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Some Infinite Product Identities

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Abstract. In this paper we derive the power series expansions of four infinite products of the form

$$\prod_{n\in S_1}(1-x^n)\prod_{n\in S_2}(1+x^n),$$

where the index sets S_1 and S_2 are specified with respect to a modulus (Theorems 1, 3, and 4). We also establish a useful formula for expanding the product of two Jacobi triple products (Theorem 2). Finally, we give nonexistence results for identities of two forms.

1. Introduction. During a computer investigation of products of the form

$$\prod_{\substack{n=1\\ n\equiv r_1,\dots,r_t\\ (\operatorname{mod} m)}}^{\infty}(1-x^n)\ (\operatorname{mod} 2),$$

we found several products with interesting modulo 2 power series expansions. For example, the expansion

$$\prod_{\substack{n=1\\ n\not\equiv 5\ (\operatorname{mod} 10)}}^{\infty}(1-x^n)\equiv\sum_{n=0}^{\infty}(x^{n(n+1)}+x^{5n(n+1)+1})\ (\operatorname{mod} 2)$$

is distinctive in that the exponents on the right are quadratic polynomials with different leading coefficients. While attempting to prove these congruences, one of us (IG) discovered with considerable surprise that some of the congruences, such as the one above, are actually equations when the plus and minus signs are chosen properly on the two sides. Four such equations with only minus signs in the factors on the left were proved in [3, Theorems 1 and 2] using standard, single-variable identities derived from the familiar triple and quintuple products. (See (9) and (20) below.)

In the present paper we establish four more equations, but this time with both plus and minus signs in their products (see [4]). For example, the identity

$$\prod_{\substack{n=1\\ n\equiv 0,\pm 3\\ (\operatorname{mod} 10)}}^{\infty}(1-x^n)\prod_{\substack{n=1\\ n\equiv \pm 1,\pm 2,\pm 4\\ (\operatorname{mod} 10)}}^{\infty}(1+x^n)=\sum_{n=0}^{\infty}(x^{n(n+1)}+x^{5n(n+1)+1})$$

is the equation standing behind the congruence above. (See Theorem 4.) We note that this equation, as well as (26), (32), and (33), have a partition theory

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interpretation as in [8, Chapter 19]. The proofs we give here employ several classical results ((8), (9), (18), (20)), as well as forms of the quintuple product identity in which both plus and minus signs occur ((21), (23)). We also develop an expansion formula for the product of two Jacobi triple products (Theorem 2). This interesting result allows us to prove the identities of this paper at quite an elementary level.

2. Preliminaries. We begin with the Jacobi triple product [8, p. 283]

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1})=\sum_{-\infty}^{\infty}x^{n^2}z^n. \tag{1}$$

Replacing (x,z) by $(x^k,(-1)^\varepsilon x^l z^p)$, $\varepsilon=0$ or 1, gives

$$\begin{aligned} T_\varepsilon(k,l;p) &\overset{\text{def}}{=} \prod_{n=1}^{\infty}(1-x^{2kn})(1+(-1)^\varepsilon x^{2kn-k+l}z^p)(1+(-1)^\varepsilon x^{2kn-k-l}z^{-p}) \\ &= \sum_{-\infty}^{\infty}(-1)^{\varepsilon n}x^{kn^2+ln}z^{pn}, \qquad k\neq 0. \end{aligned} \tag{2}$$

Replacing (x,z) by $(ix^k,(-1)^\omega ix^l z^p)$, $\omega=2$ or 3, gives

$$\begin{aligned} T_\omega(k,l;p) &\overset{\text{def}}{=} \prod_{n=1}^{\infty}(1-(-1)^n x^{2kn})(1+(-1)^{n+\omega}x^{2kn-k+l}z^p) \\ &\qquad\qquad \cdot(1+(-1)^{n-1+\omega}x^{2kn-k-l}z^{-p}) \\ &= \sum_{-\infty}^{\infty}(-1)^{\frac{n(n+1)}{2}+\omega n}x^{kn^2+ln}z^{pn}. \end{aligned} \tag{3}$$

