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Some Inequalities for Continued Fractions*

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Abstract. For some continued fractions $Q = b_0 + a_1/(b_1 + \cdots)$ with mth convergent Q_m, it is shown that relative errors are monotone in some arguments. If all the entries a_j and b_j in Q are positive, then the relative error $|Q_m/Q - 1|$ is bounded by $|Q_m/Q_{m+1} - 1|$, which is nonincreasing in the partial denominator b_j for each $j \geqslant 0$, as is $|Q_m/Q - 1|$ for $j \leqslant m+1$. If $b_j \geqslant 1$ for all $j \geqslant 1$, $b_0 \geqslant 0$, and $a_j = (-1)^{j+1}c_j$ where $c_j \geqslant 0$ and for j even, $c_j < 1$, then $|Q_m/Q - 1|$ is bounded by $|Q_m/Q_{m+2} - 1|$, and both are nonincreasing in b_j for even $j \leqslant m+2$. These facts apply to continued fractions of Euler, Gauss and Laplace used in computing Poisson, binomial and normal probabilities, respectively, giving monotonicity of relative errors as functions of the variables in suitable ranges.

For computation of various functions in suitable regions, continued fractions provide the current method of choice because of their speed of convergence for a given accuracy. Another advantage is that in certain cases error bounds are rather easily available at each stage, since one or two successive convergents are alternately above and below the final result. Thus, even in regions where continued fractions are less efficient than other methods, they may provide checks on the accuracy of those methods, which may lack such easy error bounds of their own. Then, monotonicity properties of the errors in some of the arguments are useful in reducing the amount of checking to be done. This note treats such monotonicity properties, specifically for Laplace and Gauss continued fractions useful in computing hypergeometric functions and thus probabilities of the gamma and beta families such as Poisson and binomial probabilities. For a different monotonicity property of continued fractions, see [9].

1. Continued Fractions. A *continued fraction* is given by two sequences of numbers $\{b_n\}_{n \geqslant 0}$ and $\{a_n\}_{n \geqslant 1}$, and will be written as

$$Q = b_0 + \frac{a_1}{b_1 +}\ \frac{a_2}{b_2 +} \cdots. \tag{1.1}$$

In this paper all the a_j and b_j will be real numbers. Let $T_n(z) := a_n/(b_n + z)$ for any z (the symbol " := " means "equals by definition"). Then the mth *convergent* of the continued fraction is given by

$$Q_m = b_0 + T_1(T_2(\cdots(T_m(0))\cdots))$$

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if this is defined, where $0/0$ is undefined but $a/0 = \infty$ for $a \neq 0$ and $b/(c + \infty) = 0$ for any finite b, c. Q_m is usually written as

$$Q_m = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_m}{b_m}, \qquad m \geqslant 0. \tag{1.2}$$

The continued fraction will be called *convergent* to a finite value Q if for m large enough, Q_m is defined and finite and $\lim_{m \to \infty} Q_m = Q$. A convergent continued fraction will be said to *terminate* at the mth term for the least value of m such that $a_m = 0$.

Associated with a continued fraction is the Wallis-Euler recurrence formula [15, p. 5]

$$X_m = b_m X_{m-1} + a_m X_{m-2}, \qquad m = 1, 2, \ldots . \tag{1.3}$$

For $m = 0, 1, \ldots,$ $Q_m = A_m/B_m$, where each of the sequences $\{A_m\}$ and $\{B_m\}$ satisfies (1.3) with $A_{-1} = 1$, $B_{-1} = 0$, $A_0 = b_0$, and $B_0 = 1$ [18, p. 15]. It is often convenient to combine two successive applications of (1.3), giving

$$\begin{aligned} A_{m+1} &= (b_{m+1}b_m + a_{m+1})A_{m-1} + b_{m+1}a_m A_{m-2}, \\ B_{m+1} &= (b_{m+1}b_m + a_{m+1})B_{m-1} + b_{m+1}a_m B_{m-2}. \end{aligned} \tag{1.4}$$

