MATHEMATICS OF COMPUTATION
VOLUME 51, NUMBER 183
JULY 1988, PAGES 203-218

Interpolation by Multivariate Splines

By Charles K. Chui*, Harvey Diamond, and Louise A. Raphael****

Abstract. A general interpolation scheme by multivariate splines at regular sample points is introduced. This scheme guarantees the local optimal order of approximation to sufficiently smooth data functions. A discussion on numerical implementation is included.

1. Introduction. In this paper we introduce a very general interpolation scheme by multivariate splines. Based on the quasi-interpolation formulas developed in [3], we show that the interpolating multivariate splines so obtained give the optimal order of approximation to sufficiently smooth functions.

Let ϕ be a nonnegative locally supported piecewise polynomial function symmetric with respect to the origin, and S the linear span of all the translates $\phi(\cdot - \mathbf{j})$, $\mathbf{j} \in \mathbf{Z}^s$, of ϕ. Hence, S is a multivariate spline space on a certain grid partition Δ, with certain smoothness joining conditions, and of certain total degree, induced by ϕ. Denote by $\hat{\phi}$ the Fourier transform of ϕ. We assume that ϕ is normalized, that is,

$$\sum_{\mathbf{j}} \phi(\mathbf{j}) = 1,$$

and that S contains π_ρ, the space of all polynomials in $\mathbf{R}^s$ of total degree ρ. This is equivalent to the assumption that the commutator of ϕ is of degree ρ (cf. [4]), or that ϕ satisfies the Strang and Fix condition of degree ρ (cf. [6]):

$$(1) \qquad \begin{cases} \hat{\phi}(0) = 1, \\ D^\alpha \hat{\phi}(2\pi\mathbf{j}) = 0, \qquad 0 \neq \mathbf{j} \in \mathbf{Z}^s,\ |\alpha| \le \rho. \end{cases}$$

The equivalence of these conditions is shown in [4] (see also [1]).

The interpolation problem we are going to study can be stated as follows: Let K be a compact set in $\mathbf{R}^s$ and $f \in C(K)$. For any $h > 0$, consider the sample points

$$P_h = \{\mathbf{j}h \in K : \mathbf{j} \in \mathbf{Z}^s\}$$

on K. The problem is to find an $s_{f,h}$ in

$$S_h = \{s(\cdot/h) : s \in S\}$$

such that

$$(2) \qquad s_{f,h}(\mathbf{y}) = f(\mathbf{y}), \qquad \mathbf{y} \in P_h,$$

Received July 11, 1986; revised May 19, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 41A15, 41A63.

Key words and phrases. Multivariate splines, interpolation, optimal order, box splines, fundamental function, numerical implementation.

*Partially supported by ARO Contract No. DAAG 29-84-K-0154 and NSF Grant No. DMS-8602337.

**Partially supported by ARO Contract No. DAAG 29-84-G-004.

and that the order of approximation of $s_{f,h}$ is optimal, namely

$$\|s_{f,h} - f\|_K = O(h^{\rho+1}) \tag{3}$$

for all sufficiently smooth functions f, where for convenience we only consider the supremum norm over K. Interpolation problems of this kind have been studied in particular cases, notably for the quadratic box splines $M_{1,1,1,1}$ and a class of cubic C^1 splines in [7] and [8]. Our techniques are applicable to splines in any dimensions, and our results can be applied to the cases mentioned.

Let $S_{h,K}$ denote the linear span of those functions $\phi(\cdot/h-\mathbf{j})$ whose supports have nonempty intersection with K. Then $S_{h,K}$ is a subspace of S_h, and of course, the interpolant $s_{f,h}$ is chosen from $S_{h,K}$. We note, however, that since the dimension of $S_{h,K}$, denoted by $\dim S_{h,K}$, always exceeds the cardinality of the set P_h of sample points, denoted by $\operatorname{card} P_h$, the interpolant $s_{f,h}$ is certainly not unique. It should be remarked that in most situations, such as a closed polygonal region, a sphere, etc., we have

$$\dim S_{h,K} \sim \operatorname{card} P_h.$$

Out of all interpolants of f from $S_{h,K}$ (or S_h), we are required to choose an $s_{f,h}$ that gives the optimal order of approximation to f as described by (3), when f is sufficiently smooth on some open set Ω, $K \subset \Omega$. We will see that the requirement on the smoothness of the data function f depends on the zero set of the discrete Fourier transform of ϕ defined by

$$\tilde{\Phi}(\omega) \equiv \sum_{\mathbf{j}\in\mathbf{Z}^s} \phi(\mathbf{j}) e^{i\omega\cdot\mathbf{j}}, \qquad \omega \in \mathbf{R}^s. \tag{4}$$

The importance of the positivity of $\tilde{\Phi}$ to interpolation problems is well known; indeed it is equivalent to the existence and uniqueness of cardinal interpolants for bounded data [5], [2], [4]. For our problem we show that if the function f is in the class

$$C^{\rho,1}(\Omega) \equiv \{g \in C^{\rho}(\Omega) \colon D^{\alpha} g \in \operatorname{Lip}_{\Omega}(1), |\alpha| = \rho\}, \tag{5}$$

where, hereafter, Ω is some open set containing K, then the optimal order of approximation in (3) can be obtained. There are in addition, however, important cases for which $\tilde{\Phi}$ is not strictly positive, such as Zwart's quadratic bivariate spline $M_{1,1,1,1}$ and several other bivariate splines whose supports are symmetric and whose discrete Fourier transforms vanish at isolated points. We are able to show that if $\tilde{\Phi}$ has isolated zeros, the optimal order of approximation by interpolants is still achievable provided that $f \in C^{\rho+2,1}(\Omega)$ and the interpolant is carefully chosen. Finally, we will show that if $\tilde{\Phi}$ becomes negative, vanishing on a manifold of dimension $s-1$, then the condition $f \in C^{\rho+s+1,1}(\Omega)$ is sufficient for constructing interpolants with optimal order of approximation. The results for $\tilde{\Phi} > 0$ and $\tilde{\Phi}$ having isolated zeros seem fairly sharp, given their generality and the analysis that leads to them; while the result for the remaining case is merely intended for sufficiency. We point out that little is known about the positivity of $\tilde{\Phi}$ in dimensions higher than two. In two dimensions there is a simple characterization, namely: For box splines Φ the positivity of $\tilde{\Phi}$ is equivalent to the (infinite) linear independence of the functions $\Phi(\cdot - \mathbf{j})$ (cf. [2]).

In the last section we show how interpolants can be rapidly calculated iteratively if $\tilde{\Phi}$ is positive and that, regardless of the sign of $\tilde{\Phi}$, quasi-interpolants with the optimal order of approximation and satisfying the interpolation equations to an arbitrarily high order in h can be obtained.

2. Main Results. In the above discussion and analysis that follows, elements of $\mathbf{R}^s$ and $\mathbf{Z}^s$ are denoted by boldface type, such as $\mathbf{x}$, $\boldsymbol{\omega}$ and $\mathbf{j}$, $\mathbf{k}$, respectively. Functions defined on $\mathbf{Z}^s$, or a subset of $\mathbf{Z}^s$, are denoted by upper case letters, e.g., Φ, C, I, and their point values are accessed using the customary functional notation, such as $\Phi(\mathbf{k})$. Given two functions A and B defined on $\mathbf{Z}^s$, their product is defined as convolution:

$$(AB)(\mathbf{k}) \equiv \sum_{\mathbf{j}\in\mathbf{Z}^s} A(\mathbf{k}-\mathbf{j})B(\mathbf{j}), \tag{6}$$

assuming of course that the required sums exist. The dot product of two elements of $\mathbf{R}^s$, say $\boldsymbol{\omega}$ and $\mathbf{j}$, is denoted by $\boldsymbol{\omega}\cdot\mathbf{j}$.