Except for Section 2, we will be dealing with single-variable T-functions with $p=0$:

$$\begin{aligned} T_\varepsilon(k,l) &\overset{\text{def}}{=} \prod_{n=1}^{\infty}(1-x^{2kn})(1+(-1)^\varepsilon x^{2kn-k+l})(1+(-1)^\varepsilon x^{2kn-k-l}) \\ &= \sum_{-\infty}^{\infty}(-1)^{\varepsilon n}x^{kn^2+ln}, \qquad \varepsilon=0 \text{ or } 1, \end{aligned} \tag{4}$$

and

$$\begin{aligned} T_\omega(k,l) &\overset{\text{def}}{=} \sum_{n=1}^{\infty}(1-(-1)^n x^{2kn})(1+(-1)^{n+\omega}x^{2kn-k+l}) \\ &\qquad\qquad \cdot(1+(-1)^{n-1+\omega}x^{2kn-k-i}) \\ &= \sum_{-\infty}^{\infty}(-1)^{\frac{n(n+1)}{2}+\omega n}x^{kn^2+ln}, \qquad \omega=2 \text{ or } 3. \end{aligned} \tag{5}$$

(We only use $T_\omega(k,l)$ in Section 6.) Our use of (4) in the sequel will require that the exponents in the products there (and therefore the series) be integers. This implies that $2k\in\mathbf{Z}-\{0\}$, $2l, k-l\in\mathbf{Z}$. In addition, if the exponents are to be positive, then it is necessary (taking $n=\pm1$) that $2k>0$, $k+l>0$, and $k-l>0$. These conditions imply that $|l|<k$, while this latter condition is also clearly sufficient for positivity. (These restrictions on the parameters k,l will be assumed as required without necessarily restating them each time.) Note in this

case that the exponents in the three factors will run respectively through all the positive integers in the residue classes $0, -k+l, k-l$ modulo $2k$.

In this paper we are primarily concerned with power series expansions of infinite products of the form $\prod_{n\in S}(1\pm x^n)$, where the index set S consists of the members of certain residue classes with respect to a modulus m. For this purpose, we have come to use the following simplified notation:

$$(r_1,\ldots,r_t)_m = \prod_{\substack{n=1 \\ n\equiv r_1,\ldots,r_t \pmod m}}^{\infty} (1-x^n) \tag{6}$$

and

$$[r_1,\ldots,r_t]_m = \prod_{\substack{n=1 \\ n\equiv r_1,\ldots,r_t \pmod m}}^{\infty} (1+x^n). \tag{7}$$

where $m \in \mathbf{Z}^+$ and the r_i are residues modulo m. (When an r_i is repeated in the symbol s times, the corresponding factor is multiplied into the product s times.) Accordingly, in this notation (6) and (7) yield for $|l| < k$ the formulas

$$T_0(k,l) = (0)_{2k}[\pm(k-l)]_{2k} = \sum_{-\infty}^{\infty} x^{kn^2+ln}, \tag{8}$$

$$T_1(k,l) = (0,\pm(k-l))_{2k} = \sum_{-\infty}^{\infty}(-1)^n x^{kn^2+ln}, \tag{9}$$

$$T_2(k,l) = (0,\pm(k+l))_{4k}[\pm(k-l),2k]_{4k} = \sum_{-\infty}^{\infty}(-1)^{n(n+1)/2}x^{kn^2+ln}, \tag{10}$$

and

$$T_3(k,l) = (0,\pm(k-l))_{4k}[\pm(k+l),2k]_{4k} = \sum_{-\infty}^{\infty}(-1)^{n(n+3)/2}x^{kn^2+ln}. \tag{11}$$

Two classical formulas which we shall use (with x^k replacing x), due to Euler and Gauss, respectively [8, p. 282], may be written in our notation as

$$T_1\left(\frac{3k}{2},\frac{k}{2}\right) = (0)_k = \sum_{-\infty}^{\infty}(-1)^n x^{n(3n+1)k/2} \tag{12}$$

and

$$\frac{1}{2}T_0\left(\frac{k}{2},\frac{k}{2}\right) = \frac{(0)_{2k}}{(k)_{2k}} = \sum_{n=0}^{\infty} x^{n(n+1)k/2}. \tag{13}$$