Then we have four-tuples defined for $k = 0, 1, 2, \ldots$ by recursion,

$$(A_{2k-1}, A_{2k}, B_{2k-1}, B_{2k}). \tag{1.5}$$

A_m and B_m are polynomials with integer coefficients in the $2m + 1$ variables $b_0, a_1, b_1, \ldots, a_m, b_m$. ($B_m$ does not depend on b_0, a_1.) For $j \leqslant m$ we have

$$\begin{aligned} &Q_m = Q_m(b_0, a_1, b_1, \ldots, a_m, b_m) = Q_j(b_0, a_1, \ldots, b_{j-1}, a_j, b_j + Q_{j,m}) \\ &\text{where} \\ &Q_{j,m} := Q_{m-j}(0, a_{j+1}, b_{j+1}, \ldots, a_m, b_m). \end{aligned} \tag{1.6}$$

If for a given $j = 0, 1, 2, \ldots,$ the vectors (A_{j-1}, A_j) and (B_{j-1}, B_j) are linearly independent (as is true for $j = 0$ by the definitions), then the two-dimensional space of all sequences $\{X_i\}_{i \geqslant j-1}$ satisfying (1.3) for $i \geqslant j + 1$ has a basis given by $\{A_i\}_{i \geqslant j-1}$ and $\{B_i\}_{i \geqslant j-1}$. The linear independence is equivalent to nonvanishing of the determinant $D_j := A_{j-1}B_j - A_jB_{j-1}$, where

$$D_0 = 1 \quad \text{and} \quad D_j = (-1)^j a_1 a_2 \cdots a_j, \qquad j \geqslant 1 \quad [18, \text{p. } 16]. \tag{1.7}$$

The following fact is known; for example, it follows from a special case of [13, Eq. (8)], and follows rather directly from [16, Eq. (6.1)]. It has been applied to study the propagation of errors; here it will be used in proving monotonicity properties.

1.8. THEOREM. *Suppose* $\{X_i\}_{i \geqslant -1}$ *satisfy* (1.3), $i \geqslant 1$, $Y_{-1} = X_{-1}$, *and* $\{Y_i\}_{i \geqslant -1}$ *satisfy the same relations except that either*

(a) *for some* $j \geqslant 1$, $Y_j = b_jY_{j-1} + a_jY_{j-2} + u$, $Y_0 = X_0$ *and* $D_j \neq 0$, *or*

(b) $j = 0$ *and* $Y_0 = X_0 + u$.

For any $k \geqslant j - 1$ *let*

$$T_{jk} = \frac{A_{j-1}B_k - B_{j-1}A_k}{D_j}.$$

Then $Y_k - X_k = uT_{jk}$.

Proof (based on [13]). Iterating (1.3) gives, for any $\{X_i\}$ satisfying the hypotheses, $X_k = UX_{j-1} + VX_j$ for some U and V. For $\{X_i\} = \{A_i\}$ and $\{X_i\} = \{B_i\}$ we get, by hypothesis, two linearly independent equations for U and V which can be solved by Cramer's rule, giving $V = T_{jk}$. Then replacing X_j by $Y_j = X_j + u$ gives the result. □

Note that for $k = j - 1$, $T_{jk} = 0$. To clarify that T_{jk} is a polynomial in $\{a_i, b_i\}_{j+1 \leqslant i \leqslant k}$, consider the A_n and B_n as functions of the sequences $\{a_i\}_{i \geqslant 1}$ and $\{b_i\}_{i \geqslant 0}$. So, for example,

$$B_1(\{a_i\}_{i\geqslant 1}, \{b_i\}_{i \geqslant 0}) = b_1 \quad \text{and} \quad B_1(\{a_{i+1}\}_{i\geqslant 1}, \{b_{i+1}\}_{i\geqslant 0}) = b_2.$$

