

# Weak Uniform Distribution for Divisor Functions. I

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**Abstract.** Narkiewicz (reference [3, pp. 204-205]) has proposed an algorithm for determining the moduli with respect to which a given arithmetic function (of suitable type) has weak uniform distribution. The class of functions to which this algorithm applies includes the divisor functions  $\sigma_i$ . The present paper gives an improvement to the algorithm for odd values of  $i$ , which makes computation feasible for values of  $i$  up to 200. The results of calculations for odd values of  $i$  in the range  $1 \leq i \leq 199$  are reported.

**1. Introduction.** Let  $\sigma_i(x)$  be defined for positive integers  $i, x$  by

$$\sigma_i(x) = \sum_{d|x} d^i.$$

For odd values of  $i$ , the functions  $\sigma_i$  occur as Fourier coefficients of Eisenstein series.

An arithmetic function  $f$  is defined to be weakly uniformly distributed modulo  $n$  (WUD (mod  $n$ ), for short) if the set

$$\{x \in \mathbf{Z}: x > 0, (f(x), n) = 1\}$$

is infinite and for every pair of integers  $a_1, a_2$  with  $(a_1, n) = (a_2, n) = 1$ ,

$$\#\{x: 0 < x < t, f(t) \equiv a_1 \pmod{n}\} \sim$$

$$\#\{x: 0 < x < t, f(x) \equiv a_1 \pmod{n}\}$$

as  $t \rightarrow \infty$ .

The integers  $n$  for which  $\sigma_i(x)$  is WUD (mod  $n$ ) have been determined by Sliwa [6] for  $i = 1$ , by Narkiewicz and Rayner [5] for  $i = 2$ , and by Narkiewicz [2] for  $i = 3$ . In the present paper the methods of [2] are further improved. For each odd integer  $i > 0$ , there exist two finite sets of integers  $K_1$  and  $K_2$  such that  $\sigma_i$  has WUD (mod  $n$ ) if and only if either  $n$  is odd and not divisible by an element of  $K_1$  or  $n$  is even and not divisible by an element of  $K_2$ .

Calculations of the sets  $K_1$  and  $K_2$  for  $\sigma_i$  for all odd values of  $i$  from 5 to 199 have been carried out in the University of Liverpool Computer Laboratory. The results are tabulated at the end of this paper, and the earlier results of Sliwa ( $i = 1$ ) and Narkiewicz ( $i = 3$ ) have been incorporated.

*Observation 1.* Within the range of the table, it can be seen that if  $i$  is prime and  $2i + 1$  is composite, then  $K_1$  is empty, and that if  $i$  and  $2i + 1$  are both prime, then  $K_1 = \{2i + 1\}$  for  $i \equiv 3 \pmod{4}$ , and  $K_1 = \{6i + 3\}$  for  $i \equiv 1 \pmod{4}$ .

*Observation 2.* Within the range of the table, if  $i$  is prime and  $2i + 1$  is composite, then  $K_2 = \{6\}$ , with the sole exception of  $i = 43$ , where  $K_2 = \{6, 2066\}$ . Further, if  $i$  is prime and  $2i + 1$  is prime, then  $K_2 = \{6, 4i + 2\}$ .

Received October 11, 1983; revised April 22, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 11B99; Secondary 11-04, 11A25, 11F30, 11N69, 11Y99.

*Observation 3.* The upper bound of Lemma 4 below,  $(2i + 1)^2$ , for the set of primes involved in the calculations is much higher than necessary. A value of  $(2i + 1)^{1.6}$  would be consistent with the values actually found. It would be possible to make calculations for higher values of  $i$  if this observed upper bound could be proved to hold in general.

Since this paper was originally submitted, Narkiewicz's book [4] has appeared. It describes the background and motivation for these calculations and refers to the original version of this paper in which the calculations were carried out for values of  $i \leq 107$ .

Narkiewicz records that Observation 1 concerning  $K_1$  has been shown to be true generally by E. Dobrowolski (see [4, p. 110, Theorem 6.12]). (See also Narkiewicz [2] for part of this result.)