It also follows from the product in (4) that

$$T_\varepsilon(k,-l) = T_\varepsilon(k,l) \tag{14}$$

and

$$T_1(k,rk) = 0, \qquad r \text{ odd}, \tag{15}$$

since

$$T_1(k,rk) = \sum_{-\infty}^{(-r-1)/2} (-1)^n(x^k)^{n^2+rn} + \sum_{(-r+1)/2}^{\infty} (-1)^n(x^k)^{n^2+rn},$$

and replacing n by $-n-r$ in the first sum gives the negative of the second sum. In particular,

$$T_1(k, \pm k) = 0. \tag{16}$$

We will also need the identity

$$T_0(k,l) - T_1(k,l) = 2x^{k-l}T_0(4k, 4k-2l), \tag{17}$$

which is proved by splitting the index values in the sums into even and odd parts. We derive

$$[0]_k(k)_{2k} = 1, \tag{18}$$

from the familiar Euler formula $[0]_1(1)_2 = 1$ by replacing x by x^k. Finally, we obtain four useful single-variable expansions from the familiar quintuple product formula [5]

$$\begin{aligned} &\prod_{n=1}^{\infty}(1-x^n)(1-x^n z)(1-x^{n-1}z^{-1})(1-x^{2n-1}z^2)(1-x^{2n-1}z^{-2}) \\ &\quad = \sum_{-\infty}^{\infty} x^{(3n^2+n)/2}(z^{3n} - z^{-3n-1}) \end{aligned} \tag{19}$$

by replacing (x,z) by $(x^m, x^{-k}), (-x^m, x^{-k}), (x^m, -x^{-k})$, and $(-x^m, -x^{-k})$, where $0 < 2k < m$:

$$\begin{aligned} Q(m,k) &\overset{\text{def}}{=} (0, \pm k, \pm(m-2k), \pm(m-k), m)_{2m} \\ &= \sum_{-\infty}^{\infty} x^{m(3n^2+n)/2}(x^{-3kn} - x^{3kn+k}), \end{aligned} \tag{20}$$

$$\begin{aligned} Q_1(m,k) &\overset{\text{def}}{=} (0, \pm k)_{2m}[\pm(m-2k), \pm(m-k), m]_{2m} \\ &= \sum_{-\infty}^{\infty}(-1)^{(3n^2+n)/2} x^{m(3n^2+n)/2}(x^{-3kn} - x^{3kn+k}), \end{aligned} \tag{21}$$

$$\begin{aligned} Q_2(m,k) &\overset{\text{def}}{=} (0, \pm(m-2k), m)_{2m}[\pm k, \pm(m-k)]_{2m} \\ &= (0, \pm(m-2k), m)_{2m}[\pm k]_m \\ &= \sum_{-\infty}^{\infty}(-1)^n x^{m(3n^2+n)/2}(x^{-3kn} + x^{3kn+k}), \end{aligned} \tag{22}$$

$$\begin{aligned} Q_3(m,k) &\overset{\text{def}}{=} (0, \pm(m-k))_{2m}[\pm k, \pm(m-2k), m]_{2m} \\ &= \sum_{-\infty}^{\infty}(-1)^{(3n^2-n)/2} x^{m(3n^2+n)/2}(x^{-3kn} + x^{3kn+k}). \end{aligned} \tag{23}$$

It immediately follows that

$$Q(m,k) = T_0\left(\frac{3m}{2}, \frac{m}{2} - 3k\right) - x^k T_0\left(\frac{3m}{2}, \frac{m}{2} + 3k\right) \tag{24}$$

and

$$Q_2(m,k) = T_1\left(\frac{3m}{2}, \frac{m}{2} - 3k\right) + x^k T_1\left(\frac{3m}{2}, \frac{m}{2} + 3k\right). \tag{25}$$

3. The First Identity.

THEOREM 1. *There holds*

$$(26)\qquad \prod_{\substack{n=1\\ n\equiv 0,\pm5,\pm7\\ (\mathrm{mod}\ 24)}}^{\infty}(1-x^n)\prod_{\substack{n=1\\ n\equiv\pm1,\pm4,6\\ (\mathrm{mod}\ 12)}}^{\infty}(1+x^n)=\sum_{n=0}^{\infty}(x^{2n(n+1)}+x^{6n(n+1)+1}).$$