1.9. PROPOSITION. *If $D_j \neq 0$, $j \geqslant 0$, and $k \geqslant j - 1$, then*

$$T_{jk} = B_{k-j}(\{a_{i+j}\}_{i\geqslant 1}, \{b_{i+j}\}_{i\geqslant 0}).$$

Proof. The proof of Theorem 1.8 shows that if $\{S_i\}_{i \geqslant j-1}$ and $\{T_i\}_{i \geqslant j-1}$ both satisfy (1.3) for all $m \geqslant j + 1$, and are linearly independent, then

$$T_{jk} = (S_{j-1}T_k - T_{j-1}S_k)/(S_{j-1}T_j - T_{j-1}S_j), \qquad k \geqslant j - 1.$$

Specifically, take $S_{j-1} = 1$, $T_{j-1} = 0$, $S_j = b_j$, and $T_j = 1$. Then, without loss of generality, j and k can be shifted by j, so it is enough to prove

$$T_{0r} = B_r(\{a_i\}_{i\geqslant 1}, \{b_i\}_{i \geqslant 0}) = B_r, \qquad r = -1, 0, 1, \dots$$

$(r = k - j)$, and this is clear from the definitions. □

The next fact follows directly from Theorem 1.8.

1.10. COROLLARY. *If $j \geqslant 1$, $D_j \neq 0$ and $k \geqslant j - 1$, then $\partial A_k/\partial b_j = A_{j-1}T_{jk}$, $\partial B_k/\partial b_j = B_{j-1}T_{jk}$, $\partial A_k/\partial a_j = A_{j-2}T_{jk}$, and $\partial B_k/\partial a_j = B_{j-2}T_{jk}$, while $\partial A_k/\partial b_0 = B_k$ and $\partial B_k/\partial b_0 = 0$. Also, for $j \geqslant 1$,*

$$\frac{\partial Q_k}{\partial b_j} = \frac{(B_k A_{j-1} - A_k B_{j-1})T_{jk}}{B_k^2} = \frac{(B_k A_{j-1} - A_k B_{j-1})^2}{B_k^2 D_j}$$

$$= \frac{(Q_{j-1} - Q_k)^2 B_{j-1}}{(Q_{j-1} - Q_j)B_j}, \quad \textit{and} \quad \frac{\partial Q_k}{\partial b_0} = 1.$$

1.11. THEOREM. *If $D_j \neq 0$, and the continued fraction Q converges for b_j in some open interval, then on that interval, for $j \geqslant 1$,*

$$\frac{\partial Q}{\partial b_j} = -\frac{(Q_{j-1} - Q)^2 B_{j-1}}{(Q_j - Q_{j-1})B_j} \quad \textit{and} \quad \frac{\partial Q}{\partial b_0} = 1.$$

Proof. For $j \geqslant 1$, this is the last form in (1.10) with $\partial/\partial b_j$ interchanged with the limit $Q_k \to Q$. To justify the interchange, first note that if the continued fraction has terminated by $i = j$, then simply $Q = Q_k$, $k \geqslant j$. Otherwise, the $Q_{j,k}$ defined in (1.6) are the convergents of a continued fraction converging to some $Q_{j,\infty}$, and we have

$$Q = \frac{(b_j + Q_{j,\infty})A_{j-1} + a_j A_{j-2}}{(b_j + Q_{j,\infty})B_{j-1} + a_j B_{j-2}}, \qquad j \geqslant 1; \qquad Q = b_0 + Q_{0,\infty}.$$

In this form, where the dependence on b_j is explicit, it is clear that replacing k by ∞ in the corresponding expression for Q_k, and thus replacing Q_k by Q, can be interchanged with $\partial/\partial b_j$. $\square$

Next, the derivatives of (signed) relative errors are found.

