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A Simple Homotopy Method for Determining All Isolated Solutions to Polynomial Systems

By Walter Zulehner

Abstract. A new homotopy method for solving systems of polynomial equations is presented. The homotopy equation is extremely simple: It is linear with respect to the homotopy parameter and only one auxiliary parameter is needed to regularize the problem. Within some limits, an arbitrary starting problem can be chosen, as long as its solution set is known. No restrictions on the polynomial systems are made. A few numerical tests are reported which show the influence of the auxiliary parameter, resp. the starting problem, upon the computational cost of the method.

1. Introduction. In this paper we discuss the problem of finding all isolated solutions to a system of equations

$$P(z) = 0, \tag{1.1}$$

where the components P_j of P: $\mathbf{C}^n \to \mathbf{C}^n$ ($\mathbf{C}$ denotes the complex plane) are polynomials in $z = (z_1, z_2, \ldots, z_n)$ of degree ($\deg P_j =$) $d_j \geqslant 1$ for $j = 1, 2, \ldots, n$. It is well known that there are at most $D = d_1 \cdot d_2 \cdot \cdots \cdot d_n$ isolated solutions to (1.1) (Bezout's Theorem).

Several homotopy methods for solving this problem have been proposed. In these methods a simple system of polynomial equations

$$Q(z) = 0 \tag{1.2}$$

with known solution set is continuously deformed into the system (1.1) by some homotopy mapping H: $\mathbf{C}^n \times [0,1] \to \mathbf{C}^n$, e.g.,

$$H(z,t) = tP(z) + (1-t)Q(z), \qquad t \in [0,1], \tag{1.3}$$

as the homotopy parameter t varies along the real line from 0 to 1. A solution to (1.1) is obtained by following a continuous solution path $z(t)$, $t \in [0,1]$ (i.e., $z(t) \in \mathbf{C}^n$ solves the system

$$H(z,t) = 0 \tag{1.4}$$

for each $t \in [0,1]$), starting at some solution to (1.2). Computationally efficient path following techniques often require regularity of the solution path, i.e.,

$$\det H_z(z(t),t) \neq 0 \quad \text{for } t \in [0,1), \tag{1.5}$$

where H_z denotes the Jacobian of H with respect to z.

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The homotopy mapping proposed in this paper is given by (1.3) with

$$Q(z) = aR(z),$$

where $a \in \mathbf{C}$ is a nonzero auxiliary parameter and

$$R(z) = 0 \tag{1.6}$$

is any polynomial system with

(A) $\deg R_j = d_j$ for $j = 1, 2, \ldots, n$ and exactly D easily computable distinct (simple) zeros.

Typically, system (1.6) is obtained by splitting P,

$$P(z) = R(z) + S(z),$$

where R satisfies (A) and S is regarded as a perturbation. Alternatively, a universal choice for R would be

$$R_j(z) = z_j^{d_j} - 1 \quad \text{for } j = 1, 2, \ldots, n. \tag{1.7}$$

The introduction of an auxiliary parameter $a \in \mathbf{C}$ is necessary to regularize the solution paths. A proper choice of a ensures that (1.5) is fulfilled. More precisely, it is shown that, for almost each a on the unit circle in $\mathbf{C}$, the proposed homotopy method works, that means: A regular solution path starts at each solution to (1.6) and each isolated solution to (1.1) is endpoint of such a regular path at $t = 1$. The phrase "for almost each $a \in A$" is an abbreviation of "for each $a \in A - E$, where E is finite". Therefore, it is reasonable to assume that the homotopy method works for a randomly chosen a on the complex unit circle.

The main features of the new method are its simplicity (only one auxiliary parameter), flexibility (with respect to the starting problem (1.6)), and universality (no restrictions on P). In previous papers, more auxiliary parameters were used. Under a mild condition on P the homotopy method introduced by Chow, Mallet-Paret and Yorke [3] works for $n^2 + n$ randomly chosen auxiliary parameters. These results were improved by Brunovský and Meravý [2] who showed that a slightly modified homotopy method works for n^2 auxiliary parameters out of some open and dense subset of $\mathbf{C}^{n^2}$. Only $2n$ randomly chosen auxiliary parameters are required by Wright [18], where the solution set of (1.1) is assumed to be discrete, by Li [13] under the same assumption as used in [3], and by Morgan [15]. In a recent paper, Li and Sauer [14] showed that the $2n$ auxiliary parameters can be chosen from some open and dense subset of $\mathbf{C}^{2n}$. Garcia and Zangwill [6], [7], [8] proposed a homotopy method which works for almost each system (1.1) in the measure-theoretic sense. There, the coefficients of the polynomials P_j, $j = 1, 2, \ldots, n$, play the role of randomly chosen auxiliary parameters. In all above-mentioned papers the type of the starting problem was fixed. The present paper is strongly related to the work of Drexler [4], [5] and Zulehner [19], as far as the theoretical background is concerned. However, Drexler requires a discrete solution set of (1.1), and these three papers deal with homotopies of a different sort, using a possibly complex parameter t.