Proof. Equation (26) can be rewritten as

$$[0]_{12}(0)_{12}(\pm5)_{12}[\pm1,\pm4,6]_{12}=\frac{(0)_8}{(4)_8}+x\frac{(0)_{24}}{(12)_{24}}$$

by (13), or

$$[0]_{12}Q_3(6,1)=(0)_8[0]_4+x\frac{(0)_{24}}{(12)_{24}}$$

by (23) and (18). But $[0]_{12}(12)_{24}=1$ by (18), so if we multiply the above equation by $(12)_{24}$ we obtain

$$Q_3(6,1)=(0)_8(12)_{24}[0]_4+x(0)_{24}.$$

Now

$$\begin{aligned}(0)_8(12)_{24}[0]_4&=(0)_{16}(8)_{16}(12)_{24}[0]_8[4]_8=(0)_{16}(12)_{24}[4,12,20]_{24}\quad\text{(by (18))}\\ &=(0,16,32,24)_{48}[4,20]_{24}=(0,\pm16,24)_{48}[\pm4,\pm20]_{48}=Q_2(24,4).\end{aligned}$$

We must therefore show that

$$(27)\qquad Q_3(6,1)=Q_2(24,4)+x(0)_{24}.$$

From (23) we have

$$\begin{aligned}Q_3(6,1)&=\sum_{-\infty}^{\infty}(-1)^{(3n^2-n)/2}(x^{(3n)^2}+x^{(3n+1)^2})\\ &=\sum_{-\infty}^{\infty}(x^{(12n)^2}+x^{(12n+1)^2}-x^{(12n+3)^2}-x^{(12n+4)^2}-x^{(12n+6)^2}\\ &\qquad -x^{(12n+7)^2}+x^{(12n+9)^2}+x^{(12n+10)^2})\\ &=\sum_{-\infty}^{\infty}(x^{(12n)^2}+x^{(12n+1)^2}+x^{(12n+2)^2}-x^{(12n+6)^2}-x^{(12n+7)^2}\\ &\qquad -x^{(12n+8)^2})\\ &=\sum_{-\infty}^{\infty}(-1)^n(x^{(6n)^2}+x^{(6n+2)^2})+\sum_{-\infty}^{\infty}(-1)^n x^{(6n+1)^2}\\ &=Q_2(24,4)+x(0)_{24},\end{aligned}$$

by (22) and (12). □

Remark. Replacing x by $-x$ in Theorem 1 gives the identity

$$\prod_{\substack{n=1\\ n\equiv 0,\pm1,\pm11\\ (\mathrm{mod}\ 24)}}^{\infty}(1-x^n)\prod_{\substack{n=1\\ n\equiv\pm4,\pm5,6\\ (\mathrm{mod}\ 12)}}^{\infty}(1+x^n)=\sum_{n=0}^{\infty}(x^{2n(n+1)}-x^{6n(n+1)+1}).$$

4. An Expansion Formula. We now establish a useful expansion formula for the product of two general T-functions defined in (2). The proof of this result generalizes a technique of Carlitz and Subbarao [5].

THEOREM 2. *Let* $\varepsilon_1, \varepsilon_2 \in \{0,1\}$, $k_1, l_1, p_1, k_2, l_2, p_2 \in \mathbf{Z}$ *and* $m \in \mathbf{Z}^+$. *If* a *and* b *are integers satisfying the* ***separability condition****

$$k_1 b = k_2 a(m - ab), \tag{28}$$

then

$$\begin{aligned} &T_{\varepsilon_1}(k_1, l_1; p_1) T_{\varepsilon_2}(k_2, l_2; p_2) \\ &\quad = \sum_{r \in R} (-1)^{\varepsilon_2 r} x^{k_2 r^2 + l_2 r} z^{p_2 r} T_{\delta_1}(K_1, L_1(r); P_1) T_{\delta_2}(K_2, L_2(r); P_2), \end{aligned} \tag{29}$$