The discrete Fourier transform of a function on $\mathbf{Z}^s$ is denoted by a tilde. Thus,

$$\tilde{A}(\boldsymbol{\omega}) \equiv \sum_{\mathbf{k}\in\mathbf{Z}^s} A(\mathbf{k})e^{i\boldsymbol{\omega}\cdot\mathbf{k}}.$$

For each $h > 0$, an element s_h of the spline space S_h generated by ϕ takes the form

$$s_h = \sum_{\mathbf{j}\in\mathbf{Z}^s} c_{\mathbf{j}}\phi(\mathbf{x}/h - \mathbf{j}).$$

The interpolation conditions (2) become

$$\sum_{\mathbf{j}\in\mathbf{Z}^s} c_{\mathbf{j}}\phi(\mathbf{k}-\mathbf{j}) = f(h\mathbf{k}), \qquad h\mathbf{k}\in K. \tag{7}$$

We define the functions C and Φ on $\mathbf{Z}^s$ by $C(\mathbf{k}) = c_{\mathbf{k}}$ and $\Phi(\mathbf{k}) = \phi(\mathbf{k})$. The function F_h is defined by

$$F_h(\mathbf{k}) = \begin{cases} f(h\mathbf{k}) & \text{if } h\mathbf{k} \text{ is in the domain of } f, \\ 0 & \text{otherwise.} \end{cases}$$

The equations (7) can then be written (using commutativity of convolution) as

$$(\Phi C)(\mathbf{k}) = F_h(\mathbf{k}), \qquad h\mathbf{k}\in K. \tag{8}$$

Our analysis proceeds from here, following the operator method introduced in [3]. We decompose Φ as $I - M$, where

$$I(\mathbf{k}) = \begin{cases} 1 & \text{if } \mathbf{k} = \mathbf{0}, \\ 0 & \text{otherwise,} \end{cases}$$

so that I is the multiplicative identity. Since $\sum \Phi(\mathbf{k}) = 1$ and Φ is symmetric, $\sum M(\mathbf{k}) = 0$ and M is symmetric. It then follows that M can be written as a linear combination of second difference operators (thinking of M as an operator via multiplication). Indeed, we have

$$M = \frac{-1}{2}\sum_{\mathbf{j}\neq\mathbf{0}} \Phi(\mathbf{j})\Delta_{\mathbf{j}}^2, \tag{9}$$

where

$$\Delta_{\mathbf{j}}^2(\mathbf{k}) = \begin{cases} 1 & \text{if } \mathbf{k} = \mathbf{j} \text{ or } -\mathbf{j}, \\ -2 & \text{if } \mathbf{k} = \mathbf{0}, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that in the present context, $\Delta_{\mathbf{j}}^2$ is a function on $\mathbf{Z}^s$ and acts as a second difference operator via multiplication.)

Our interpolation scheme is in the form of a quasi-interpolant plus a remainder:

$$C = (I + M + M^2 + \cdots + M^n)F_h + E \equiv Q_n F_h + E, \tag{10}$$

for some appropriately chosen $n \geq (\rho - 1)/2$. It was proved in [3] that

$$\sum_{\mathbf{j} \in \mathbf{Z}^s} (Q_n F_h)(\mathbf{k})\phi(\mathbf{x}/h - \mathbf{k})$$

reproduces all polynomials $f \in \pi_\rho(\mathbf{R}^s)$ if $2n \geq \rho - 1$. We have the following simple lemma concerning $E(\mathbf{k})$.

LEMMA 1. *If $(\Phi C)(\mathbf{k}) = F_h(\mathbf{k})$, $\mathbf{k} \in \mathbf{Z}^s$, and $h\mathbf{k} \in K$, and if $C = Q_n F_h + E$, then E satisfies $(\Phi E)(\mathbf{k}) = M^{n+1} F_h(\mathbf{k})$, $h\mathbf{k} \in K$. Moreover, if $f \in C^{m,1}(\Omega)$, $\Omega \supset K$ and Ω is open, then $M^{n+1}F_h(\mathbf{k}) = O(h^{\min(m+1, 2n+2)})$, $h\mathbf{k} \in K$.*

Proof. Multiplying both sides of (10) by $\Phi = I - M$, we obtain

$$F_h(\mathbf{k}) = (\Phi C)(\mathbf{k}) = (I - M)(Q_n F_h + E)(\mathbf{k}) = F_h(\mathbf{k}) - M^{n+1}F_h(\mathbf{k}) + (\Phi E)(\mathbf{k}),$$

and the first part of the lemma follows. As for the second part, if $K \subset \Omega$ and $h\mathbf{k} \in K$, then $M^{n+1}F_h(\mathbf{k})$ represents a $(2n+2)$nd-order difference operator applied to the function f and evaluated at $\mathbf{x} = h\mathbf{k}$. The estimate $O(h^{\min(m+1,2n+2)})$ then follows. □

From the fact that

$$\sum (Q_n F_h)(\mathbf{k})\phi(\mathbf{x}/h - \mathbf{k})$$

reproduces polynomials in π_ρ, it is a standard proof to show that if $f \in C^{\rho,1}(\Omega)$ then

$$\left\| f - \sum_{\mathbf{k}} (Q_n F_h)(\mathbf{k})\phi(\mathbf{x}/h - \mathbf{k}) \right\|_K = O(h^{\rho+1}).$$

Since the actual coefficients $C(\mathbf{k})$ differ from $Q_n F_h(\mathbf{k})$ by $E(\mathbf{k})$, we see that if $E(\mathbf{k}) = O(h^{\rho+1})$ for all $\mathbf{k}$ such that $\phi(\mathbf{x}/h - \mathbf{k})$ is in $S_{h,K}$, then the optimal order of approximation will be attained by

$$\sum C(\mathbf{k})\phi(\mathbf{x}/h - \mathbf{k}).$$

We are thus faced with the problem of estimating the norm $\|E(\mathbf{k})\|_{\infty, h\mathbf{k} \in \Omega}$. We approach this problem by studying the fundamental functions $C^*(\mathbf{k})$ that solve $\Phi C^* = I$.

LEMMA 2. *If $\tilde{\Phi} > 0$, then the fundamental function C^* satisfying $\Phi C^* = I$ exists, is unique, and has the following properties:*

(a) $|C^*(\mathbf{k})| < a\exp(-b|\mathbf{k}|)$, *$a$ and $b > 0$, and*

(b) $C^* = I + M + M^2 + \cdots$, *where M^n satisfies the inequality*

$$|M^n(\mathbf{k})| \leq \left\{1 - \min_\omega \tilde{\Phi}(\omega)\right\}^n. \tag{11}$$

Proof. Existence/uniqueness and property (a) were observed in [2]. To verify Eq. (11), we simply note that $(\tilde{M^n}) = (\tilde{M})^n$, so that

$$
\begin{aligned}
|M^n(\mathbf{k})| &= (2\pi)^{-s}\left|\int_{[-\pi,\pi]^s} \tilde{M}^n(\omega)e^{-i\omega\cdot\mathbf{k}}\,d\omega\right| \\
&\le (2\pi)^{-s}\int_{[-\pi,\pi]^s}\left\{1-\min_{\omega}\tilde{\Omega}(\omega)\right\}^n d\omega.
\end{aligned}
\tag{12}
$$

The representation of C^* as the Neumann series in M follows from

$$(I-M)\left(\sum_{m=0}^{n} M^m\right) = I - M^{n+1}$$

and taking the limit as $n\to\infty$. □

Actually, only property (a) is necessary to provide the order of approximation of our interpolation scheme. However, (b) provides an efficient numerical scheme for computing C^*, and the same technique will be used next to construct a fundamental function if $\tilde{\Phi}$ has isolated zeros.

If $\tilde{\Phi}$ has zeros, the fundamental function C^* is obviously not unique. Nevertheless, interpolation can be effected on bounded regions and the optimal order of approximation attained. The construction of our fundamental function begins with an asymptotic expansion of M^n for large n which is uniform in $\mathbf{k}$.