Many different algorithms for numerically following the solution paths were studied; see, e.g., Garcia and Zangwill [6], [7], Kojima and Mizuno [10], Kojima, Nishino and Arima [11], Kuhn [12] for simplicial methods, which are based on piecewise linear approximation of H, and Drexler [4], [5], Garcia and Zangwill [8],

Wright [18] for continuation methods where path following is done by numerically integrating an exact differential equation. In this paper we adopt the second approach, using a standard predictor-corrector procedure. In order to deal with poles, which frequently occur for solution paths at $t = 1$, the differential equation was set up in $\mathbf{C}^{n+1}$ (including a scaling condition) rather than in $\mathbf{C}^n$. This allows one to compute proper as well as improper isolated zeros of P.

2. Setting Up the Problem in Projective n-Space. For theoretical reasons, the problem (1.1) is set up in projective n-space rather than in $\mathbf{C}^n$.

Notation. For any nonconstant polynomial, say F, in $z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$ of degree d, there is a homogeneous polynomial, denoted by the corresponding script letter, here $\mathscr{F}$, in $z = (z_0, z_1, \ldots, z_n) \in \mathbf{C}^{n+1}$ of the same degree, uniquely determined by

$$\mathscr{F}(z) = z_0^d F\left(\frac{1}{z_0} z\right) \quad \text{for } z_0 \neq 0.$$

The original polynomial is recovered from $\mathscr{F}$ by setting $z_0 = 1$:

$$F(z) = \mathscr{F}(1, z).$$

$\mathscr{F}$ is called the *homogenization* of F.

With these notations, the homogenized version of (1.1) is written as

$$\mathscr{P}(z) = 0, \tag{2.1}$$

a system of n homogeneous equations in $n + 1$ variables. Obviously, if z solves (2.1), any scalar multiple ρz, $\rho \in \mathbf{C}$, is also a solution to (2.1). Therefore, it is natural to identify points in $\mathbf{C}^{n+1}$ which lie on the same straight line through the origin. This motivates the

Definition (Projective n-space $\mathbf{P}^n$). $\mathbf{P}^n$ is the collection of all one-dimensional complex subvectorspaces in $\mathbf{C}^{n+1}$.

Any solution $z = (z_1, z_2, \ldots, z_n)$ to (1.1) in $\mathbf{C}^n$ produces a solution to (2.1) in $\mathbf{P}^n$, namely the linear span of $(1, z_1, \ldots, z_n)$ in $\mathbf{C}^{n+1}$. On the other hand, each solution to (2.1) in $\mathbf{P}^n$ is either a straight line through the origin containing a point $(1, z_1, \ldots, z_n)$, which corresponds to the (proper) zero $(z_1, z_2, \ldots, z_n)$ of P, or a straight line in $\mathbf{C}^{n+1}$ lying in the hyperplane $z_0 = 0$. In the second case the solution in $\mathbf{P}^n$ is called an improper zero of P.

For analyzing the homogenized homotopy mapping $\mathscr{H}$, we study the closely related mapping

$$\mathsf{H}(z, \lambda) = \lambda_0 \mathscr{P}(z) + \lambda_1 \mathscr{R}(z) \quad \text{with } \lambda = (\lambda_0, \lambda_1) \in \mathbf{C}^2,$$

where $\mathscr{R}$ denotes the homogenization of R, see (1.6). Next we introduce the set

$$X = \{(z, \lambda) \in \mathbf{C}^{n+1} \times \mathbf{C}^2 \colon \mathsf{H}(z, \lambda) = 0\}.$$

Each solution $(z, t) \in \mathbf{C}^n \times [0, 1]$ to (1.4) corresponds to a point $((1, z), (t, a(1 - t))) \in \mathbf{C}^{n+1} \times \mathbf{C}^2$ in X. The set $X \subset \mathbf{C}^{n+1} \times \mathbf{C}^2$ can also be viewed as a subset of $\mathbf{P}^n \times \mathbf{P}^1$, since H is homogeneous in $z \in \mathbf{C}^{n+1}$ and in $\lambda \in \mathbf{C}^2$. In the next section the set X is analyzed.

