \geq 1$ for all $x \geq 0$, all relative maxima of $|\hat{p}_{2n}|$ as well as $\hat{p}_{2n}(1)$ have a value not less than 1. Therefore, $p_{2n}$ has all properties required in (ii).

To establish the Chebyshev quadrature rules $Q_{2n-1}^4$ for the weight function $w_4$ for all positive $n$, we consider first the Radau rules for the Chebyshev weight function $w_1$. They are given by (see, e.g., Ghizzetti and Ossicini [11, p. 101 ff.])

$$
Q_n^+ [f] = \frac{2\pi}{2n - 1} \left\{ \frac{1}{2} f(1) + \sum_{\nu=1}^{n-1} f\left( \cos \frac{2\nu}{2n - 1} \pi \right) \right\},
$$

and

$$
Q_n^- [f] = \frac{2\pi}{2n - 1} \left\{ \frac{1}{2} f(-1) + \sum_{\nu=1}^{n-1} f\left( \cos \frac{2\nu - 1}{2n - 1} \pi \right) \right\}.
$$
We note that the nodes of $Q^+_n$ are the zeros of $T_n - T_{n-1}$ and that the nodes of $Q^-_n$ are the zeros of $T_n + T_{n-1}$. Both quadrature rules have degree of precision $2n - 2$. Transforming $w_1$ and $Q^+_n$ to the interval $[-1, 0]$ and $w_1$ and $Q^-_n$ to the interval $[0, 1]$ and compounding the two resulting weight functions and rules to the interval $[-1, 1]$ yields the weight function $w_4$ together with an equally weighted quadrature rule $Q_2^{4n-1}$ having $2n - 1$ nodes and degree of precision not less than $2n - 2$. Because of the symmetry of $w_4$ and the symmetry of $Q_2^{4n-1}$, this quadrature rule has degree of precision $2n - 1$ and is therefore a Chebyshev quadrature rule. We have thus proven the following theorem.

**Theorem 1.** The weight function $w_4(x) = (1 - x^2)^{-1/2}|x|^{-1/2}(1 + |x|)^{1/2}$ admits Chebyshev quadrature. The nodes of the corresponding Chebyshev rules $Q^+_n$ are given by the zeros of $p^4_n$, where

\begin{equation}
 p^4_{2m}(x) = T_m(2x - 1)T_m(2x + 1) - \frac{1}{2},
\end{equation}

\begin{equation}
 x p^4_{2m-1}(x) = [T_m(2x - 1) + T_{m-1}(2x - 1)] [T_m(2x + 1) - T_{m-1}(2x + 1)].
\end{equation}

To establish Chebyshev quadrature for the weight functions $w_5$ and $w_6$, the lemma below is helpful (cf. also Gautschi [8, p. 482]).

**Lemma 1.** Let $w$ be a weight function on $[-1, 1]$ with $w(x) = w(-x)$ for all $x \in [-1, 1]$ and let $Q_{2n+1}$ be a quadrature rule with respect to $w$ given by

\begin{equation}
 Q_{2n+1}[f] = a_0f(0) + \sum_{\nu=1}^{n} a_{\nu} f(x_{\nu}) + f(-x_{\nu})
\end{equation}

with $0 < x_1 < \cdots < x_n$. Let $\tilde{w}(x) := w(\sqrt{x})/\sqrt{x}$ be a weight function on $[0, 1]$ and let $\tilde{Q}_{n+1}$ be the quadrature rule with respect to $\tilde{w}$ given by

\begin{equation}
 \tilde{Q}_{n+1}[f] = a_0f(0) + \sum_{\nu=1}^{n} 2a_{\nu} f(x_{\nu}^2).
\end{equation}

Then $Q_{2n+1}$ has degree of precision $2m + 1$ if and only if $\tilde{Q}_{n+1}$ has degree of precision $m$.

This lemma is well known for $w(x) = 1$ and is used to derive Gauss rules on $[0, 1]$ with respect to the weight function $x^{-1/2}$. (Note that it is possible for $a_{\nu}$ to be zero.)