1.12. Corollary. *If $k \geqslant j-1$, $m \geqslant j-1$, $D_j \neq 0$, and Q_k and Q_m are defined and finite, then for $j \geqslant 1$,*

$$\frac{\partial}{\partial b_j}\left(\frac{Q_m}{Q_k}-1\right)=\frac{B_{j-1}(Q_m-Q_k)\left(Q_{j-1}^2-Q_kQ_m\right)}{B_jQ_k^2(Q_j-Q_{j-1})}, \quad \text{or} \quad \frac{Q_k-Q_m}{Q_k^2} \quad \text{if } j=0.$$

If the continued fraction Q converges for b_j in an open interval, then Q_m can be replaced by Q on that interval.

Proof. One need only apply 1.10 and 1.11 and a little algebra. $\square$

2. Inequalities for Fractions with Positive Terms. The following fact is rather easily proved. It was stated, or very nearly so, by Euler [3, pp. 103–105] = [6, Vol. 14, pp. 191–192], cf. [11, Theorem 2].

2.1. Theorem. *If $a_m \geqslant 0$ and $b_m > 0$ for all $m \geqslant 1$, and if Q converges, then*

$$Q_0 \leqslant Q_2 \leqslant Q_4 \leqslant \cdots \leqslant Q \leqslant \cdots \leqslant Q_5 \leqslant Q_3 \leqslant Q_1.$$

If Q does not converge, the inequalities remain true if "$\leqslant Q \leqslant$" is deleted.

If (2.1) applies, and Q converges, one can stop calculating Q when Q_{m-1}/Q_m is as close to 1 as desired. In this case it may be better to use (1.3) individually rather than "two terms at a time" as in (1.4). Next, here is a first monotonicity result.

2.2. Theorem. *In a continued fraction as in (2.1) with $a_j \geqslant 0$ for all j, $b_j > 0$ for all $j \geqslant 1$, $b_0 \geqslant 0$, and either $b_0 > 0$ or $a_1 > 0$, the magnitude of the relative error, given by*

$$|r_{m,1}(Q)| := |Q_m/Q_{m+1} - 1|,$$

is nonincreasing in b_j for each $j \geqslant 0$ (for any fixed a_i and the other b_k). Also, if $j \leqslant m+1$ and Q is convergent for an open interval of values of b_j, then in that interval, $|Q_m/Q - 1|$ is also nonincreasing in b_j.

Proof. If $a_1 = 0$, then $Q_m = b_0 > 0$ for all m, and the result holds trivially; so assume $a_1 > 0$. We have $B_j > 0$ for all $j \geqslant 0$ by (1.3). First consider the statement about $r_{m,1}$. We can assume that $j \leqslant m+1$ and $D_j \neq 0$, since otherwise Q_m/Q_{m+1} does not depend on b_j (using (1.7)). By Corollary 1.12 we have, if $j \geqslant 1$, or $j = 0$,

$$\frac{\partial(Q_m/Q_{m+1})}{\partial b_j}=\frac{B_{j-1}(Q_m-Q_{m+1})\left(Q_{j-1}^2-Q_mQ_{m+1}\right)}{B_jQ_{m+1}^2(Q_j-Q_{j-1})}, \quad \text{or} \quad \frac{Q_{m+1}-Q_m}{Q_{m+1}^2},$$

respectively. Now by Theorem 2.1, $Q_j - Q_{j-1}$ has the sign of $(-1)^{j+1}$. Also, since $j-1 \leqslant m$, $Q_{j-1}^2 - Q_{m+1}Q_m$ has the sign $(-1)^j$. So, the displayed expressions have sign opposite to that of $Q_m - Q_{m+1}$ (or are 0), which implies the first result.

For the case of Q_m/Q, with $j \leqslant m+1$, the proof is essentially the same, using Theorem 1.11. $\square$

3. Alternating Continued Fractions. As will be seen in Section 5, some useful continued fractions have the following property.