In [4, p. 112, Problem V] Narkiewicz asks for a characterization of those odd integers  $i$  such that  $\sigma_i$  fails to have WUD (mod  $n$ ) if and only if 6 divides  $n$ . Since for composite  $i$  the set of moduli for which WUD fails is at least the union of the corresponding sets for the factors of  $i$ , one might first consider prime values for  $i$ . However, even for prime  $i$ , there seems to be no easily observed pattern of behavior of  $K_2$ . As in Observation 2 above, in the case in which  $i$  is prime and  $2i + 1$  is composite, while  $K_1$  is always empty it is not always true that  $K_2 = \{6\}$ , since  $\sigma_{43}$  is not WUD (mod 2066), although this seems to be a rare exception. Calculations for prime values of  $i$  are easier than for composite ones, and a search beyond the limits of the present tables, assuming a reduced upper bound as in Observation 3, shows that the next primes  $i$  for which  $K_2$  behaves in this way are

$$\begin{aligned} i &= 467, & \text{where } K_2 &= \{6, 24286\}, \\ i &= 503, & \text{where } K_2 &= \{6, 24146\}, \text{ and} \\ i &= 883, & \text{where } K_2 &= \{6, 38854\}. \end{aligned}$$

It is worth noticing in connection with Observation 2 and Dobrowolski's result cited above from [4] that for  $i = 809$  we have  $K_2 = \{6, 3338, 38834\}$ . Thus, although here  $i$  and  $2i + 1$  are both prime, it is not always true that under these conditions  $K_2 = \{6, 4i + 2\}$ , 809 being the first exception.

Because of the reduced upper bound assumed here, these results for  $i > 200$  may possibly be incomplete in the sense that the sets  $K_2$  might be larger than stated (and therefore similar results might hold for smaller values of  $i$ ), but this is extremely unlikely.

*Observation 4.* Ramanujan's  $\tau$  function has WUD (mod  $n$ ) if and only if either  $n$  is odd and not divisible by 7 (Serre) or even and divisible neither by 6 nor 46 (Narkiewicz). (See [4, p. 89, Theorem 5.18].) Thus  $\tau$  behaves with respect to weak uniform distribution in the same way as  $\sigma_3$  for odd  $n$  and in the same way as  $\sigma_{11}$  for even  $n$ .

**2. Narkiewicz's Algorithm.** For a fixed value of  $i > 2$ , let

$$V_j(x) = 1 + x^i + x^{2i} + \cdots + x^{ji}.$$

Thus, for a prime  $p$ ,  $\sigma_i(p^j) = V_j(p)$ . Let

$$R_j(n) = \{V_j(a) \bmod n : a \in \mathbf{Z}, (aV_j(a), n) = 1\},$$

regarded as a subset of the multiplicative group  $G(n)$  of residue classes prime to  $n$ . Let  $\Lambda_j(n)$  be the subgroup of  $G(n)$  generated by  $R_j(n)$ . Let  $d(n)$  be the smallest  $j \geq 1$  for which  $R_j(n) \neq \emptyset$ .

The following Lemmas 1–4 are special cases of results proved by Narkiewicz [2], [3].

LEMMA 1.  $\sigma_i$  has WUD (mod  $n$ ) for  $i > 2$  if and only if  $\Lambda_{d(n)}(n) = G(n)$ .

Note that for odd  $i > 2$ ,  $d(n) = 1$  if  $n$  is odd, and  $d(n) = 2$  if  $n$  is even. Lemma 1 gives a means of calculating whether  $\sigma_i$  is WUD (mod  $n$ ) for any particular values of  $i$  and  $n$ .

LEMMA 2. Let  $n = q_1 \cdots q_r$ , where  $q_1, \dots, q_r$  are powers of distinct primes. Suppose for each  $q_s$ ,  $\Lambda_j(q_s) = G(q_s)$ . Then  $\Lambda_j(n) \neq G(n)$  if and only if

- (i) there exist characters  $\chi_s$  of  $G(q_s)$  ( $s = 1, \dots, r$ ) such that  $\chi_s$  takes a constant value  $c_s$  (say) on  $R_j(q_s)$ ;
- (ii)  $\prod_{s=1}^r c_s = 1$ ; and
- (iii) not all the characters  $\chi_s$  are trivial.