Application of Lemma 1 to the weight function $w_4$ and considering the rule $Q_{2n+1} := Q^{4n}_2$ (i.e., $a_0 = 0$) which, because of symmetry, has degree of precision $2n + 1$, gives the weight function

\begin{equation}
 \tilde{w}(x) = x^{-3/4}(1 - x)^{-1/2}(1 + \sqrt{x})^{1/2}
\end{equation}

on the interval $[0, 1]$ and a corresponding equally weighted quadrature rule $\tilde{Q}_n$, whose nodes are the zeros of

\begin{equation}
 \tilde{p}^4_n(x) = T_n(2\sqrt{x} - 1)T_n(2\sqrt{x} + 1) - \frac{1}{2}.
\end{equation}

By Lemma 1 the quadrature rule $\tilde{Q}_n$ has degree of precision $n$. The nodes of $Q^{4n}_n$ are all pairwise distinct and contained in $(-1, 1)$. So by (20), the $n$ nodes of $\tilde{Q}_n$ are also pairwise distinct and contained in $(0, 1)$. Transforming to the interval $[-1, 1]$ yields the following theorem.
Theorem 2. The weight function
\[ w_5(x) = (1 - x^2)^{-1/2}(1 + x)^{-1/4}(\sqrt{2} + (1 + x)^{1/2})^{1/2} \]
admits Chebyshev quadrature. The nodes for the corresponding Chebyshev rules \( Q_n^5 \) are given by the zeros of \( p_n^5 \), where
\[ p_n^5(x) = T_n(\sqrt{2}x + 2 - 1)T_n(\sqrt{2}x + 2 + 1) - \frac{1}{2}. \]

Using (21), let \( \tilde{w} \) be defined by
\[ \tilde{w}(x) = \tilde{w}(1 - x) = x^{-1/2}(1 - x)^{-3/4}(1 + \sqrt{1 - x})^{1/2}. \]
Since \( \tilde{w} \) is a weight function on \([0, 1]\) admitting Chebyshev quadrature, so is \( \tilde{w} \). By (22), the nodes of the corresponding Chebyshev rules \( \tilde{Q}_n \) are the zeros of
\[ p_n(x) = T_n(2\sqrt{1 - x} - 1)T_n(2\sqrt{1 - x} + 1) - \frac{1}{2}. \]
Applying Lemma 1 (\( a_0 = 0 \)) to \( \tilde{w} \) and \( \tilde{Q}_n \) gives the weight function \( w_6 \) on \([-1, 1]\) and the corresponding Chebyshev rule \( Q_{2n}^6 \).

To establish the Chebyshev quadrature rules \( Q_{2n-1}^6 \), we consider again the weight function \( w_1 \) and the corresponding Radau rules \( Q_n^+ \) and \( Q_n^- \) in (15) and (16). Transforming \( w_1 \) and \( Q_n^- \) to the interval \([-1, 0]\) and \( w_1 \) and \( Q_n^+ \) to the interval \([0, 1]\) and compounding yields the weight function \( w_6 \) on \([-1, 1]\) together with a quadrature rule \( Q_{2n}^+ \). Owing to symmetry, the rule \( Q_{2n}^+ \) has degree of precision \( 2n - 1 \). The \( 2n - 2 \) nodes in \((-1, 1)\) are equally weighted, for the nodes -1 and 1 the weights are half as large as the weight of the other nodes. Applying Lemma 1 again gives on \([0, 1]\) the weight function \( \tilde{w} \) in (21) and a quadrature rule \( \tilde{Q}_n^+ \) having degree of precision \( n - 1 \). \( \tilde{Q}_n^+ \) has in \((0, 1)\) \( n - 1 \) equally weighted nodes; for the node 1 the weight is half as large as the weight of the other nodes. With the help of the transformation \( y = 1 - x \) we obtain on \([0, 1]\) the weight function \( \tilde{w} \) in (24) together with a corresponding quadrature rule \( Q_{2n}^+ \). Applying Lemma 1 again now yields the weight function \( w_6 \) and the Chebyshev quadrature rule \( Q_{2n-1}^6 \).