Definition. A continued fraction (1.1) will be called *alternating* if for all $j \leqslant J$, the least i with $a_i = 0$,

(a) $a_j \equiv (-1)^{j+1}c_j$ where $c_j \geqslant 0$ and for j even, $c_j < 1$, and

(b) $b_0 \geqslant 0$ and $b_j \geqslant 1$ for $j \geqslant 1$.

The following fact is known, at least in some cases [12, p. 108]; [14, p. 1452]. It follows directly from (1.2).

3.1. THEOREM. *For any alternating continued fraction Q, if Q converges, we have*

$$Q_1 \leqslant Q_4 \leqslant Q_5 \leqslant Q_8 \leqslant \cdots \leqslant Q \leqslant \cdots \leqslant Q_7 \leqslant Q_6 \leqslant Q_3 \leqslant Q_2.$$

If Q fails to converge, the inequalities are true with "$\leqslant Q \leqslant$" deleted.

For an alternating continued fraction, in view of (3.1), Q is between Q_m and Q_{m+2} for any m, so it is natural to consider the error after two more terms,

$$r_{m,2}(Q) := (Q_m/Q_{m+2}) - 1.$$

To compute Q to a given relative error (neglecting rounding errors), we can iterate (1.3) and (1.4), stopping when $|r_{m,2}(Q)|$ is as small as desired.

3.2. THEOREM. *For any alternating continued fraction Q, any $m \geqslant 1$, and even $j \leqslant m+2$, the magnitude $|r_{m,2}(Q)|$ of the relative error is nonincreasing in b_j, and if Q is convergent for an open interval of values of b_j, then in that interval, $|(Q_m/Q) - 1|$ is nonincreasing in b_j.*

Proof. Let $J := \min\{i\colon\ c_i = 0\}$, or $+\infty$ if there is no such i. If $j \geqslant J$, then nothing depends on b_j and the results are clear. Suppose $1 \leqslant j < J$. Then (since j is even) $J \geqslant 3$, $A_1 = b_1 b_0 + c_1 > b_0 \geqslant 0$, and $B_1 = b_1 \geqslant 1 > 0$, so $Q_1 > 0$. Thus $Q_m > 0$ for all $m \geqslant 1$ by (3.1). Then $D_j \neq 0$ and $Q_{j-1} \neq Q_j$ by (1.7).

Inductively, it will be shown that $B_k > 0$ for all $k \geqslant 0$, as is true for $k = 0, 1$, and that for k odd, $B_k \geqslant B_{k-1}$, as is true for $k = 1$. In fact, for k odd, $B_k = b_k B_{k-1} + c_k B_{k-2} \geqslant B_{k-1} > 0$ by the induction hypothesis. For k even, $k \geqslant 2$, $B_k = b_k B_{k-1} - c_k B_{k-2} > 0$ since $B_{k-1} \geqslant B_{k-2} > 0$ and $c_k < 1 \leqslant b_k$, proving the claims. Thus also $A_m > 0$ for all $m \geqslant 1$.

Now consider Q_m/Q_{m+2}. By Corollary 1.12, if $1 \leqslant j \leqslant m+1$, or $j = 0$,

$$\frac{\partial(Q_m/Q_{m+2})}{\partial b_j} = \frac{B_{j-1}(Q_m - Q_{m+2})(Q_{j-1}^2 - Q_m Q_{m+2})}{B_j Q_{m+2}^2 (Q_j - Q_{j-1})}, \quad \text{or} \quad \frac{Q_{m+2} - Q_m}{Q_{m+2}^2},$$

respectively. Now by (3.1), for $j \equiv 2 \mod 4$, $Q_j > Q_{j-1}$, and since $j - 1 \leqslant m$, we have $Q_{j-1}^2 \leqslant Q_m Q_{m+2}$. For $0 \neq j \equiv 0 \mod 4$, both the last two inequalities are reversed. Thus in any case, the above derivative and $Q_m/Q_{m+2} - 1$ have opposite signs or are 0, giving the desired result.