3. Properties of the set $X \subset \mathbf{P}^n \times \mathbf{P}^1$. For simplicity we do not distinguish between an element of $\mathbf{P}^n$, which is a straight line in $\mathbf{C}^{n+1}$, and any nonzero point on this straight line. So $0 \neq z = (z_0, z_1, \ldots, z_n)$ denotes a point in $\mathbf{C}^{n+1}$ as well as the straight line in $\mathbf{C}^{n+1}$ through z.

In order to introduce topological and differential-geometric concepts for projective n-space, we now represent $\mathbf{P}^n$ as a complex analytic manifold: Let U_i be the set of all one-dimensional subspaces of $\mathbf{C}^{n+1}$ not lying in the hyperplane $z_i = 0$. Naturally,

$$\mathbf{P}^n = \bigcup_{i=0}^{n} U_i.$$

A one-dimensional subspace in $\mathbf{C}^{n+1}$ through $(z_0, z_1, \ldots, z_n)$ not lying in the hyperplane $z_i = 0$ is uniquely determined by the point of intersection $(z_0/z_i, \ldots, z_{i-1}/z_i, 1, z_{i+1}/z_i, \ldots, z_n/z_i)$ with the hyperplane $z_i = 1$, which is just a copy of $\mathbf{C}^n$. Hence, the set $U_i \subset \mathbf{P}^n$ can be identified with $\mathbf{C}^n$ by the mapping

$$D_i\colon U_i \to \mathbf{C}^n \quad \text{with } D_i(z_0, z_1, \ldots, z_n) = \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right).$$

The system $\{(U_i, D_i)\colon i = 0, 1, \ldots, n\}$ is a set of charts covering $\mathbf{P}^n$ by a union of $n + 1$ copies of $\mathbf{C}^n$. Thus $\mathbf{P}^n$ becomes a complex analytic manifold.

Notation. Let $p \in \mathbf{P}^n$, $S \subset \mathbf{P}^n$. If $p \in U_i$, resp. $U_i \cap S \neq \emptyset$, for some $i = 0, 1, \ldots, n$, the point $D_i(p) \in \mathbf{C}^n$, resp. the set $D_i(U_i \cap S) \subset \mathbf{C}^n$, are called *affine representations* of p resp. S, briefly denoted by the corresponding primed letters p', resp. S'. In a natural way, this notation is extended to points and subsets of $\mathbf{P}^n \times \mathbf{P}^1$.

Topological concepts can easily be introduced for $\mathbf{P}^n$, for example:

Definition. Let $S \subset \mathbf{P}^n$, $p \in S$. The point p is an *isolated point of* S if and only if it is an isolated point in some affine representation, i.e., $p' \in \mathbf{C}^n$ is isolated in $S' \subset \mathbf{C}^n$ for the natural topology in $\mathbf{C}^n$.

In a similar way, concepts from differential geometry are carried over to $\mathbf{P}^n \times \mathbf{P}^1$:

Definition. Let $(z, \lambda) \in X \subset \mathbf{P}^n \times \mathbf{P}^1$. X is *smooth at* (z, λ) if and only if there is an affine representation $(z', \lambda') \in X' \subset \mathbf{C}^n \times \mathbf{C}$ such that, within some neighborhood $U_{z'} \times U_{\lambda'}$ of (z', λ'), the set X' coincides with the graph of some analytic mapping $\varphi\colon U_{z'} \to U_{\lambda'}$.

Strongly related to smoothness is the concept of regularity:

Definition. Let $(z, \lambda) \in X \subset \mathbf{P}^n \times \mathbf{P}^1$. The point (z, λ) is *regular* if and only if $\operatorname{rank} \mathsf{H}_z(z, \lambda) = n$.

A straightforward application of the Implicit Function Theorem to some affine representation gives

LEMMA 1. *Let* $(z, \lambda) \in X \subset \mathbf{P}^n \times \mathbf{P}^1$. *If* (z, λ) *is a regular point, then* X *is smooth at* (z, λ).

The proof is omitted.

Next we study the fibers of X_λ with respect to $\lambda \in \mathbf{P}^1$:

Definition. $X_\lambda = \{z \in \mathbf{P}^n\colon \mathsf{H}(z, \lambda) = 0\}$ for $\lambda \in \mathbf{P}^1$.