where

$$\begin{array}{lll} K_1 = k_1 + k_2 a^2, & L_1(r) = l_1 - l_2 a - 2k_2 a r, & P_1 = p_1 - p_2 a, \\ K_2 = k_2 m(m - ab), & L_2(r) = (2k_2 r + l_2)(m - ab) + l_1 b, & P_2 = p_1 b + p_2(m - ab), \end{array}$$

$$\delta_1 = \begin{cases} 0 & \text{if } \varepsilon_1 - \varepsilon_2 a \text{ is even}, \\ 1 & \text{if } \varepsilon_1 - \varepsilon_2 a \text{ is odd}, \end{cases} \tag{30}$$

$$\delta_2 = \begin{cases} 0 & \text{if } \varepsilon_1 b + \varepsilon_2(m - ab) \text{ is even}, \\ 1 & \text{if } \varepsilon_1 b + \varepsilon_2(m - ab) \text{ is odd}, \end{cases} \tag{31}$$

and R *is a complete residue system* $(\bmod\, m)$.

Proof. Let

$$\begin{aligned} S &= T_{\varepsilon_1}(k_1, l_1; p_1) T_{\varepsilon_2}(k_2, l_2; p_2) \\ &= \sum_i (-1)^{\varepsilon_1 i} x^{k_1 i^2 + l_1 i} z^{p_1 i} \sum_j (-1)^{\varepsilon_2 j} x^{k_2 j^2 + l_2 j} z^{p_2 j}. \end{aligned}$$

The change of index $j = n - ai$, which sets up a 1-1 correspondence between $\{(i,j)\}$ and $\{(i,n)\}$, gives

$$\begin{aligned} S = \sum_i &(-1)^{(\varepsilon_1 - \varepsilon_2 a) i} x^{(k_1 + k_2 a^2) i^2 + (l_1 - l_2 a) i} z^{(p_1 - p_2 a) i} \\ &\cdot \sum_n (-1)^{\varepsilon_2 n} x^{k_2 n^2 + (l_2 - 2k_2 a i) n} z^{p_2 n}. \end{aligned}$$

For each $r \in R$, let S_r denote the subseries obtained from S as n runs through the set of values congruent to r modulo m, so that

$$S = \sum_{r \in R} S_r.$$

To determine S_r, put $n = sm + r$, where s is a new summation index, obtaining

$$\begin{aligned} S_r = (-1)^{\varepsilon_2 r} x^{k_2 r^2 + l_2 r} z^{p_2 r} &\sum_i (-1)^{(\varepsilon_1 - \varepsilon_2 a) i} x^{(k_1 + k_2 a^2) i^2 + (l_1 - l_2 a - 2k_2 a r) i} z^{(p_1 - p_2 a) i} \\ \cdot \sum_s &(-1)^{\varepsilon_2 m s} x^{k_2 m^2 s^2 + (2k_2 m r + l_2 m - 2k_2 a m i) s} z^{p_2 m s}. \end{aligned}$$

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*This condition assures that a certain doubly indexed sum in the proof will factor into a product of two singly indexed sums.

Making a last change of index $i = t + bs$ and setting $d = m - ab$, we obtain

$$S_r = (-1)^{\varepsilon_2 r} x^{k_2 r^2 + l_2 r} z^{p_2 r} \cdot \sum_s (-1)^{(\varepsilon_1 b + \varepsilon_2 d)s} x^{(k_1 b^2 + k_2 d^2)s^2 + ((2k_2 r + l_2)d + l_1 b)s} z^{(p_1 b + p_2 d)s} \cdot \sum_t (-1)^{(\varepsilon_1 - \varepsilon_2 a)t} x^{(k_1 + k_2 a^2)t^2 + (l_1 - l_2 a - 2k_2 ar)t + 2(k_1 b - k_2 ad)st} z^{(p_1 - p_2 a)t}.$$

Using the separability condition (28), the coefficient of s^2 simplifies and the st term drops out, giving a sum which separates into the product of two sums. Reversing the order of the sums and using (30) and (31) yields the theorem. □

Remarks. 1. It is worth mentioning that (28) can always be satisfied, e.g., $a = k_1$, $b = k_2$, $m = k_1 k_2 + 1$. If a certain value of m is desired, however, then (28) may not be solvable. It can also happen, because of the asymmetry of (28) in a and b, that condition (28) may be solvable for some value of m when the factors on the left of (29) are taken in one order but not in the other.