Theorem 3. The weight function \( w_6(x) = (1 - x^2)^{-3/4}(1 + (1 - x^2)^{1/2})^{1/2} \) admits Chebyshev quadrature. The nodes of the corresponding Chebyshev rules \( Q_n^6 \) are given by the zeros of \( p_n^6 \), where
\[ p_n^6(x) = T_n(2\sqrt{1 - x^2} - 1)T_n(2\sqrt{1 - x^2} + 1) + \frac{1}{2}, \]
\[ xp_n^6(x) = \left[ T_n(2\sqrt{1 - x^2} - 1) - T_{n-1}(2\sqrt{1 - x^2} - 1) \right] \]
\[ \times \left[ T_n(2\sqrt{1 - x^2} + 1) + T_{n-1}(2\sqrt{1 - x^2} + 1) \right]. \]

In connection with the open problem of characterizing all weight functions admitting Chebyshev quadrature, we mention that the Jacobi weight function \( (1 - x^2)^{-3/4} \) does not admit Chebyshev quadrature \([5, 6]\).

Remarks. (a) Note that the polynomial \( p_n^5 \) is symmetric on the interval \([-1, 1]\) and has even degree \( 2n \). Hence \( p_n^4, p_n^5, \tilde{p}_n \), and \( p_{2n}^6 \) are also polynomials because of the identities \( \tilde{p}_n^4(x) = p_n^4(\sqrt{x}), \tilde{p}_n^5(x) = p_{2n}^5(\sqrt{2x + 2}), \tilde{p}_n = p_n^4(\sqrt{1 - x}), \) and \( p_{2n}^6(x) = p_{2n}^4(\sqrt{1 - x^2}) \). Applying the same reasoning, \( p_{2n-1}^6 \) can also be shown to be a polynomial by virtue of the identity \( xp_{2n-1}^6(x) = q_{2n}^6(\sqrt{1 - x^2}) \), where \( q_{2n}^6 \) is defined by
\[ q_{2n}^6(x) = \left[ T_n(2x - 1) - T_{n-1}(2x - 1) \right] \left[ T_n(2x + 1) + T_{n-1}(2x + 1) \right], \]
since again, \( q_{2n}^6 \) is a symmetric polynomial of even degree \( 2n \).
(b) Since $T_n(x) = \cos(n \arccos x)$, the zeros of $p_{2m-1}^4$ and $p_{2m-1}^5$ can be obtained explicitly. Using the identity (see, e.g., [13, p. 22])

$$T_n(2x - 1) + T_{n-1}(2x - 1) = T_{2n}(\sqrt{x}) + T_{2n-2}(\sqrt{x}) = 2\sqrt{x} T_{2n-1}(\sqrt{x}),$$

we have that the zeros of the symmetric polynomial $p_{2m-1}^4$ agree with those of the function $t_{2m-1}$, where $t_{2m-1}$ is defined by

$$t_{2m-1}(x) = T_{2m-1}(|x|);$$

and by similar argument, that the zeros of $p_{2m-1}^5$ agree with those of $i_{2m-1}$, where $i_{2m-1}$ is defined by

$$i_{2m-1}(x) = T_{2m-1}(\sqrt{1 - (1 - x^2)^{1/2}}), \quad |x| \leq 1.$$  

(c) Let $W$ be defined by $W(x) = w(-x)$. If $w$ admits Chebyshev quadrature, then so does $W$. This follows by transforming $w$ and $Q_n$ by a reflection at the origin on the interval $[-1,1]$. Applying this argument to $w_5$ shows that $\tilde{w}_5$ also admits Chebyshev quadrature, where $\tilde{w}_5$ is defined by

$$\tilde{w}_5(x) = w_1(x)(1 - x)^{-1/4}(\sqrt{2} + (1 - x)^{1/2})^{1/2}.$$  

3. Construction of Weight Functions Having Infinitely Many Chebyshev Quadrature Rules. Weight functions admitting Chebyshev quadrature are rare (Gautschi [7, p. 109]). Therefore, one may seek weight functions having infinitely many Chebyshev quadrature rules. Apart from the weight functions $w_1$, $w_2$, and $w_3$, the author has found in the literature only three other weight functions having this property. They are given by Geronimus [10] as follows:

$$w_A(x) = w_1(x) \frac{1 - a + 2ax^2}{(1 - a)^2 + 4ax^2}, \quad |a| < \frac{1}{2},$$