Example.

$$X_{(0,1)} = \{z \in \mathbf{P}^n\colon \mathscr{R}(z) = 0\},$$

$$X_{(1,0)} = \{z \in \mathbf{P}^n\colon \mathscr{P}(z) = 0\}.$$

We have the important

LEMMA 2. *For almost each* $\lambda \in \mathbf{P}^1$ (i.e., for each $\lambda \in \mathbf{P}^1 - E$, E is finite), *in particular for* $\lambda = (0,1)$, *the set* X_λ *consists of exactly* D *distinct points and* (z, λ) *is regular for each* $z \in X_\lambda$.

Proof. See appendix.

Of special interest for the homotopy method is the limit of X_λ as λ approaches $(1,0)$:

Definition. $\lim_{\lambda \to (1,0)} X_\lambda$ is the set of all limit points of sequences $\{(z^{(k)}, \lambda^{(k)})\}$ with $z^{(k)} \in X_{\lambda^{(k)}}$ as $\lambda^{(k)}$ approaches $(1,0)$.

LEMMA 3. $\lim_{\lambda \to (1,0)} X_\lambda$ *consists of at most* D *points, including all isolated points of* $X_{(1,0)}$.

Proof. See appendix.

4. The Solution Set of the Homotopy Equation. We now apply the results of the previous section to the homotopy equation

$$\mathscr{H}(z,t) = t\mathscr{P}(z) + a(1-t)\mathscr{R}(z) \quad \text{for } t \in [0,1],\ z \in \mathbf{P}^n$$

for some nonzero parameter $a \in \mathbf{C}$. Obviously,

$$\mathscr{H}(z,t) = \mathsf{H}(z, \lambda_a(t))$$

with $\lambda_a(t) = (t, a(1-t))$, $t \in [0,1]$. An immediate consequence of this representation and the results of the previous section is

THEOREM 1. *For almost each parameter* $a \in \mathbf{C}$ *on the unit circle, there are* D *functions* $z^{(l)}$: $[0,1] \to \mathbf{P}^n$, *continuous on* $[0,1]$ *and analytic on* $[0,1)$, $l = 1, 2, \ldots, D$, *such that*

(a) $\{z^{(l)}(t)\colon l = 1, 2, \ldots, D\}$ *is the set of all solutions to*

$$\mathscr{H}(z,t) = 0 \quad \text{for } t \in [0,1)$$

and rank $\mathscr{H}_z(z^{(l)}(t), t) = n$ *for* $l = 1, 2, \ldots, D$, $t \in [0,1)$.

(b) $\{z^{(l)}(1)\colon l = 1, 2, \ldots, D\}$ *contains all isolated solutions of*

$$\mathscr{P}(z) = 0.$$

Proof. In order to apply the results of the previous section, we first prove that, for almost each $a \in \mathbf{C}$ on the unit circle, $\lambda_a(t) \notin E$ for $t \in [0,1)$; see Lemma 2. This is trivial for $t = 0$, since $(0,1) \notin E$. For $0 < t < 1$, we have $D_1(\lambda_a(t)) = a(1-t)/t \in \mathbf{C}$; see Section 2. On the other hand, $D_1(E)$ is a finite subset of $\mathbf{C}$, say $\{e_1, e_2, \ldots, e_N\}$. For almost each $a \in \mathbf{C}$ on the unit circle, namely for each a with $\arg(a) \neq \arg(e_i)$, $i = 1, 2, \ldots, N$, we have $D_1(\lambda_a(t)) \notin D_1(E)$ and therefore $\lambda_a(t) \notin E$ if $0 < t < 1$. Now Lemma 2 and Lemma 1 directly imply the existence and analyticity of the functions $z^{(l)}$ on $[0,1)$. For any $l = 1, 2, \ldots, D$, there is at least one cluster point of $z^{(l)}(t)$ as t approaches 1, since $\mathbf{P}^n$ is a compact topological space. Assume that there is a second cluster point. Then, for some $i = 0, 1, \ldots, n$, the function $D_i \circ z^{(l)}$: $[0,1) \to \mathbf{C}^n$ has two cluster points as t approaches 1. This easily implies that there are infinitely many cluster points of $D_i(z^{(l)}(t))$ as t tends to 1, which contradicts Lemma 3. This shows that $\lim_{t \to 1} z^{(l)}(t)$ exists in $\mathbf{P}^n$, which completes the proof. □