LEMMA 3. Let  $q = p^t$ , where  $p$  is an odd prime. Then there is a nontrivial character of  $G(q)$  taking a constant value on  $R_j(q)$  if and only if there is such a character of  $G(p^u)$  taking a constant value on  $R_j(p^u)$ , where  $u = \min\{t, 2\}$ . For  $p = 2$  a similar result holds with  $u = \min\{t, 3\}$ .

LEMMA 4. For any prime  $p$ , if there is a nontrivial character of  $G(p^t)$  taking a constant value on  $R_j(q)$ , then  $p < (e_j + 1)^2$  where  $e_j$  is the degree of  $V_j(x)$ .

*Remark.* A slightly stronger result is due to Fomenko [1].

Let  $i$  now denote an odd integer greater than 1. It is easily seen that if  $\Lambda_j(n) \neq G(n)$ , then  $\Lambda_j(mn) \neq G(mn)$  for any integer  $m > 1$ . It follows that there are finite sets of integers  $K_1$  and  $K_2$  such that  $\sigma_i$  is WUD (mod  $n$ ) if and only if  $n$  is odd and not divisible by an element of  $K_1$  or  $n$  is even and not divisible by an element of  $K_2$ . The sets  $K_1$  and  $K_2$  can be found in the following way, as follows from Lemmas 1–4.

For  $j = 1, 2$ , let  $H_j$  be the set of primes  $p$  satisfying the inequality of Lemma 4 (in which  $e_1 = i$  and  $e_2 = 2i$ ).

Let  $I_j = H_j \cup \{p^2: p \in H_j\} \cup \{8\}$ , and let

$$J_j = \{m \in I_j: \text{there exists a nontrivial character on } G(m) \text{ constant on } R_j(m)\},$$

including cases in which  $\Lambda_j(m)$  is a proper subgroup of  $G(m)$ .

Then  $K_j$  is the set of all products  $r$  of elements of  $J_j$  (no element being taken more than once in each product) for which  $\Lambda_j(r) \neq G(r)$ .

Narkiewicz [2] has determined  $K_1$  and  $K_2$  for  $i = 3$ . Because it may be necessary to examine primes  $p$  up to  $(2i + 1)^2$  and to calculate values of  $R_2(p^2)$  in  $G(p^2)$ , the calculations become difficult with increasing  $i$ . The Propositions in Section 3 below make it unnecessary to consider squares of most odd primes and reduce the number of primes which need to be included in the sets  $H_j$ , although the upper bounds are not altered.

**3. Some Improvements.** Throughout this paragraph, let  $W(x)$  be a polynomial with integer coefficients, and let

$$R(n) = \{W(a) \bmod n: a \in \mathbf{Z}, (aW(a), n) = 1\},$$

regarded as a subset of  $G(n)$ .

For any prime  $q$ , let  $\phi: G(q^2) \rightarrow G(q)$  be defined, for  $x \in \mathbf{Z}$ , by  $\phi(x \bmod q^2) = x \bmod q$ , and let  $\psi: G(q) \rightarrow G(q^2)$  be defined, for  $x \in \mathbf{Z}$ , by  $\psi(x \bmod q) = x^q \bmod q^2$ . It is easy to see that  $\phi$  and  $\psi$  are homomorphisms of abelian groups, that  $\psi(\phi(z)) = z$  for all  $z \in G(q)$  (so that  $\phi$  is an epimorphism and  $\psi$  is a monomorphism) and that  $\phi(R(q^2)) = R(q)$ .

**LEMMA 5.** *Let  $\chi$  be any nontrivial character on  $G(q)$  which is constant on  $R(q)$ . Then  $\chi \circ \phi$  is a nontrivial character on  $G(q^2)$  which is constant on  $R(q^2)$ .*

*Proof.* Immediate.