2. The Carlitz and Subbarao proof [5] of the quintuple product formula (19) is a special case of Theorem 2: Replacing x by x^2 on the left-hand side of (19) and multiplying by $\prod_{n=1}^{\infty}(1 - x^{4n})$ gives $T_1(2,0;2)T_1(1,1;1)$. Applying Theorem 2 with $k_1 = 2$, $l_1 = 0$, $p_1 = 2$ and $k_2 = l_2 = p_2 = 1$, the separability condition is satisfied with $a = 2$, $b = 1$, and $m = 3$. Taking $R = \{0, 1, -1\}$, we find, after some simplification, that

$$\begin{aligned} T_1(2,0;2)T_1(1,1;1) &= T_1(6,2)[T_0(3,1;3) - z^{-1}T_0(3,-1;3)] \\ &= (0)_4 \sum_{-\infty}^{\infty} x^{3n^2+n}(z^{3n} - z^{-3n-1}). \end{aligned}$$

Cancelling $(0)_4 = \prod_{n=1}^{\infty}(1 - x^{4n})$ and replacing x by $x^{1/2}$ yields (19).

3. Formula (29) gives an immediate proof of Theorem 1 in [6] and identity (2) in [6], viz., with $m = 2$, $a = b = 1$, and $R = \{0, 1\}$, we have

$$\begin{aligned} T_0^2(1,0;1) &= \sum_{r \in R} x^{r^2} z^t T_0(2,-2r;0) T_0(2,2r;2) \\ &= \sum_{-\infty}^{\infty} x^{2n^2} \sum_{-\infty}^{\infty} x^{2n^2} z^{2n} + xz \sum_{-\infty}^{\infty} x^{2n^2+2n} \sum_{-\infty}^{\infty} x^{2n^2+2n} z^{2n}. \end{aligned}$$

5. Three More Identities.

THEOREM 3. *We have*

$$(32) \quad \prod_{\substack{n=1 \\ n \equiv 0, \pm 1 \\ (\mathrm{mod}\ 10)}}^{\infty} (1 - x^n) \prod_{\substack{n=1 \\ n \equiv \pm 12 \\ (\mathrm{mod}\ 40)}}^{\infty} (1 - x^n) \prod_{\substack{n=1 \\ n \equiv \pm 4 \\ (\mathrm{mod}\ 20)}}^{\infty} (1 + x^n) = 1 + \sum_{n=1}^{\infty} (-1)^n (x^{n^2} + x^{5n^2})$$

and

$$(33) \quad \prod_{\substack{n=1 \\ n \equiv 0, \pm 3 \\ (\mathrm{mod}\ 10)}}^{\infty} (1 - x^n) \prod_{\substack{n=1 \\ n \equiv \pm 4 \\ (\mathrm{mod}\ 40)}}^{\infty} (1 - x^n) \prod_{\substack{n=1 \\ n \equiv \pm 8 \\ (\mathrm{mod}\ 20)}}^{\infty} (1 + x^n) = \sum_{n=1}^{\infty} (-1)^n (x^{5n^2-1} - x^{n^2-1}).$$

Proof. First rewrite (32) and (33) as

$$\sum_{-\infty}^{\infty}(-1)^n(x^{n^2}+x^{5n^2}) = 2(0,\pm 1)_{10}(\pm 12)_{40}[\pm 4]_{20}$$

and

$$\sum_{-\infty}^{\infty}(-1)^n(x^{5n^2}-x^{n^2}) = 2x(0,\pm 3)_{10}(\pm 4)_{40}[\pm 8]_{20}.$$

Subtracting and adding these equations gives

$$T_1(1,0) = (0,\pm 1)_{10}(\pm 12)_{40}[\pm 4]_{20} - x(0,\pm 3)_{10}(\pm 4)_{40}[\pm 8]_{20},$$
$$T_1(5,0) = (0,\pm 1)_{10}(\pm 12)_{40}[\pm 4]_{20} + x(0,\pm 3)_{10}(\pm 4)_{40}[\pm 8]_{20}.$$