Theorem 1 shows that all isolated solutions to $\mathscr{P}(z) = 0$ are obtained by following the solution paths $z^{(l)}(t)$, $t \in [0,1]$ for $l = 1, 2, \ldots, D$ in $\mathbf{P}^n$. However, the actual computation does not take place in $\mathbf{P}^n$. One rather represents an element in $\mathbf{P}^n$, i.e., a straight line through the origin in $\mathbf{C}^{n+1}$, by some characteristic point on that line. This leads to

THEOREM 2. *Assume the notations and hypothesis of Theorem* 1. *For* $l \in \{1, 2, \ldots, D\}$ *let* $a^{(l)} \in \mathbf{C}^{n+1}$ *be a nonzero point on* $z^{(l)}(0) \in \mathbf{P}^n$. *Then the initial value problem*

$$\begin{bmatrix} \mathscr{H}_z(y(t), t) \\ y(t)^H \end{bmatrix} y_t(t) + \begin{bmatrix} \mathscr{H}_t(y(t), t) \\ 0 \end{bmatrix} = 0, \qquad t \in [0,1), \tag{4.1}$$

$$y(0) = a^{(l)}$$

(with $y^H = (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n)$, $\bar{y}_i$ being the complex conjugate to y_i) *has a unique solution* $y^{(l)}$: $[0,1] \to \mathbf{C}^{n+1}$, *continuous on* $[0,1]$ *and analytic on* $[0,1)$, *which represents* $z^{(l)}$, *i.e.*, $y^{(l)}(t) \in \mathbf{C}^{n+1}$ *is a nonzero point on the straight line* $z^{(l)}(t) \in \mathbf{P}^n$ *for each* $t \in [0,1]$.

Proof. It is easy to show that, for each $t \in [0,1]$, a nonzero point $x^{(l)}(t) \in \mathbf{C}^{n+1}$ on the line $z^{(l)}(t) \in \mathbf{P}^n$ can be selected such that the resulting function $x^{(l)}$: $[0,1] \to \mathbf{C}^{n+1}$ is continuous on $[0,1]$, analytic on $[0,1)$ and $x^{(l)}(0) = a^{(l)}$: First one shows that such a selection function exists locally by working with a proper affine representation. Then these pieces are glued together. Now we consider the function $y^{(l)}(t) = \rho^{(l)}(t)\, x^{(l)}(t)$ with

$$\rho^{(l)}(t) = \exp\left(-\int_0^t \langle x^{(l)}(s), x_t^{(l)}(s) \rangle / \langle x^{(l)}(s), x^{(l)}(s) \rangle \, ds \right) \qquad (\langle y, z \rangle = y^H z).$$

Observe that $\rho^{(l)}(1)$ exists, although $x_t^{(l)}(s)$ may diverge for $s \to 1$. (From Wainberg and Trenogin [17, p. 75] it follows that $x^{(l)}(s) = \varphi^{(l)}((1-s)^{1/m^{(l)}})$, with some analytic mapping $\varphi^{(l)}$ and a positive integer $m^{(l)}$, in some neighborhood of $s = 1$. This ensures the existence of the integral at $t = 1$.) Easy calculations show that $y^{(l)}$ satisfies (4.1). To show that the solution of (4.1) is unique, it suffices to prove the regularity of the matrix in (4.1). Assume that, for fixed $t \in [0,1)$,

$$\mathscr{H}_z(y^{(l)}(t), t)v = 0, \qquad y^{(l)}(t)^H v = 0$$

with $v \in \mathbf{C}^{n+1}$. Euler's differential equation for the componentwise homogeneous mapping $\mathscr{H}$,

$$\mathscr{H}_z(z, t)z = \operatorname{diag}(d_1, d_2, \ldots, d_n)\mathscr{H}(z, t) \quad \text{at } z = y^{(l)}(t),$$

and the condition $\operatorname{rank} \mathscr{H}_z(y^{(l)}(t), t) = n$, see Theorem 1(a), imply that the null space of $\mathscr{H}_z(y^{(l)}(t), t)$ is the one-dimensional subspace of $\mathbf{C}^{n+1}$ spanned by $y^{(l)}(t)$.

Hence, $v = \rho y^{(l)}(t)$, $\rho \in \mathbf{C}$. With $y^{(l)}(t)^H v = \rho \| y^{(l)}(t) \|^2 = 0$, it follows that $\rho = 0$ and finally $v = 0$, which shows the regularity of the matrix in (4.1). This completes the proof. □

Remark. The scaling condition $y(t)^H y_t(t) = 0$ is essentially the same condition as used in Brunovský and Meravý [2].