If $j = m+2$, then by 1.10,

$$\frac{\partial(Q_m/Q_{m+2})}{\partial b_j} = -\frac{(Q_{j-1} - Q_{m+2})^2 B_{j-1} Q_m}{Q_{m+2}^2 (Q_{j-1} - Q_j) B_j}$$

which has the sign of $Q_j - Q_{j-1}$. Recalling that j is even, this sign is $+1$ for $j \equiv 2 \mod 4$ and -1 for $j \equiv 0 \mod 4$, which is opposite to the sign of $Q_m - Q_{m+2} = Q_{j-2} - Q_j$, completing the proof for the Q_m/Q_{m+2} case.

Now for the Q_m/Q case, the same proofs as above apply with Q in place of Q_{m+2} for each case $j \leqslant m+1$ or $j = m+2$. $\square$

4. Even Parts of Continued Fractions. Starting from an arbitrary continued fraction Q given in (1.1), one can form another continued fraction V such that the convergents of V equal the even convergents of Q, $V_k = Q_{2k}$ for $k = 0, 1, 2, \ldots,$ by

$$V = b_0 + \frac{a_1 b_2}{b_1 b_2 + a_2 +} \; \frac{-a_2 a_3 b_4}{(b_2 b_3 + a_3) b_4 + b_2 a_4 +} \; \frac{-a_4 a_5 b_2 b_6}{(b_4 b_5 + a_5) b_6 + b_4 a_6 +} \; \frac{-a_6 a_7 b_4 b_8}{(b_6 b_7 + a_7) b_8 + b_6 a_8 +} \cdots,$$

provided that for all $k = 1, 2, \ldots,$ $b_{2k} \neq 0$ and (for Q_{2k} to be defined and finite) $B_{2k} \neq 0$ [15, pp. 200–201]. From this it is easily seen that:

4.1. THEOREM. *For an alternating continued fraction Q, with even part continued fraction $V = b_0 + s_1/(t_1 + s_2/(t_2 + \cdots))$, the entries s_j and t_j are nonnegative, at least until some $s_j = 0$.*

It is clear from (2.1) that for any continued fraction such that the entries in its even part are nonnegative, with denominators strictly positive, we have

$$Q_0 \leqslant Q_4 \leqslant Q_8 \leqslant \cdots \leqslant Q_{10} \leqslant Q_6 \leqslant Q_2. \tag{4.2}$$

This includes some of the inequalities in Theorem 3.1. The "alternating" property is not necessary for the conclusion of Theorem 4.1, so that Theorem 3.1 can be extended if desired.

5. Continued Fractions for Normal, Poisson and Binomial Probabilities. For the standard normal probability density function

$$\phi(x) := (2\pi)^{-1/2} \exp(-x^2/2)$$

and cumulative distribution function

$$\Phi(x) := \int_{-\infty}^{x} \phi(t)\, dt,$$

we have Laplace's continued fraction [18, formula (92.15)]

$$\Phi(-x) = \phi(x) \left\{ \frac{1}{x +} \; \frac{1}{x +} \; \frac{2}{x +} \; \frac{3}{x +} \cdots \right\}, \quad x > 0. \tag{5.1}$$

For $x > 0$, Theorem 2.2 implies that the number of terms needed to attain a given relative error in (5.1) decreases as x increases. Thus (5.1) is more useful for larger x, say $x \geqslant 2$ or 3. See, for example, [1] and [2]. Also, $\Phi(x) \equiv 1 - \Phi(-x)$.