LEMMA 3. *Suppose that in the 2π-cube W in $\mathbf{R}^s$, $\tilde{\Phi}(\omega)$ is nonnegative and has zeros at the isolated interior points $\{\omega_j\}$ and $\tilde{\Phi}$ has positive definite Hessian H_j at ω_j. Then*

$$
\begin{aligned}
M^n(\mathbf{k}) = (2\pi n)^{-s/2}&\sum_j \frac{e^{i\mathbf{k}\cdot\omega_j}}{\sqrt{|H_j|}}\exp\left(\frac{-\mathbf{k}^T H_j^{-1}\mathbf{k}}{2n}\right) \\
&\times\left(1+\sum_{|\alpha|=3}^{m} P_{\alpha,j}(n)Q_{\alpha,j}\left(\frac{\mathbf{k}}{\sqrt{n}}\right)n^{-|\alpha|/2}\right) \\
&+O(n^{[(m+1)/3]-(m+1+s)/2}),
\end{aligned}
\tag{13}
$$

where $P_{\alpha,j}$ is a polynomial in n of degree at most $[|\alpha|/3]$, and $Q_{\alpha,j}$ is a polynomial in s variables of degree $|\alpha|$. The approximation (13) *for M^n is uniform in* $\mathbf{k}$.

Proof. We estimate $M^n(\mathbf{k})$ by estimating the integral

$$\int_W \tilde{M}^n(\omega)e^{i\omega\cdot\mathbf{k}}\,d\omega = (2\pi)^s M^n(\mathbf{k}),$$

using the ideas of Laplace's method in asymptotic analysis. The maximum value of $\tilde{M}(\omega)$ is 1 at precisely the roots ω_j of $\tilde{\Phi}$. Outside of any fixed neighborhood of the ω_j, $\tilde{M}^n(\omega)$ is exponentially small (as $n\to\infty$) and the contribution to the integral

$$\int_W \tilde{M}^n(\omega)e^{i\omega\cdot\mathbf{k}}\,d\omega$$

from that portion of W lying outside that neighborhood is likewise exponentially small. We consider then the contribution to the integral

$$\int_W \tilde{M}^n(\omega)e^{i\omega\cdot\mathbf{k}}\,d\omega$$

from a ball $B(\varepsilon)$ of radius ε about the root ω_j. This leads to consideration of the integral

$$\int_{B(\varepsilon)} \tilde{M}^n(\omega+\omega_j)e^{i\mathbf{k}\cdot(\omega+\omega_j)}\,d\omega,$$

where $B(\varepsilon)$ is now a ball of radius ε about the origin. It is not difficult to show that with exponentially small error, we can scale ε with n, choosing $\varepsilon = n^{-2/5}$. This is because in a neighborhood of ω_j, we have

$$\tilde{M}^n(\omega+\omega_j) < (1-c|\omega|^2)^n < \exp(-ncn^{-4/5}) = \exp(-cn^{1/5})$$

for some positive constant c, whenever $|\omega| > n^{-2/5}$. We note that the exponent $-2/5$ was chosen so that $n\varepsilon^2 \to \infty$ as $n \to \infty$ but $n\varepsilon^3 \to 0$ as $n \to \infty$. Consider now a Taylor polynomial approximation for $\tilde{M}^n(\omega+\omega_j)$ for $\omega \in B(n^{-2/5})$. We have

$$\begin{aligned}\tilde{M}^n(\omega+\omega_j) &= \exp\left\{n\log\left(1-\frac{\omega^T H\omega}{2}+\sum_{|\alpha|=3}^{m}\frac{D^\alpha\tilde{M}(\omega_j)}{\alpha!}\omega^\alpha+O(|\omega|^{m+1})\right)\right\}\\ &= \exp\left(\frac{-n\omega^T H\omega}{2}\right)\left\{1+\sum_{|\alpha|=3}^{m}P_\alpha(n)\omega^\alpha+O(n^{[(m+1)/3]}|\omega|^{m+1})\right\},\end{aligned} \tag{14}$$

where $P_\alpha(n)$ is a polynomial in n of degree no larger than $[|\alpha|/3]$, and m is an arbitrary integer. The above expression must be integrated against $e^{i\mathbf{k}\cdot(\omega+\omega_j)}$ over the ball $B(\varepsilon)$ about the origin. If we consider integrating term by term, we see that the integrals may be extended to all of $\mathbf{R}^s$ with exponentially small error in n. This leads naturally to consideration of integrals of the form

$$\int_{\mathbf{R}^s}\exp\left(\frac{-n\omega^T H\omega}{2}\right)e^{i\omega\cdot\mathbf{k}}\omega^\alpha\,d\omega. \tag{15}$$

A standard result in probability theory, specifically multivariate Gaussian distributions, is that

$$\int_{\mathbf{R}^s}\exp\left(\frac{-n\omega^T H\omega}{2}\right)e^{i\omega\cdot\mathbf{k}}\,d\omega = \frac{(2\pi)^{s/2}}{\sqrt{|H|}n^{s/2}}\exp\left(-\frac{\mathbf{k}^T H^{-1}\mathbf{k}}{2n}\right).$$

The integrals (15) can then be obtained by differentiation with respect to $\mathbf{k}$. These derivatives have the form

$$\begin{aligned}&\int_{\mathbf{R}^s}\exp\left(\frac{-n\omega^T H\omega}{2}\right)e^{i\omega\cdot\mathbf{k}}\omega^\alpha\,d\omega\\ &\quad= \frac{(2\pi)^{s/2}}{\sqrt{|H|}n^{s/2}}\exp\left(-\frac{\mathbf{k}^T H^{-1}\mathbf{k}}{2n}\right)\frac{1}{n^{|\alpha|/2}}Q_\alpha(\mathbf{k}/\sqrt{n}),\end{aligned} \tag{16}$$

where Q_α is a polynomial of degree $|\alpha|$, with coefficients independent of n and $\mathbf{k}$. Finally, we consider the contribution of the remainder term in (14) involving $O(|\omega|^{m+1})$ to the integral

$$\int_{B(\varepsilon)}\tilde{M}^n(\omega+\omega_j)e^{ik\cdot(\omega+\omega_j)}\,d\omega.$$

Since $|e^{i\omega\cdot\mathbf{k}}| = 1$, we can extend the integral back to $\mathbf{R}^s$ with exponentially small error, obtaining

$$\int_{\mathbf{R}^s} \exp\left(\frac{-n\omega^T H\omega}{2}\right) e^{i\omega\cdot\mathbf{k}} O(|\omega|^{m+1})\, d\omega = O(n^{-(m+1+s)/2}). \tag{17}$$

Combining the previous estimates (14)–(17) we have

$$\begin{aligned}
&\int_{B(\varepsilon)} \tilde{M}^n(\omega+\omega_j) e^{i\mathbf{k}\cdot(\omega+\omega_j)}\, d\omega \\
&\quad = e^{i\mathbf{k}\cdot\omega_j} \frac{(2\pi)^{s/2}}{\sqrt{|H|}n^{s/2}} \exp\left(-\frac{\mathbf{k}^T H^{-1}\mathbf{k}}{2n}\right)\left\{1 + \sum_{|\alpha|=3}^{m} P_\alpha(n) Q_\alpha\left(\frac{\mathbf{k}}{\sqrt{n}}\right) n^{-|\alpha|/2}\right\} \\
&\qquad + O\left(n^{[(m+1)/3]} n^{-(m+1+s)/2}\right),
\end{aligned}$$

where H, P_α, Q_α all implicitly depend on j. The above estimate is uniform in $\mathbf{k}$. This completes the proof of Lemma 3, for one only needs sum over j and divide by $(2\pi)^s$ to obtain the desired result (13). □

If $s > 2$, then

$$\sum_{n=0}^{\infty} M^n$$

converges to a fundamental function C^* by virtue of (13). The asymptotic behavior of C^* is calculated in the next lemma.