Theorem 2 reduces the problem of finding all isolated zeros of $\mathscr{P}$ to the numerical integration of the initial value problem (4.1) for D different initial values. In the next section a few numerical experiments on this are described.

5. Numerical Experiments. In principle, any efficient and reliable path following algorithm for solving (4.1) can be used; see, e.g., the review article by Allgower and Georg [1]. For no particular reason we chose the well-known Euler-predictor Newton-corrector algorithm. All computations were performed on an IBM 4341 computer in double precision. The algorithm stops at a solution $z^{(k)} \in \mathbf{C}^{n+1}$ with $\|z^{(k)}\| = 1$ if

$$\text{either } \left\|\mathscr{P}(z^{(k)})\right\| \leqslant 10^{-10} \quad \text{or} \quad \left\|z^{(k)} - z^{(k-1)}\right\| \leqslant 10^{-10},$$

where $z^{(k)}$ and $z^{(k-1)}$ denote the two last iterates of Newton's method for (2.1).

In order to test the influence of the auxiliary parameter a, we considered

Problem 1 (see Wright [18]):

$$(z_1 - z_2 - z_0)^2 = 0, \qquad z_1^2 - z_2^2 = 0.$$

Solutions:

$$A_1 = A_2 = (0,1,1) \quad \text{(improper zero of multiplicity 2)},$$

$$B_1 = B_2 = \left(1, -\tfrac{1}{2}, \tfrac{1}{2}\right) \quad \text{(proper zero of multiplicity 2)}.$$

This simple system was solved for different values of a on the unit circle in $\mathbf{C}$, using (1.7) as starting problem.

Theoretically, only 8 values of $a \in \mathbf{C}$ on the unit circle (marked by crosses on the $\arg(a)$-axis in Figure 1) lead to nonregular solution paths. As expected, the method also fails to work if $a \in \mathbf{C}$ is sufficiently close to one of those 8 values. For 86 percent of the unit circle the method produces all 4 solutions. On the rest of the unit circle only two or three solutions are obtained. The diagram of Figure 1 shows the computational cost versus the auxiliary parameter for each of the 4 solutions. The computational cost is measured in terms of Jacobian evaluations, N, of $\mathscr{H}$ and the parameter $a \in \mathbf{C}$ on the unit circle is identified by its angle $\arg(a)$.

If, as usually suggested, the problem is run with a randomly chosen auxiliary parameter, an average of 94 Jacobian evaluations is needed to compute all four solutions. (A standard random number generator was used and the average was taken over 100 runs.) For 15 percent of the runs the method failed to produce all zeros. This high failure rate is due to the rather optimistic step length strategy, which decreases the computational cost but also increases the probability for jumping from one path to another near the 8 marked values of $\arg(a)$, where two paths almost touch. If some internal parameters of the path following algorithm are decreased by a factor 0.1, which forces the algorithm to follow the paths more closely, the method works for $a \in \mathbf{C}$ on more than 99.9 percent of the unit circle. However, the computational cost increases.

Remark. Although the problem has only two different solutions, they are approached by four different solution paths. This makes it possible to decide whether all "four" zeros are obtained or some zero is "lost" by jumping to another path.