Individual Poisson probabilities are defined by

$$p(k, \lambda) := e^{-\lambda} \lambda^k / k!, \qquad k = 0, 1, \ldots, \text{ for } \lambda > 0,$$

and cumulative probabilities by

$$P(k, \lambda) := \sum_{0 \leqslant j \leqslant k} p(j, \lambda), \qquad Q(k, \lambda) := \sum_{k \leqslant j < \infty} p(j, \lambda).$$

For ratios of cumulative to individual probabilities we have

$$\begin{aligned} P(k, \lambda)/p(k, \lambda) &= 1 + k/\lambda + k(k-1)/\lambda^2 + \cdots + k!/\lambda^k \\ &= \frac{1}{1 -} \; \frac{k/\lambda}{1 +} \; \frac{1/\lambda}{1 -} \; \frac{(k-1)/\lambda}{1 +} \; \frac{2/\lambda}{1 -} \; \frac{(k-2)/\lambda}{1 +} \cdots \end{aligned} \tag{5.2}$$

[15, p. 313]; [18, p. 350]

$$= \frac{1}{1 -} \; \frac{k}{\lambda +} \; \frac{1}{1 -} \; \frac{k-1}{\lambda +} \; \frac{2}{1 -} \; \frac{k-2}{\lambda +} \; \frac{3}{1 -} \cdots. \tag{5.3}$$

Sometimes called Legendre's continued fraction [10, Chapter 17], this was given by Euler [4, p. 232] and [5, pp. 40–41] = [6, Vol. 14, p. 612 and Vol. 16, Part 1, pp. 36, 41]. For alternate forms of (5.2), see [8, Section 4.3]. Next,

$$Q(k,\lambda)/p(k,\lambda) = 1 + \lambda/(k+1) + \lambda^2/((k+1)(k+2)) + \cdots \tag{5.4}$$

$$= \frac{1}{1-}\,\frac{\lambda/(k+1)}{1+}\,\frac{\lambda/((k+1)(k+2))}{1-}\,\frac{(k+1)\lambda/((k+2)(k+3))}{1+}\cdots \tag{5.5}$$

[15, p. 312, (18)]

$$= \frac{1}{1-}\,\frac{\lambda}{k+1+}\,\frac{\lambda}{k+2-}\,\frac{(k+1)\lambda}{k+3+}\,\frac{2\lambda}{k+4-}\,\frac{(k+2)\lambda}{k+5+}\,\frac{3\lambda}{k+6-}\cdots$$

[18, p. 348].

There is another continued fraction for $Q(k,\lambda)/p(k,\lambda)$ [8, Section 5] but its convergents are just the partial sums of the series (5.4) of positive terms. Thus, it cannot have such properties as (2.1) or (3.1); for large k and λ, for example $k \approx \lambda + 3\lambda^{1/2}$, (5.4) tends to converge more slowly than (5.5).

Individual binomial probabilities are defined by

$$b(k,n,p) := \binom{n}{k} p^k q^{n-k}, \quad \text{where } 0 \leqslant p \leqslant 1,\ q := 1-p,$$

and $k = 0, 1, \ldots, n$, where 0^0 is replaced by 1 in this case. Cumulative probabilities are defined by

$$B(k,n,p) := \sum_{0\leqslant j\leqslant k} b(j,n,p), \qquad E(k,n,p) := \sum_{k\leqslant j\leqslant n} b(j,n,p).$$

For the ratios of cumulative to individual probabilities we again have continued fractions. In terms of the hypergeometric function F we have

$$E(k,n,p)/b(k,n,p) = F(k-n, 1, k+1, -p/q) \qquad [18, \text{p. } 335, (89.1)]$$

$$= 1 + \frac{(n-k)p}{(k+1)q} + \frac{(n-k)(n-k-1)p^2}{(k+1)(k+2)q^2} + \cdots$$

$$= \frac{1}{1-}\,\frac{(n-k)p/((k+1)q)}{1+}\,\frac{(n+1)p/(q(k+1)(k+2))}{1-}$$
$$\frac{(k+1)(n-k-1)p/(q(k+2)(k+3))}{1+}$$
$$\frac{2(n+2)p/(q(k+3)(k+4))}{1-}$$
$$\frac{(k+2)(n-k-2)p/(q(k+4)(k+5))}{1+} \tag{5.6}$$
$$\frac{3(n+3)p/(q(k+5)(k+6))}{1-}\cdots$$