**LEMMA 6.** *Let  $\chi$  be any nontrivial character on  $G(q^2)$  taking the constant value 1 on  $R(q^2)$ , and suppose that  $\chi \circ \psi$  is the trivial character on  $G(q)$ . Then  $R(q^2)$  and  $R(q)$  have the same cardinal number.*

*Proof.* First,  $R(q^2) \subset \ker \chi$ . Again,  $\text{im } \psi \subset \ker \chi$ . Now  $\text{im } \psi$  is a subgroup of  $G(q^2)$  of prime index  $q$ , so, since  $\chi$  is not the trivial character,  $\text{im } \psi = \ker \chi$ . Thus  $R(q^2) \subset \text{im } \psi$ . The restriction of  $\phi$  to  $\text{im } \psi$  is bijective, and  $\phi(R(q^2)) = R(q)$ . Hence the result.

**LEMMA 7.** *Suppose that the prime number  $q$  and polynomial  $W(x)$  are such that  $\psi(R(q)) \subset R(q^2)$ . Let  $\chi$  be any nontrivial character on  $G(q^2)$  which is constant on  $R(q^2)$ . Then  $\chi \circ \psi$  is a nontrivial character on  $G(q)$  which is constant on  $R(q)$ .*

*Proof.* Since  $\psi(R(q)) \subset R(q^2)$ ,  $\chi \circ \psi$  is a character constant on  $R(q)$ , and it will be enough to show that it is nontrivial. If it is trivial, then  $\chi(\psi(R(q))) = 1$ , and so the constant value of  $\chi$  on  $R(q^2)$  is 1. The result now follows from Lemma 6.

**PROPOSITION 1.** *Let  $W(x) = 1 + x^i$ , where  $i$  is odd and not divisible by the odd prime  $q$ . Then there is a nontrivial character on  $G(q^2)$  constant on  $R(q^2)$  if and only if there is a nontrivial character on  $G(q)$  constant on  $R(q)$ .*

*Proof.* It is enough to show that Lemma 7 applies. Let  $x \in \mathbf{Z}$  be such that  $x \bmod q \neq 0$ , and let  $y_\lambda = x + \lambda q$  for  $\lambda = 0, 1, \dots, q-1$ . Then

$$\phi((1 + y_\lambda^i) \bmod q^2) = (1 + x^i) \bmod q$$

and  $1 + y_\lambda^i \equiv 1 + y_\mu^i \pmod{q^2}$  if and only if  $\lambda \equiv \mu \pmod{q}$ . Thus  $R(q^2)$  contains every element of  $G(q^2)$  which is mapped into  $R(q)$  by  $\phi$ . Hence  $\#R(q^2) = q\#R(q)$  and  $\psi R(q) \subset R(q^2)$ . Since  $\psi$  is a monomorphism and  $q > 2$ , Lemmas 5 and 7 now give the result.

**PROPOSITION 2.** *Let  $W(x) = 1 + x^i + x^{2i}$ , where  $i$  is odd and not divisible by the odd prime  $q$ . Then there is a nontrivial character on  $G(q^2)$  constant on  $R(q^2)$  if and only if there is a nontrivial character on  $G(q)$  constant on  $R(q)$ .*

*Proof.* For  $q = 3$ , it is easily seen that such characters exist both on  $R(q)$  and on  $R(q^2)$ . Now suppose  $q \geq 5$ . It is enough to show that if  $\chi$  is a nontrivial character

on  $G(q^2)$  taking a constant value  $a$  on  $R(q^2)$ , then  $\chi \circ \psi$  is a nontrivial character on  $G(q)$  taking a constant value on  $R(q)$ . Putting  $x = q - 1$ , we see that  $1 \in R(q^2)$ , so that  $a = \chi(1) = 1$ . Now let  $x$  be such that  $x \bmod q \neq 0$ , and put  $y_\lambda = x + \lambda q$  for  $\lambda = 0, 1, \dots, q - 1$ . Clearly,  $W(y_\lambda) \equiv W(y_\mu) \bmod q^2$  if and only if