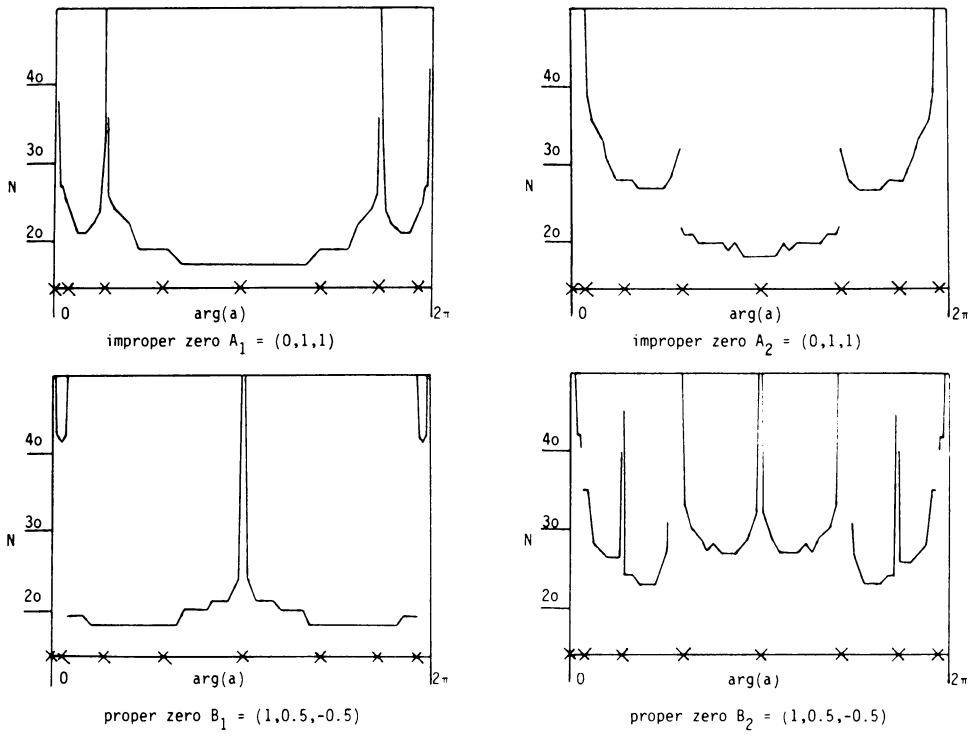

FIGURE 1

Increasing computational cost near 8 *auxiliary parameters* $a \in \mathbf{C}$ (*marked by* $\times$).

The second test problem.

Problem 2 (see Kojima and Mizuno [10], Wright [18]):

$$z_j^2 + \sum_{i=1}^{5} z_0 z_i - 2 z_0 z_j - C z_0^2 = 0 \quad \text{for } j = 1, 2, \ldots, 5$$

with $C = 10$, 4.1 or 4 gives some indication on the importance of the starting problem $\mathscr{R}(z) = 0$. For $C = 10$ and $C = 4.1$ the system has 32 zeros of multiplicity 1, for $C = 4$ there is one zero of multiplicity 16 and 16 zeros of multiplicity 1. As mentioned before, a starting system $\mathscr{R}(z) = 0$ can be obtained by splitting $\mathscr{P}$, e.g., there is always an "artificial" splitting with $\mathscr{R}$ given by (1.7), no matter how $\mathscr{P}$ looks like. For Problem 2 this leads to the starting system

$$\text{(R1)} \qquad z_j^2 - z_0^2 = 0 \quad \text{for } j = 1, 2, \ldots, 5.$$

On the other hand, the polynomials $\mathscr{P}_j$ of Problem 2 "naturally" split into terms in z_j and z_0 only and the rest of $\mathscr{P}_j$. Then the starting problem becomes

$$\text{(R2)} \qquad z_j^2 - z_0 z_j - C z_j^2 = 0 \quad \text{for } j = 1, 2, \ldots, 5.$$

Table 1 shows that the total number N of Jacobian evaluations is considerably smaller for the "natural" splitting ($a \in \mathbf{C}$ randomly chosen on the unit circle).

TABLE 1

Computational cost for different starting problems.

N	$C = 10$	$C = 4.1$	$C = 4$
(R1)	462	383	485
(R2)	155	257	367

This is no proof that, in general, the natural splitting (whatever this means) is better than an "artificial" splitting. But at least this test problem shows that flexibility with respect to the starting problem is a desirable feature of a homotopy method, a fact that has not been appreciated in previous papers, except in Drexler [4], [5].

Appendix. This section contains sketches of the proofs for Lemma 2 and Lemma 3, using algebraic-geometric arguments.

Solution sets in $\mathbf{P}^n$ to systems of homogeneous polynomials in $z = (z_0, z_1, \ldots, z_n)$ are called closed algebraic sets in $\mathbf{P}^n$. Solution sets in $\mathbf{P}^n \times \mathbf{P}^1$ to systems of polynomials in $(z, \lambda) = (z_0, z_1, \ldots, z_n, \lambda_0, \lambda_1)$, homogeneous in z and in λ, are called closed algebraic sets in $\mathbf{P}^n \times \mathbf{P}^1$.

Example. X is a closed algebraic set in $\mathbf{P}^n \times \mathbf{P}^1$, X_λ is a closed algebraic set in $\mathbf{P}^n$ for each $\lambda \in \mathbf{P}^1$.