LEMMA 4. *Let $s > 2$. If $\tilde{\Phi}(\omega)$ is nonnegative and has isolated zeros at a set of points ω_j where the Hessians H_j of $\tilde{\Phi}(\omega)$ are positive definite, then the series*

$$\sum_{n=0}^{\infty} M^n$$

converges uniformly to a function C^* which satisfies $\Phi C^* = I$ and has the asymptotic behavior

$$C^*(\mathbf{k}) = (2\pi)^{-s/2} \sum_j \frac{e^{i\mathbf{k}\cdot\omega_j}}{\sqrt{|H_j|}} \left(\frac{\mathbf{k}^T H_j^{-1}\mathbf{k}}{2}\right)^{1-s/2} \Gamma\left(\frac{s}{2}-1\right)(1 + O(1/|\mathbf{k}|)) \tag{18}$$

for large values of $|\mathbf{k}|$.

Proof. We need sum the series (13) over n. We know a priori that $M^n(\mathbf{k}) = 0$ for $n < c|\mathbf{k}|$ for some constant $c > 0$. The last term, $O(n^{[(m+1)/3]-(m+1+s)/2})$ in (13), when summed from $n = c|\mathbf{k}|$ to ∞, contributes $O(|\mathbf{k}|^{[(m+1)/3]+(1-m-s)/2})$. The rest of the terms in (13) give rise to a generic sum of the form

$$\sum_{n=1}^{\infty} n^{-\beta} \exp\left(-\frac{\mathbf{k}^T H^{-1}\mathbf{k}}{2n}\right),$$

where we have extended the sum down to $n = 1$ with exponentially small error in $|\mathbf{k}|$. Using the Euler-Maclaurin summation formula, one can show that

$$\sum_{n=1}^{\infty} n^{-\beta} \exp\left(-\frac{\mathbf{k}^T H^{-1}\mathbf{k}}{2n}\right) = \int_0^\infty x^{-\beta} \exp\left(-\frac{\mathbf{k}^T H^{-1}\mathbf{k}}{2x}\right) dx + \text{e.s.t.}(|\mathbf{k}|).$$

Here and throughout, e.s.t.$(\cdot)$ denotes a term which exponentially decays in $(\cdot)$. Transforming the integral via $u = 1/x$, we have

$$\sum_{n=1}^{\infty} n^{-\beta} \exp\left(-\frac{\mathbf{k}^T H^{-1}\mathbf{k}}{2n}\right) = \left(\frac{\mathbf{k}^T H^{-1}\mathbf{k}}{2}\right)^{1-\beta} \Gamma(\beta - 1) + \text{e.s.t.}(|\mathbf{k}|).$$

The result (18) then follows upon retaining the first term and estimating the remainder. Indeed, if desired, one can compute a complete asymptotic expansion of $C^*(\mathbf{k})$ by including additional terms. □

If $s = 2$, then the series

$$\sum_{n=0}^{\infty} M^n$$

does not converge. Instead, a fundamental function is obtained by subtracting from M^n an appropriate solution of the homogeneous system $\Phi C = 0$ so as to produce a convergent series.

LEMMA 5. *For $s = 2$ and $\tilde{\Phi}(\omega)$ having isolated zeros, the series*

$$\sum_{n=0}^{\infty} \left(M^n(\mathbf{k}) - \frac{1}{2\pi(n+1)} \sum_j \frac{e^{i\mathbf{k}\cdot\omega_j}}{\sqrt{|H_j|}} \right) \tag{19}$$

converges to a fundamental function $C^*(\mathbf{k})$ with asymptotic behavior

$$C^*(\mathbf{k}) = \frac{-1}{2\pi} \sum_j \frac{e^{i\mathbf{k}\cdot\omega_j}}{\sqrt{|H_j|}} \left(2\gamma + \ln\left(\frac{\mathbf{k}^T H_j^{-1}\mathbf{k}}{2}\right)\right) + O(1/|\mathbf{k}|), \tag{20}$$

where γ is Euler's constant.

Proof. Given the asymptotic behavior of M^n in Lemma 3, for large n the terms in (19) are $O(n^{-3/2})$ so that the series converges. On the other hand, the series in (19) represents a fundamental function, since applying Φ to the partial sum from 0 to m, we obtain

$$(\Phi)\left(\sum_{n=0}^{m} \left(M^n - \frac{1}{2\pi(n+1)} \sum_j \frac{e^{i\mathbf{k}\cdot\omega_j}}{\sqrt{|H_j|}} \right)\right) = (\Phi)\left(\sum_{n=0}^{m} M^n\right) = I - M^{m+1} \to I$$

as $m \to \infty$. It remains to find the asymptotic behavior. The estimation of C^* requires the estimation of the sum

$$\sum_{n=0}^{\infty} \left(\frac{1}{n} \exp\left(\frac{-\mathbf{k}^T H^{-1}\mathbf{k}}{2n}\right) - \frac{1}{1+n}\right)$$

or

$$\sum_{n=0}^{\infty} ((1/n)e^{-a/n} - 1/(1+n))$$

for large a. Using the Euler-Maclaurin summation formula we can estimate

$$\begin{aligned} \sum_{n=0}^{m} (1/n)e^{-a/n} &= \int_0^m (1/x)e^{-a/x}\,dx + O(1/m) + \text{e.s.t.}(a) \\ &= \int_{a/m}^{\infty} (1/u)e^{-u}\,du + O(1/m) + \text{e.s.t.}(a), \end{aligned} \tag{21}$$

and integrating by parts, we have

$$\sum_{n=0}^{m}(1/n)e^{-a/n} = \ln(m/a)(1+O(1/m)) + \int_{a/m}^{\infty} \ln u e^{-u}\,du + O(1/m) + \text{e.s.t.}(a).$$

On the other hand,

$$\sum_{n=0}^{m} 1/(n+1) = \gamma + \ln(m) + O(1/m).$$

We then obtain

$$\begin{aligned}\lim_{m\to\infty}&\sum_{n=0}^{m}\{(1/n)e^{-a/n} - 1/(n+1)\}\\ &= -\gamma - \ln a + \int_0^{\infty} \ln u e^{-u}\,du + \text{e.s.t.}(a)\\ &= -2\gamma - \ln a + \text{e.s.t.}(a).\end{aligned} \tag{22}$$

Using (22), we obtain

$$\begin{aligned}C^*(\mathbf{k}) &= \sum_{n=0}^{\infty}\left\{M^n(\mathbf{k}) - \frac{1}{2\pi(n+1)}\sum_j \frac{e^{i\mathbf{k}\cdot\omega_j}}{\sqrt{|H_j|}}\right\}\\ &= \sum_j -e^{i\mathbf{k}\cdot\omega_j}\frac{1}{2\pi\sqrt{|H_j|}}\left(2\gamma + \ln\left(\frac{\mathbf{k}^T H_j^{-1}\mathbf{k}}{2}\right)\right) + O(1/|\mathbf{k}|).\end{aligned} \tag{23}$$

Again, higher-order terms in an asymptotic series for C^* can be easily calculated. □

The next case to consider is when $\tilde{\Phi}(\omega)$ becomes negative for some values of ω. Our bounds on the fundamental function in this case are not sharp and we do not have as pleasing a representation for the fundamental function as in the other two cases. Nevertheless, our bounds will be adequate to furnish a smoothness requirement of f which leads to the optimal approximation order upon interpolation.