$$(\lambda - \mu)ix^{i-1}(1 + 2x^i) \equiv 0 \bmod q.$$

If  $x$  is such that  $1 + 2x^i \bmod q \neq 0$ , it follows that  $q$  distinct elements of  $R(q^2)$  are mapped onto  $W(x) \bmod q$  by  $\phi$ . On the other hand, if  $x$  is such that  $1 + 2x^i \bmod q = 0$ , then exactly one element of  $R(q^2)$  is mapped onto  $W(x) \bmod q$  by  $\phi$ . Note that in this case  $W(x) \bmod q$  is uniquely determined. Thus, provided  $R(q)$  has at least two elements, we can conclude that  $\#R(q^2) > \#R(q)$ . But  $q$  is a prime greater than 3, and  $1 \in R(q)$ ,  $3 \in R(q)$ . Lemma 6 now shows that  $\chi \circ \phi$  is nontrivial. Now let  $z \bmod q$  be any element of  $R(q)$ , so that  $z = W(x) \bmod q$  for suitable  $x \in \mathbf{Z}$ . Then  $z \bmod q^2 \in R(q^2)$ , and

$$\chi(\phi(z \bmod q)) = \chi(z^q \bmod q^2) = (\chi(z \bmod q^2))^q = 1^q = 1.$$

Thus  $\chi \circ \phi$  is constant on  $R(q)$ , and the proposition is proved.

**PROPOSITION 3.** *Let  $i$  be odd, and let  $q$  be a prime greater than 3, and let  $W(x)$  be either  $1 + x^i$  or  $1 + x^i + x^{2i}$ . Suppose that there is a nontrivial character on  $G(q)$  which is constant on  $R(q)$ . Then  $(i, q - 1) \neq 1$ .*

*Proof.* Suppose that  $(i, q - 1) = 1$ . Then  $x \rightarrow x^i$  is an automorphism of  $G(q)$ .

For  $W(x) = 1 + x^i$  we have  $R(q) = \{2, 3, \dots, q - 1\}$  and the only character constant on this set is trivial, so that the proposition holds in this case.

For  $W(x) = 1 + x^i + x^{2i} = (x^i + \alpha)^2 + \beta$ , where  $\alpha$  and  $\beta$  are calculated in the finite field  $\mathbf{Z}_q$ , we have  $1 = W(-1) \in R(q)$ , so that there will only be a nontrivial character constant on  $R(q)$  if  $R(q)$  generates a proper subgroup of  $G(q)$ . As  $x^i$  runs through all the nonzero elements of  $\mathbf{Z}_q$ ,  $x^i + \alpha$  runs through all except  $\alpha$  (but including 0 and  $-\alpha$ ), so that  $(x^i + \alpha)^2$  runs through all the quadratic residues, and also takes the value 0. Thus  $(x^i + \alpha)^2 + \beta$  takes  $(q - 1)/2$  values in  $G(q)$  if  $-\beta$  is a quadratic residue, and  $(q + 1)/2$  values otherwise. If  $R(q)$  generates a proper subgroup of  $G(q)$ , this can only be the subgroup of order  $(q - 1)/2$ , that is, the group of quadratic residues. Thus, for every quadratic residue  $r^2$ ,  $r^2 + \beta$  is also a quadratic residue. It follows that every element of  $G(q)$  is a quadratic residue. This contradiction completes the proof of the proposition.

**4. Results.** With the help of Propositions 1, 2 and 3, the algorithm of Section 2 can be simplified as follows.

For an odd integer  $i > 1$ , let  $H_1$  (respectively,  $H_2$ ) be the set consisting of the primes  $p$  of the form  $1 + \lambda r$  (where  $r$  is a nontrivial divisor of  $i$  and  $\lambda$  is an integer) for which  $p < (i + 1)^2$  (respectively,  $p < (2i + 1)^2$ ), together with the prime divisors of  $i$  and their squares.