$$w_B(x) = \begin{cases} 
0 & \text{for all } x \in (-\alpha, \alpha), 0 < \alpha < 1, \\
 w_1(x)(x^2 - \alpha^2)^{-1/2}\sqrt{|x|} \frac{(1 - \alpha^2)(1 - a) + 2a(x^2 - \alpha^2)}{(1 - \alpha^2)(1 - a) + 4a(x^2 - \alpha^2)}, & |a| < \frac{1}{2},
\end{cases}$$

$$w_C(x) = w_1(x) \frac{1 + a^2 - 2ax}{1 + b^2 - 2bx},$$

$$a = \frac{2 + b^2 - \sqrt{(1 - b^2)(4 - b^2)}}{3b}, \quad |b| < 1, |a| < 1.$$

For each of these three weight functions, Chebyshev quadrature rules $Q_n$ exist for every even $n$. In the case of $w_B$ with $a = 0$ see also Gautschi [8, p. 483], where the Gaussian degree $2n - 1$ for even $n$ has been proved.

We now indicate a simple method for constructing other weight functions having infinitely many Chebyshev quadrature rules. Given a weight function $w_a$ on $[-1,1]$ of the form

$$w_a(x) = w_1(x)v_a(x)$$
having a Chebyshev quadrature rule $Q_n^a$, we transform $w_a$ and $Q_n^a$ to the interval $[0,1]$ and apply Lemma 1. We obtain on $[-1,1]$ the weight function

$$w_h(x) = w_1(x)v_a(2x^2 - 1) = w_1(x)v_a(T_2(x))$$

and a corresponding Chebyshev quadrature rule $Q_{2n}^h$. Repeating this procedure and noting that

$$T_n(T_m) = T_{nm},$$

we obtain the following theorem.

**Theorem 4.** Let $w_a(x) = w_1(x)v_a(x)$ be a weight function having a Chebyshev quadrature rule with $n$ nodes and let $k \in \mathbb{N}$. Then the weight function

$$w_h(x) = w_1(x)v_a(T_{2k}(x))$$

has a Chebyshev quadrature rule with $2^{kn}$ nodes.

As a first example, we apply Theorem 4 to the weight function $w_2$ of Ullman given in (6). In the special case $k = 1$ we arrive at the weight function $w_A$ in (28) and the corresponding result of Geronimus [10].

Applying Theorem 4 to the weight function $w_3$ of Byrd and Stalla [2] for $k = 1$, we obtain that the weight function

$$w_D(x) = \frac{1}{a + x^2}, \quad a > 1,$$

has a Chebyshev quadrature rule with $2^{kn}$ nodes.

In the case of weight functions $w_4$ and $w_6$ we arrive at the weight functions

$$w_F(x) = w_1(x)\left|U_{2k}^a(x)\right|^{-1/2}\left|1 + \left|U_{2k}^a(x)\right|\right|^{1/2}, \quad k \in \mathbb{N},$$

$$w_F(x) = (1 - x^2)^{-3/4}\left|U_{2k-1}^a(x)\right|^{-1/2}\left[1 + \left(1 - x^2\right)^{1/2}\left|U_{2k-1}^a(x)\right|\right]^{1/2},$$

$k \in \mathbb{N}$,

having for every $n \in \mathbb{N}$ a Chebyshev quadrature rule with $2^{kn}$ nodes. ($U_m$ denotes the Chebyshev polynomial of the second kind of degree $m$.) With the help of $w_F$ resp. $w_E$ we see that for every $k \in \mathbb{N}$ there exists a weight function admitting infinitely many Chebyshev quadrature rules, which has $2^k$ resp. $2^k - 1$ pairwise distinct singularities in $(-1,1)$.

Finally, we mention two generalizations of the above principle for the construction of weight functions admitting Chebyshev quadrature. Instead of transforming $w_a$ and $Q_n^a$ in (31) to the interval $[0,1]$ we transform both to the interval $[a,1]$, $0 \leq a < 1$, and then apply Lemma 1. For example, in the case of the weight function $w_2$, we arrive at the weight function $w_B$ in (29) and the corresponding result of Geronimus [10]. A further variation is given by the additional transformation $y = a + 1 - x$ before applying Lemma 1. If $w_s$ is nonsymmetric on $[-1,1]$, this also leads to new weight functions admitting Chebyshev quadrature.

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