LEMMA 6. *Suppose that the set $\{\omega\colon \tilde{\Phi}(\omega) = 0\}$ is a manifold of dimension $s-1$ on which the gradient of $\tilde{\Phi}$ does not vanish. Define the function C^* by*

$$\begin{aligned}C^*(\mathbf{k}) = (2\pi)^{-s}\sum_{n=0}^{\infty}\Bigg(&\int_{\{\tilde{\Phi}(\omega)>0\}} \tilde{M}^n(\omega)e^{i\mathbf{k}\cdot\omega}\,d\omega\\ &-\int_{\{\tilde{\Phi}(\omega)<0\}} (2-\tilde{M}(\omega))^n e^{i\mathbf{k}\cdot\omega}\,d\omega\Bigg),\end{aligned} \tag{24}$$

where the regions of integration are subsets of the 2π-cube W. Then C^* is a fundamental function satisfying $\Phi C^* = I$ and $C^*(k) = O(|\mathbf{k}|)$.

Proof. When $\tilde{\Phi}(\omega) = 0$, $M(\omega) = 1$, and if $\tilde{\Phi} < 0$, then we have $\tilde{M} > 1$. Consider first the behavior of

$$\int_{\{\tilde{\Phi}(\omega)>0\}} \tilde{M}^n(\omega)e^{i\mathbf{k}\cdot\omega}\,d\omega$$

for large n. $\tilde{M}$ is less than one over the region of integration, so that with exponentially small error the region of integration may be changed to $\{0 < \tilde{\Phi}(\omega) < \varepsilon\}$.

The integration is performed by making a change of variable so as to integrate first over the level surface $\tilde{M}(\omega) = u$ and then over $1-\varepsilon < u < 1$. This leads to the iterated integral

$$\int_{u=1-\varepsilon}^{1} u^n \left\{ \int_{\tilde{M}(\omega)=u} e^{i\mathbf{k}\cdot\omega} J(\omega, u)\, d\omega_{s-1} \right\} du,$$

where J is the Jacobian of the transformation. Let the inside integral be denoted by $F(\mathbf{k}, u)$. Because of our assumption that the level set $\tilde{\Phi} = 0$ is an $(s-1)$-dimensional manifold on which the gradient of $\tilde{\Phi}$ does not vanish,

$$|F(\mathbf{k},1) - F(\mathbf{k},u)| = O(|\mathbf{k}|)|1-u| \quad \text{or} \quad F(\mathbf{k},u) = F(\mathbf{k},1) + O(|\mathbf{k}|)(1-u).$$

Substituting and using integration by parts, we obtain

$$\begin{aligned} \int_{u=1-\varepsilon}^{1} u^n F(\mathbf{k},u)\, du &= \frac{F(\mathbf{k},1)}{n+1} - \frac{O(|\mathbf{k}|)}{n+1} \int_{1-\varepsilon}^{1} u^{n+1}\, du + \text{e.s.t.}(n) \\ &= \frac{F(\mathbf{k},1)}{n+1} + O(|\mathbf{k}|/n^2). \end{aligned} \tag{25}$$

Likewise, consider the integral over the region $\tilde{\Phi}(\omega) < 0$ in (24). If we set $2-\tilde{M} = u$ and again integrate first over level surfaces of $\tilde{\Phi}$, we obtain the integral

$$\int_{u=1-\varepsilon}^{1} u^n \left\{ \int_{\tilde{M}(\omega)=2-u} e^{i\mathbf{k}\cdot\omega} J(\omega, 2-u)\, d\omega_{s-1} \right\} du = \int_{u=1-\varepsilon}^{1} u^n F(\mathbf{k}, 2-u)\, du,$$

and upon integration by parts it becomes

$$\frac{F(\mathbf{k},1)}{n+1} + O(|\mathbf{k}|/n^2). \tag{26}$$

Consider now the definition of C^* in (24). The leading terms in the two integrals cancel, so that for large n the terms in the sum are $O(n^{-2})$. Therefore, the series converges and the convergence is uniform on compact subsets of $\mathbf{k}$. Applying $\Phi = I - M$ to the partial sum of the series, we can take it inside the integrals and apply it to the exponentials, obtaining

$$\begin{aligned} &\int_{\{\tilde{\Phi}>0\}} \left(\sum_{n=0}^{m} \tilde{M}^n(\omega) \right) (1 - \tilde{M}(\omega))\, d\omega \\ &\qquad - \int_{\{\tilde{\Phi}<0\}} \left(\sum_{n=0}^{m} (2 - \tilde{M}(\omega))^n \right) \{(2 - \tilde{M}(\omega)) - 1\}\, d\omega \\ &\quad = \int_{\{\tilde{\Phi}>0\}} (1 - \tilde{M}^{m+1}(\omega))\, d\omega + \int_{\{\tilde{\Phi}<0\}} \{1 - (2 - \tilde{M}(\omega))^{m+1}\}\, d\omega \\ &\quad \to (2\pi)^s \quad \text{as } m \to \infty, \end{aligned}$$

showing that (24) does indeed define a fundamental function.

As for the asymptotic behavior of C^*, the terms in the sum in (24) were shown to be $O(|\mathbf{k}|/n^2)$, so that the sum is $O(|\mathbf{k}|)$ as desired. □

Having obtained the required estimates for the asymptotic behavior of the fundamental functions associated with the different classes of $\tilde{\Phi}$, we can now prove our main result.

THEOREM 1. *Let f be a function of class $C^{m,1}(\Omega)$ as defined in* (5), *where Ω is an open set in $\mathbf{R}^s$ and K a compact subset of Ω. Suppose that f is interpolated by the spline s_h on K as in* (7), *where the coefficients $C(\mathbf{k}) = c_{\mathbf{k}}$ are given by the interpolation scheme $C = Q_n F_h + E$ and E is chosen as follows:*

case A) *If $\tilde{\Phi}(\omega) > 0$ for all ω, then*

$$E(\mathbf{k}) = \sum_{h\mathbf{j}\in K} C^*(\mathbf{k}-\mathbf{j}) M^{n+1} F_h(\mathbf{j}), \tag{27}$$

where C^* is given by $\sum_{j=0}^{\infty} M^j$.

case B) *If $\tilde{\Phi}(\omega) \geq 0$ and zero only at isolated points where its Hessian is positive definite, then for $s > 2$, E is given by* (27) *and C^* is again given by $\sum_{j=0}^{\infty} M^j$.*

case B$'$) *If $\tilde{\Phi}(\omega) \geq 0$ and zero only at isolated points ω_l where its Hessian is positive definite, then for $s = 2$,*

$$E(\mathbf{k}) = \sum_{h\mathbf{j}\in K} \left(2\ln N \sum_l e^{i\mathbf{k}\cdot\omega_l} \frac{1}{2\pi\sqrt{|H_l|}} + C^*(\mathbf{k}-\mathbf{j}) \right) M^{n+1} F_h(\mathbf{j}),$$

where C^* is given by (19) *and $N = h^{-1} \cdot \operatorname{diam} K$.*

case C) *If $\tilde{\Phi}(\omega) = 0$ on a manifold of dimension $s-1$ where the gradient of $\tilde{\Phi}$ does not vanish, then E is given by* (27), *where C^* is defined by* (24).