If a closed algebraic set is irreducible, i.e., it is not the union of two proper closed algebraic subsets, then it is said to be a variety. Each closed algebraic set is a finite union of varieties which are called the components of the closed algebraic set (see Mumford [16, p. 22 and Proposition (2.12)]).

The natural projection p_2 of a variety in $\mathbf{P}^n \times \mathbf{P}^1$ onto the second factor $\mathbf{P}^1$ is a variety in $\mathbf{P}^1$ (see Mumford [16, Main Theorem of elimination theory (2.23)]). The only varieties in $\mathbf{P}^1$ are the empty set, the one-element subsets and $\mathbf{P}^1$ itself (which follows from the fundamental theorem of algebra).

Proof of Lemma 2. Those components of X whose projection onto $\mathbf{P}^1$ is $\mathbf{P}^1$ are denoted by Y^i, $i = 1, 2, \ldots, r$; for the rest we write Z^i, $i = 1, 2, \ldots, s$. Then we have

$$X = Y \cup Z \quad \text{with } Y = \bigcup_{i=1}^{r} Y^i \text{ and } Z = \bigcup_{i=1}^{s} Z^i.$$

Obviously, $p_2(Z)$ is a finite subset of $\mathbf{P}^1$, and

$$X_\lambda = Y_\lambda$$

for $\lambda \notin p_2(Z)$, where $Y_\lambda = \{z \in \mathbf{P}^n : (z, \lambda) \in Y\}$.

Let Y^i be any component of X with $p_2(Y^i) = \mathbf{P}^1$. Y^i must contain a point of the form $(z^0, (0, 1)) \in \mathbf{P}^n \times \mathbf{P}^1$, with z^0 being a simple zero of $\mathscr{R}$. Condition (A), see Section 1, guarantees that $(z^0, (0, 1))$ is a regular point. Then by Lemma 1 it follows that $\dim Y^i = 1$ (for the concept of dimension see Mumford [16]).

Next, consider the subset $\operatorname{sing}(Y^i) \subset Y^i$ of nonregular points in Y^i. $\operatorname{sing}(Y^i)$ is a proper closed algebraic subset of Y^i because $(z^0, (0, 1)) \notin \operatorname{sing}(Y^i)$ and nonregularity can be described in terms of vanishing subdeterminants of $\mathscr{H}_z(z, \lambda)$ which lead to polynomial equations. It is known that the dimension of a proper closed algebraic subset of a variety is smaller than the dimension of that variety (see Mumford [16, Proposition (1.14)]). Therefore, $\dim \operatorname{sing}(Y^i) < 1$, which means that $\operatorname{sing}(Y^i)$ is finite. Consequently, $\operatorname{sing}(Y) = \bigcup_{i=1}^{r} \operatorname{sing}(Y^i)$ is finite and the projection $p_2(\operatorname{sing}(Y))$ is also finite.

So, finally, we have that, for $\lambda \notin p_2(\text{sing}(Y)) \cup p_2(Z) = E$, $X_\lambda = Y_\lambda$ consists of only regular points. In particular, Condition (A) implies that $X_{(0,1)}$ is a set of exactly D distinct points. Lemma 1 and a simple continuity argument show that X_λ also consists of exactly D distinct points as long as $\lambda \notin E$. This completes the proof of Lemma 2. □

Proof of Lemma 3. Observe that

$$\lim_{\lambda \to (1,0)} X_\lambda \subset Y_{(1,0)}$$

because $X_\lambda = Y_\lambda$ for each $\lambda \neq (1,0)$ in some neighborhood of $(1,0) \in \mathbf{P}^1$, see proof of Lemma 2, and Y is closed in the natural topology of $\mathbf{P}^n \times \mathbf{P}^1$.

The closed algebraic set $X \subset \mathbf{P}^n \times \mathbf{P}^1$ is defined as the solution set of n polynomials. Hence, each component of X is at least one-dimensional, see Kendig [9, Theorem IV.2.23 and Theorem IV.3.8]. In particular, $\dim Z^i \geqslant 1$. Since $p_2(Z^i)$ is a one-element subset of $\mathbf{P}^1$, $\dim Z^i_\lambda = \dim Z^i \geqslant 1$ or Z^i_λ is empty for $\lambda \in \mathbf{P}^1$. This implies that the variety $Z^i_{(1,0)}$ cannot contain an isolated point. Thus the isolated zeros of $\mathscr{P}$ lie in $Y_{(1,0)}$. The rest of Lemma 3 follows by simple continuity arguments. This completes the proof of Lemma 3. □

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