Let

$$I_1 = H_1 \cup \{p^2: p \in H_1 \text{ is prime and there exists } q \in H_1 \text{ with } q \equiv 1 \pmod{p}\}$$

and let

$$I_2 = H_2 \cup \{p^2: p \in H_2 \text{ is prime and there exists } q \in H_2 \text{ with } q \equiv 1 \pmod{p}\} \\ \cup \{2, 4, 8\}.$$

As before, let  $J_1$  be the subset of  $I_1$  consisting of those elements  $m$  for which there is a nontrivial character modulo  $m$  constant on the set  $R(m)$  of values of the polynomial  $1 + x^i$ , and let  $J_2$  be calculated similarly from  $I_2$  using  $1 + x^i + x^{2i}$ . The sets  $K_1$  and  $K_2$  consist of the products  $r$  (say) of elements of  $J_1$  and  $J_2$ , respectively, with no repeated factor, for which  $\Lambda_1(r) \neq G(r)$  (respectively,  $\Lambda_2(r) \neq G(r)$ ), but omitting from  $K_1$  and  $K_2$  any  $r$  which is strictly divisible by another element already known to lie in  $K_1$  or  $K_2$ , respectively. It follows from the results of Section 3 that, with  $K_1$  and  $K_2$  found from these smaller sets  $I_1$  and  $I_2$ ,  $\sigma_i$  fails to have WUD (mod  $n$ ) if and only if  $n$  is odd and divisible by an element of  $K_1$  or  $n$  is even and divisible by an element of  $K_2$ .

The results tabulated below include the cases  $i = 1$ , due to Sliwa [6] and  $i = 3$  due to Narkiewicz [2].

#### TABLE OF RESULTS

The notation is as in Section 2.  $\sigma_i$  has WUD (mod  $n$ ) if and only if  $n$  is odd and not divisible by an element of  $K_1$  or  $n$  is even and not divisible by any element of  $K_2$ .

$i$	$K_1$	$K_2$
1	–	6
3	7	6
5	33	6 22
7	–	6
9	7 57	6 146
11	23	6 46
13	–	6
15	7 31 33	6 22 122 302
17	–	6
19	–	6
21	7 43	6
23	47	6 94
25	33	6 22
27	7 57 109	6 146
29	177	6 118
31	–	6
33	7 23 201	6 46 134
35	33 71	6 22 142
37	–	6
39	7 79 157	6 1874
41	249	6 166
43	–	6 2066
45	7 31 33 57 209	6 22 122 146 302
47	–	6
49	–	6
51	7 103 307	6 206 614
53	321	6 214
55	23 33	6 22 46
57	7 229	6
59	–	6

61	–	6
63	7 43 57 127	6 146
65	33 393 1441	6 22 262
67	–	6
69	7 47 277 417	6 94
71	–	6
73	–	6
75	7 31 33 151	6 22 122 302 1202 2402
77	23	6 46
79	–	6
81	7 57 109 489 3097	6 146
83	167	6 334
85	33	6 22 3742
87	7 177	6 118
89	537	6 358
91	–	6
93	7	6
95	33 191	6 22 382
97	–	6
99	7 23 57 199 201 397 1273	6 46 134 146
101	–	6
103	–	6
105	7 31 33 43 71 633 2321	6 22 122 142 302
107	–	6
109	–	6
111	7 223	6
113	681	6 454
115	33 47	6 22 94
117	7 57 79 157	6 146 1874
119	239	6 478
121	23	6 46
123	7 249	6 166
125	33 251	6 22 502
127	–	6
129	7	6 2066
131	263	6 526
133	–	6
135	7 31 33 57 109 209 271	6 22 122 146 302 542
137	–	6
139	–	6
141	7 283	6
143	23	6 46
145	33 177 649	6 22 118
147	7 43	6
149	–	6
151	–	6
153	7 57 103 307 919	6 146 206 614 1226 1838 7346
155	33 311	6 22 622
157	–	6
159	7 321	6 214

(continues)

(continued)

161	47	6 94
163	–	6
165	7 23 31 33 201 331 737	6 22 46 122 134 302 1322
167	–	6
169	–	6
171	7 57 229	6 146
173	1041	6 694
175	33 71	6 22 142
177	7	6
179	359	6 718
181	–	6
183	7 367 733	6 734
185	33	6 22
187	23	6 46
189	7 43 57 109 127 1137 7201	6 146 1514
191	383	6 766
193	–	6
195	7 31 33 79 157 393 1441	6 22 122 262 302 1874
197	–	6
199	–	6

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