Then,

$$\left\| f - \sum_{\mathbf{k}} c_{\mathbf{k}} \phi(\mathbf{x}/h - \mathbf{k}) \right\|_{K,\infty} = O(h^{\rho+1}),$$

provided that f satisfies the following smoothness conditions and n is as given:

For case A, *$f \in C^{\rho,1}$ and $2n \geq \rho - 1$; for cases* B *and* B$'$, *$f \in C^{\rho+2,1}$ and $2n \geq \rho + 1$; for case* C, *$f \in C^{\rho+s+1,1}$ and $2n \geq \rho + s$.*

Proof. In view of the remarks following the proof of Lemma 1, we need only show that $E(\mathbf{k}) = O(h^{\rho+1})$ in each of the four cases. In case A, owing to the exponential decay of the fundamental function, $M^{n+1}F_h = O(h^{\rho+1})$ is sufficient to guarantee this. In case B for $s > 2$, $C^*(\mathbf{k}) = O(|\mathbf{k}|^{2-s})$, so that if $M^{n+1}F_h = O(h^{\rho+3})$, then we have

$$\begin{aligned} E(\mathbf{k}) &= \sum_{h\mathbf{j}\in K} C^*(\mathbf{k}-\mathbf{j}) M^{n+1} F_h(\mathbf{j}) = O(h^{\rho+3}) \sum_{h\mathbf{j}\in K} |\mathbf{k}-\mathbf{j}|^{2-s} \\ &= O(h^{\rho+3}) \sum_{|\mathbf{j}|\leq N} |\mathbf{j}|^{2-s} = O(h^{\rho+3}) O(N^2) = O(h^{\rho+1}). \end{aligned}$$

In case B$'$ we obtain, by similar inequalities,

$$\begin{aligned} E(\mathbf{k}) &= O(h^{\rho+3}) \sum_{|\mathbf{j}|\leq N} O\left(\ln \frac{N}{|\mathbf{j}|}\right) = O(h^{\rho+3}) \int_0^N \ln\left(\frac{N}{r}\right) r\,dr \\ &= N^2 O(h^{\rho+3}) = O(h^{\rho+1}). \end{aligned}$$

In case C the relevant estimate becomes

$$\begin{aligned} E(\mathbf{k}) &= O(h^{\rho+s+2}) \sum_{|\mathbf{j}|\leq N} O(|\mathbf{j}|) = O(h^{\rho+s+2}) \int_0^N (r) r^{s-1}\,dr \\ &= O(h^{\rho+s+2}) N^{s+1} = O(h^{\rho+1}) \end{aligned}$$

as required. This completes the proof. $\square$

In case A, the interpolation problem is exceptionally stable. Basically, any bounded interpolation operator gives the optimal order of interpolation as shown in the next theorem.

THEOREM 2. *Suppose* $\tilde{\Phi}(\omega) > 0$, $f \in C^{\rho,1}(\Omega) \cap C(\overline{\Omega})$, Ω *open. Furthermore, suppose that the coefficients* $C_h(\mathbf{k})$ *satisfy* $(\Phi C_h)(\mathbf{k}) = f(h\mathbf{k})$, $h\mathbf{k} \in \overline{\Omega}$, *and* $|C_h(\mathbf{k})| = O(|\mathbf{k}|^m)$, *where* $m > 0$ *is a constant independent of* h; *for instance,* $C_h(\mathbf{k}) = C^* F_h(\mathbf{k})$ (*when* $f(\mathbf{x})$ *is polynomially bounded*). *Then, on any compact* $K \subset \Omega$,

$$\left\| f - \sum_{\mathbf{k}} C_h(\mathbf{k})\phi(\mathbf{x}/h - \mathbf{k}) \right\|_{\infty,K} = O(h^{\rho+1}).$$

Remark. We do not require that Ω be bounded.

Proof. For compact $K \subset \Omega$, we need only show that the coefficients $C_h(\mathbf{k})$ differ by $O(h^{\rho+1})$ from those of the quasi-interpolant, $Q_n F_h(\mathbf{k})$, for all $\mathbf{k}$ such that

$$h\mathbf{k} \in K^+ \equiv \{h\mathbf{k}\colon \phi(\mathbf{x}/h - \mathbf{k}) \neq 0 \text{ for some } \mathbf{x} \in K\}.$$

We may assume that C_h is defined for all $\mathbf{k}$; and where it is not defined, we set it to be zero. Since C_h is the cardinal interpolant of ΦC_h, we have

$$C_h(\mathbf{k}) = \sum_{\mathbf{j}} C^*(\mathbf{k} - \mathbf{j})\Phi C_h(\mathbf{j}).$$

Now $C^* = I + M + M^2 + \cdots + M^n + E^* = Q_n + E^*$, where $E^* = C^* M^{n+1}$. We compare $C_h(\mathbf{k})$ with $Q_n F_h(\mathbf{k})$ for $h\mathbf{k}$ in K^+ and obtain

$$C_h(\mathbf{k}) = \sum_{\mathbf{j}} Q_n(\mathbf{k} - \mathbf{j})\Phi C_h(\mathbf{j}) + \sum_{\mathbf{j}} C^*(\mathbf{k} - \mathbf{j}) M^{n+1}\Phi C_h(\mathbf{j}).$$

If $h\mathbf{k} \in K^+$, then the first sum may be restricted to $h\mathbf{j} \in \Omega$, since Q_n is local, and then for $h\mathbf{j} \in \Omega$, $\Phi C_h(\mathbf{j}) = F_h(\mathbf{j})$, so that

$$C_h(\mathbf{k}) = Q_n F_h(\mathbf{k}) + \sum_{\mathbf{j}} C^*(\mathbf{k} - \mathbf{j}) M^{n+1}\Phi C_h(\mathbf{j}), \qquad h\mathbf{k} \in K^*.$$

Define $\varepsilon = \inf\{|\mathbf{x} - \mathbf{y}|\colon x \in K,\ y \notin \Omega\}$. Let Ω' be a covering of K with balls of radius $\varepsilon/2$. Then in the above sum, we have

$$M^{n+1}\Phi C_h(\mathbf{j}) = O(h^{\rho+1}) \quad \text{for } h\mathbf{j} \in \Omega'.$$

For $h\mathbf{j} \notin \Omega'$, $h\mathbf{k} \in K^+$,

$$\left| \sum_{h\mathbf{j} \notin \Omega'} C^*(\mathbf{k} - \mathbf{j}) M^{n+1}\Phi C_h(\mathbf{j}) \right| < \sum_{|\mathbf{k} - \mathbf{j}| \geq \varepsilon/2h} O(e^{-b|\mathbf{k}-\mathbf{j}|}) O(|\mathbf{j}|^m) = O(e^{-b\varepsilon/4h}).$$

We then obtain

$$C_h(\mathbf{k}) = Q_n F_h(\mathbf{k}) + \sum_{h\mathbf{j} \in \Omega'} C^*(\mathbf{k} - \mathbf{j}) O(h^{\rho+1}) + O(e^{-b\varepsilon/4h}) = Q_n F_h(\mathbf{k}) + O(h^{\rho+1})$$

for $h\mathbf{k} \in K^+$. This completes the proof of the theorem. □

Of course, a simple method of choosing interpolation coefficients to satisfy the hypotheses of Theorem 2 is to define $f(x) = 0$ outside its original domain and then choose for C_h the cardinal interpolant. Naturally, only those coefficients whose corresponding splines Φ have support with nonempty intersection with $\overline{\Omega}$ need be calculated. In Section 3 we present an iterative interpolation scheme which does not require precomputation of $C^*(\mathbf{k})$.

We close this section with a final remark concerning the smoothness requirements in Theorem 1 leading to the optimal order of approximation. In cases B and B′ of the theorem, where cardinal interpolation is not unique, it may be the case that not only is

$$\sum_{\mathbf{k}} e^{i\omega_j \cdot \mathbf{k}} \phi(\mathbf{x}/h - \mathbf{k}) = 0$$

when $\mathbf{x}$ is at a grid point, but the sum may be identically zero. If this is the case at all the zeros ω_j of $\tilde{\Phi}$, then the smoothness requirement can be decreased by one derivative. We have the following

THEOREM 3. *In cases* B *and* B′ *of Theorem* 1, *if*

$$\sum_{\mathbf{k}} e^{i\omega_j \cdot \mathbf{k}} \phi(\mathbf{x}/h - \mathbf{k}) = 0$$

for all $\mathbf{x}$, *then the condition* $f \in C^{\rho+1,1}(\Omega)$ *is sufficient to guarantee that the interpolation scheme gives the optimal approximation order.*