[15, p. 312]; [18, p. 340]

$$= \frac{1}{1-}\,\frac{(n-k)p/q}{k+1+}\,\frac{(n+1)p/q}{k+2-}\,\frac{(k+1)(n-k-1)p/q}{k+3+}$$
$$\frac{2(n+2)p/q}{k+4-}\,\frac{(k+2)(n-k-2)p/q}{k+5+}\,\frac{3(n+3)p/q}{k+6-}\cdots.$$

The lower cumulative binomial $B(k, n, p)$ need not be treated separately, since it equals $E(n-k, n, q) = 1 - E(k+1, n, p)$.

Now the relative errors $r_{m,2}(Q) := Q_m/Q_{m+2} - 1$ in some of the above continued fractions will be treated as functions of λ or p.

5.7. THEOREM. *For any fixed integers k and $m \geqslant 0$, the relative error $|r_{m,2}(Q)|$ in* (5.2) *is decreasing in λ for $\lambda > k$ and converges to* 0 *as $\lambda \uparrow +\infty$. Likewise in* (5.5), *$|r_{m,2}(Q)|$ is increasing in λ for $\lambda < k+1$ and converges to* 0 *as $\lambda \downarrow 0$. In* (5.6), *for fixed k, n and m, $|r_{m,2}(Q)|$ is increasing in p for $p < (k+1)q/(n-k)$ and converges to* 0 *for $p \downarrow 0$.*

Proof. In all three of these continued fractions we have $b_0 = 0$, $a_1 = 1$ and $b_r = 1$ for all $r \geqslant 1$. All are alternating, for the ranges of variables being considered, so that (3.1) and (3.2) apply. We consider pairs of variables (a_{2j}, a_{2j+1}) for $j = 1, 2, \ldots$. In (5.2), replacing λ by $\mu > \lambda$ multiplies both a_{2j} and a_{2j+1} by λ/μ. Equivalently, one can leave a_{2j} and a_{2j+1} fixed and replace $b_{2j} = 1$ by $b_{2j} = \mu/\lambda$. By (3.2), this can only decrease $|r_{m,2}(Q)|$. Doing this for $j = 1, 2, \ldots, [(m+1)/2]$ gives the result for m odd. For m even, we still have the $j = m+2$ term, where increasing λ is equivalent to increasing b_j, so again (3.2) applies, giving that $|r_{m,2}(Q)|$ is decreasing in λ for $\lambda > k$ in (5.2). The monotonicity of $|r_{m,2}(Q)|$ is proved likewise for (5.4) and (5.6). We have $Q_m \to 1$ for all m, and so $r_{m,2}(Q) \to 0$, if $\lambda \to +\infty$ in (5.2), $\lambda \downarrow 0$ in (5.5), or $p \downarrow 0$ in (5.6). $\square$

The restrictions on λ or p in (5.7) are needed to obtain the alternating property and thus (3.1). The condition on p is equivalent to $k > np - q$. If it fails, the continued fraction (5.6) still converges (if $0 < p < 1$: [18, p. 339]), but it may first hover around an incorrect value. For example if $k = 150$, $n = 319$, and $p = .63$, then $Q_{22}, Q_{24}, \ldots, Q_{42}$ are all between -1.047804 and -1.047805, but $Q = \lim_{m \to \infty} Q_m$ is $4.362 \cdot 10^8$ to the given accuracy, as is Q_m for $m \geqslant 132$. On such "deceptive" convergence see also [7].

Waadeland [17] considers partial derivatives $\partial Q/\partial a_k$ when $b_n \equiv 1$. His results do not seem strongly related to those in (1.10) and (1.11) above.

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