Proof. In the proof of Theorem 1, the smoothness requirements were obtained by requiring that the remainder $E(\mathbf{k})$ was $O(h^{\rho+1})$ for all relevant $\mathbf{k}$. Given the hypotheses of this theorem, we can do better by considering the size of

$$e(\mathbf{x}) = \sum_{\mathbf{k}} E(\mathbf{k}) \phi(\mathbf{x}/h - \mathbf{k})$$

for $\mathbf{x} \in K$. For case B, we make use of the definition of $E(\mathbf{k})$ in (27) and interchange summations of $\mathbf{k}$ and $\mathbf{j}$, obtaining

$$e(\mathbf{x}) = \left(\sum_{h\mathbf{j} \in K} M^{n+1} F_h(\mathbf{j}) \sum_{\mathbf{k}} C^*(\mathbf{k} - \mathbf{j}) \phi(\mathbf{x}/h - \mathbf{k}) \right). \tag{28}$$

The inner sum may be rewritten as

$$\sum_{\mathbf{k}} C^*(\mathbf{k} - \mathbf{j}) \phi(\mathbf{x}/h - \mathbf{k}) = \sum_{\mathbf{k}} C^*(\mathbf{k}) \phi\left(\frac{\mathbf{x} - \mathbf{j}h}{h} - \mathbf{k} \right).$$

From (18), giving the asymptotic behavior of $C^*(\mathbf{k})$ in case B, we see that $C^*(\mathbf{k})$ is a sum of terms of the form

$$a e^{i\omega \cdot \mathbf{k}} (\mathbf{k}^T H^{-1} \mathbf{k})^{1-s/2} [1 + O(1/|\mathbf{k}|)],$$

where ω is a root of $\tilde{\Phi}$, and the sum is over the roots ω of $\tilde{\Phi}$. Consider then the sum

$$\sum_{\mathbf{k}} (\mathbf{k}^T H^{-1} \mathbf{k})^{1-s/2} e^{i\omega \cdot \mathbf{k}} \phi\left(\frac{\mathbf{x} - \mathbf{j}h}{h} - \mathbf{k} \right).$$

The number of nonzero terms is finite and occur for $\mathbf{k}$ in a neighborhood of $(\mathbf{x}/h - \mathbf{j})$. This would lead to an estimate of $O(|\mathbf{x}/h - \mathbf{j}|^{2-s})$ upon first glance. However, by the assumption of linear dependence of the translates of ϕ, we have that for each $\mathbf{x}$,

$$\sum_{\mathbf{k}} e^{i\omega\cdot\mathbf{k}}\phi\left(\frac{\mathbf{x}-h\mathbf{j}}{h}-\mathbf{k}\right)=0.$$

From this it follows that

$$\sum_{\mathbf{k}}(\mathbf{k}^T H^{-1}\mathbf{k})^{1-s/2}e^{i\omega\cdot\mathbf{k}}\phi\left(\frac{\mathbf{x}-h\mathbf{j}}{h}-\mathbf{k}\right)=O(|\mathbf{x}/h-\mathbf{j}|^{1-s})$$

and finally

$$\sum_{\mathbf{k}} C^*(\mathbf{k}-\mathbf{j})\phi(\mathbf{x}/h-\mathbf{j})=O(|\mathbf{x}/h-\mathbf{j}|^{1-s}).$$

Using this estimate in the expression (28), we have

$$e(\mathbf{x})=O(h^{\rho+2})O\left(\sum_{|\mathbf{j}|\le N}|\mathbf{j}|^{1-s}\right)=O(h^{\rho+2})O(N)=O(h^{\rho+1})$$

as required. A similar analysis applies in case B′ and leads to the same result for $e(\mathbf{x})$. □

We remark that the quadratic C^1 spline $M_{1,1,1,1}$ is an example for which the theorem applies, since the fact that

$$\sum_{\mathbf{k}}(-1)^{|\mathbf{k}|}\phi(\mathbf{x}-\mathbf{k})=0$$

for all $\mathbf{x}$ corresponds to the existence of the root $\omega=(\pi,\pi)$ of $\tilde{\Phi}$.

3. Some Remarks on Numerical Implementation. Owing to their stability when used to interpolate, splines for which $\tilde{\Phi}(\omega)>0$ seem most desirable from a computational viewpoint, given Theorem 2. One simple implementation of an interpolation scheme is to precompute C^* from the infinite series $\sum M^n$ or from the recursion $C^*_{n+1}=I+MC^*_n$, $C^*_0=0$. Since $C^*(\mathbf{k})$ decreases exponentially in $|\mathbf{k}|$, its effective support is local and, in any case, one need only store $C^*(\mathbf{k})$ for a set of values of k sufficient to calculate the interpolant, with coefficients $C(\mathbf{j})=\sum_{h\mathbf{k}\in\overline{\Omega}}C^*(\mathbf{j}-\mathbf{k})f(h\mathbf{k})$, for those $\mathbf{j}$ affecting the approximation.

A more direct iterative method for computing an interpolant is given by the recursion $C_{(n+1)}=F_h+MC_{(n)}$, $C_{(0)}=0$, where $C_{(n)}$ denotes the nth iterate and $C_{(n)}=\sum_0^n M^m F_h\to C$, the interpolating coefficients. Of course we are calculating the cardinal interpolant of F_h here, and while the values of the coefficients in the interpolating region K will in practice converge quickly, this method still requires that values of coefficients be maintained for splines $\phi(\mathbf{x}/h-\mathbf{k})$ whose supports do not intersect $\overline{\Omega}$ and in fact may be a considerable number of grid nodes away. Instead, we will show that an optimal interpolant (i.e., one that satisfies Theorem 2) can be obtained by defining $C(\mathbf{k})=0$ for $h\mathbf{k}\notin\overline{\Omega}$ and performing the iteration $C_{(n+1)}(\mathbf{k})=F_n(\mathbf{k})+MC_{(n)}(\mathbf{k})$ only at those $\mathbf{k}$ such that $h\mathbf{k}\in\overline{\Omega}$.

THEOREM 4. *Suppose $\tilde{\Phi}(\omega)>0$, Ω is an open set and f satisfies the hypotheses of Theorem 2 and is bounded on Ω. Then the recursion*

$$\begin{aligned}&C_{(0)}(\mathbf{k})=0 \textit{ for all } \mathbf{k},\\ (29)\qquad &C_{(n+1)}(\mathbf{k})=f(h\mathbf{k})+MC_{(n)}(\mathbf{k}),\qquad h\mathbf{k}\in\overline{\Omega},\\ &C_{(n+1)}(\mathbf{k})=0,\qquad h\mathbf{k}\notin\overline{\Omega}\end{aligned}$$

converges to a vector C of coefficients satisfying $\Phi C(\mathbf{k}) = f(h\mathbf{k})$, $h\mathbf{k} \in \overline{\Omega}$, and on any compact subset K of Ω, $\|f - \sum_{\mathbf{k}} C(\mathbf{k})\phi(\mathbf{x}/h - \mathbf{k})\|_{\infty,K} = O(h^{\rho+1})$.

Proof. Consider the linear operator M_Ω defined as follows: If A is a function on $\mathbf{Z}^s$, then

$$M_\Omega(A)(\mathbf{k}) \equiv \begin{cases} MA(\mathbf{k}) & \text{if } h\mathbf{k} \in \overline{\Omega}, \\ 0 & \text{if } h\mathbf{k} \notin \overline{\Omega}. \end{cases}$$

We calculate an upper bound on the norm of M_Ω relative to the $l_2(\mathbf{Z}^s)$-norm. We have

$$\|M_\Omega(A)\|_{l_2(\mathbf{Z}^s)} \leq \|MA\|_{l_2(\mathbf{Z}^s)} = \|\tilde{M}\tilde{A}\|_{L_2[-\pi,\pi]^s} \leq r\|A\|_{l_2(\mathbf{Z}^s)}, \tag{30}$$

where $r = \max \tilde{M}(\omega)$ and we have used Parseval's equality to equate $\|\cdot\|_{l_2(\mathbf{Z}^s)}$ and $\|\tilde{\cdot}\|_{L_2[-\pi,\pi]^s}$ (under an appropriate normalization).

Let $C_{(n)}^{(\mathbf{j})}$ denote the nth iterate of (29) when

$$f(h\mathbf{k}) = \begin{cases} 1 & \text{if } \mathbf{k} = \mathbf{j} \\ 0 & \text{otherwise} \end{cases} \equiv I_{\mathbf{j}}(\mathbf{k}).$$

Then, for $h\mathbf{j} \in \overline{\Omega}$,

$$C_{(n)}^{(\mathbf{j})} = I_{\mathbf{j}} + M_\Omega(I_{\mathbf{j}}) + M_\Omega^2(I_{\mathbf{j}}) + \cdots + M_\Omega^n(I_{\mathbf{j}}).$$

Now, $\|M_\Omega^n(I_{\mathbf{j}})\|_\infty \leq \|M_\Omega^n(I_{\mathbf{j}})\|_{l_2(\mathbf{Z}^s)} \leq r^n$, and since M is locally supported, we have for some constant a

$$|M_\Omega^n(I_{\mathbf{j}})(\mathbf{k})| \leq \begin{cases} r^n & \text{if } |\mathbf{j} - \mathbf{k}| \leq an, \\ 0 & \text{if } |\mathbf{j} - \mathbf{k}| > an. \end{cases} \tag{31}$$

For general data $f(h\mathbf{k})$ we can write for the iterates $C_{(n)}$

$$C_{(n)}(\mathbf{k}) = \sum_{h\mathbf{j} \in \overline{\Omega}} C_{(n)}^{(\mathbf{j})}(\mathbf{k}) f(h\mathbf{j}),$$

so, if $m > n$,

$$\begin{aligned} |C_{(m)}(\mathbf{k}) - C_{(n)}(\mathbf{k})| &\leq \|f\|_{\infty,\Omega} \sum_{h\mathbf{j} \in \overline{\Omega}} |C_{(m)}^{(\mathbf{j})}(\mathbf{k}) - C_{(n)}^{(\mathbf{j})}(\mathbf{k})| \\ &\leq \|f\|_{\infty,\Omega} \sum_{h\mathbf{j} \in \overline{\Omega}} \sum_{t=n}^{m} |M^t(I_{\mathbf{j}})(\mathbf{k})| \leq \|f\|_{\infty,\Omega} \sum_t \sum_{\mathbf{j}} |M^t(I_{\mathbf{j}})(\mathbf{k})| \\ &\leq \|f\|_{\infty,\Omega} \sum_{t=n}^{m} (2at)^s r^t \quad \text{(using (31))} \\ &\leq \|f\|_{\infty,\Omega} \alpha (n+1)^s r^n, \quad \text{where } \alpha \text{ is a constant.} \end{aligned}$$

Thus, $C_{(n)}(\mathbf{k})$ converges uniformly in $\mathbf{k}$, geometrically in n to a solution C of $\Phi C(\mathbf{k}) = f(h\mathbf{k})$, $h\mathbf{k} \in \overline{\Omega}$. Moreover, the previous calculation shows also that $|C(\mathbf{k})| \leq c\|f\|_{\infty,\Omega}$, where c is a constant independent of h and Ω and $\mathbf{k}$. Theorem 2 can then be applied and shows that the spline with coefficients $C(\mathbf{k})$ gives the optimal approximation order. This completes the proof. □

Actually, it is not difficult to modify the proof to show that if $f(\mathbf{x})$ is merely polynomially bounded, the recursion (29) will converge to a vector of coefficients

$C(\mathbf{k})$ interpolating f on Ω, yielding a spline with optimal approximation order on compact sets.

The assumption of zero coefficients for B-splines whose centers lie outside $\overline{\Omega}$ will probably lead to poor performance in the approximation near the boundary of $\overline{\Omega}$. If one desires an interpolant giving a uniformly good approximation on $\overline{\Omega}$, the following procedure could be used:

(a) "Pad" the data around the boundary of $\overline{\Omega}$, using either known values of $f(\mathbf{x})$ or extrapolated values giving $O(h^{\rho+1})$ error. (This of course assumes that f is sufficiently smooth on Ω to allow the extrapolation.) The "padding" should be sufficient that the quasi-interpolant $Q_nF_h(\mathbf{k})$ (see (10)) can be computed using only original or padded function values at all those $\mathbf{k}$ whose corresponding B-splines $\phi(\mathbf{x}/h-\mathbf{k})$ affect the approximation on $\overline{\Omega}$.

(b) Writing the desired coefficients in the form $C = Q_nF_h + E$, we have $\Phi E(\mathbf{k}) = f(h\mathbf{k}) - \Phi Q_nF_h(\mathbf{k})$, $h\mathbf{k} \in \overline{\Omega}$, as the equation to be satisfied for interpolation. One then sets $E(\mathbf{k}) = 0$ for $h\mathbf{k} \notin \overline{\Omega}$ and applies the recursion (29) to compute a solution. Since the proof of Theorem 4 shows that $|E(\mathbf{k})| \le c\|f(h\mathbf{k}) - \Phi Q_nF_h(\mathbf{k})\|_\infty$ for $h\mathbf{k} \in \overline{\Omega}$, and this quantity in turn is $O(h^{\rho+1})$, the coefficients $C(\mathbf{k})$ differ by $O(h^{\rho+1})$ from those of the quasi-interpolant and so yield an approximation uniformly $O(h^{\rho+1})$ in error on $\overline{\Omega}$.

As a final remark, we observe that the recursion $C_{(n+1)}(\mathbf{k}) = f(h\mathbf{k}) + MC_{(n)}(\mathbf{k})$ can be rewritten as $\Phi C_{(n)}(\mathbf{k}) = f(h\mathbf{k}) + C_{(n)}(\mathbf{k}) - C_{(n+1)}(\mathbf{k})$. We then identify $C_{(n)}(\mathbf{k}) - C_{(n+1)}(\mathbf{k})$, the difference in two consecutive iterates, as the interpolation error at the grid point $h\mathbf{k}$. This makes monitoring the convergence of the iterates a simple matter.

Department of Mathematics
Texas A & M University
College Station, Texas 77843

Department of Mathematics
West Virginia University
Morgantown, West Virginia 26506

Department of Mathematics
Howard University
Washington, DC 20059

1. C. DE BOOR, *The Polynomials in the Linear Span of Integer Translates of a Compactly Supported Function*, MRC Report #2930, April 1986.

2. C. DE BOOR, K. HÖLLIG & S. RIEMENSCHNEIDER, "Bivariate cardinal interpolation by splines on a three-directional mesh," *Illinois J. Math.*, v. 29, 1985, pp. 533–566.

3. C. K. CHUI & H. DIAMOND, "A natural formulation of quasi-interpolation by multivariate splines," *Proc. Amer. Math. Soc.*, v. 99, 1987, pp. 643–646.

4. C. K. CHUI, K. JETTER & J. WARD, "Cardinal interpolation by multivariate splines," *Math. Comp.*, v. 48, 1987, pp. 711–724.

5. I. J. SCHOENBERG, *Cardinal Spline Interpolation*, SIAM Regional Conference Series in Appl. Math., 1973.

6. G. STRANG & G. FIX, "A Fourier analysis of the finite element variational method," C.I.M.E. II Ciclo 1971, *Constructive Aspects of Functional Analysis* (G. Geymonat, ed.), 1973, pp. 793–840.

7. SHA ZHEN, "On interpolation by $S_2^1(\Delta_{mn}^{(2)})$," *Approx. Theory Appl.*, v. 1, no. 2, 1985, pp. 71–82.

8. SHA ZHEN, "On interpolation by $S_3^1(\Delta_{mn}^{(1)})$," *Approx. Theory Appl*, v. 1, no. 4, 1985, pp. 1–18.