Multiplying these by $(0)_4$ ($= T_1(6,2)$ by (12)) and manipulating the factors on the right puts the identities into the form in which we will prove them, namely

$$T_1(6,2)T_1(1,0) = T_1(10,2)Q(10,1) - xT_1(10,6)Q(10,3), \tag{34}$$

$$T_1(6,2)T_1(5,0) = T_1(10,2)Q(10,1) + xT_1(10,6)Q(10,3). \tag{35}$$

To prove (34), we use Theorem 2 with $p_1 = p_2 = 0$, $k_1 = 6$, $l_1 = 2$, $k_2 = 1$, $l_2 = 0$, $\varepsilon_1 = \varepsilon_2 = 1$. In this case the choice of parameters $a = 2$, $b = 1$, $m = 5$, satisfies the separability condition (28), from which we find $\delta_1 = 1$, $\delta_2 = 0$. It is also convenient to take $R = \{0, 1, 2, -1, -2\}$. Then

$$\begin{aligned} T_1(6,2)T_1(1,0) &= \sum_{r\in R}(-1)^r x^{r^2} T_1(10, 2-4r)T_0(15, 2+6r) \\ &= T_1(10,2)T_0(15,2) - xT_1(10,-2)T_0(15,8) + x^4T_1(10,-6)T_0(15,14) \\ &\quad - xT_1(10,6)T_0(15,-4) + x^4T_1(10,10)T_0(15,-10) \\ &= T_1(10,2)[T_0(15,2) - xT_0(15,8)] - xT_1(10,6)[T_0(15,4) - x^3T_0(15,14)] \\ &= T_1(10,2)Q(10,1) - xT_1(10,6)Q(10,3), \end{aligned}$$

using (14), (16), and (24). This verifies (34).

We next establish the equation

$$(0)_{20}T_1(5,0) = T_1(5,4)Q_2(20,4) + xT_1(5,2)Q_2(20,8). \tag{36}$$

(Note that (35) follows directly from (36) by multiplying each term of (36) by $(\pm 4, \pm 8)_{20}$.) To do this, we first replace x by x^5 in (34). Since the left-hand sides of the resulting equation and (36) are the same, to prove (36) it suffices to show their right-hand sides are equal, namely

$$\begin{aligned} &T_1(5,4)Q_2(20,4) + xT_1(5,2)Q_2(20,8) \\ &\qquad = T_1(50,10)Q(50,5) - x^5T_1(50,30)Q(50,15). \end{aligned} \tag{37}$$

If in the expression $T_1(5,4)Q_2(20,4) + xT_1(5,2)Q_2(20,8)$ we expand the two Q_2's by (25), we obtain

$$\begin{aligned} &T_1(30,2)T_1(5,4) + x^4T_1(30,22)T_1(5,4) \\ &\qquad + xT_1(30,-14)T_1(5,2) + x^9T_1(30,-34)T_1(5,2). \end{aligned} \tag{38}$$

(Here the order of the factors in each term is carefully chosen, as are the signs in the second components. (Cf. (14).)) We next expand the four $T_1 \cdot T_1$ pairs by Theorem 2 with $p_1 = p_2 = 0$, noting that $k_1 = 30$, $k_2 = 5$, and $\varepsilon_1 = \varepsilon_2 = 1$ in each pair. In addition, in each of these expansions we take $m = 5$, $R = \{0, 1, 2, -1, -2\}$, and $a = 3$, $b = 1$ so that (28) is satisfied and $\delta_1 = 0$, $\delta_2 = 1$. It is convenient to exhibit (29) with these values inserted, namely

$$\begin{aligned} &T_1(30, l_1)T_1(5, l_2) \\ &\quad = \sum_{r \in R} (-1)^r x^{5r^2 + l_2 r} T_0(75, l_1 - 3l_2 - 30r) T_1(50, 20r + l_1 + 2l_2). \end{aligned} \tag{39}$$

Now in (39), if we take (l_1, l_2) in turn to be $(2, 4), (22, 4), (-14, 2)$, and $(-34, 2)$, we get 20 products of the form $x^n T_0 T_1$ of which 16 miraculously cancel when they are properly grouped. To see this, we first note that four terms are 0, using (16). If the sum of the 16 remaining terms is then separated into 5 subsums S_i, $0 \le i \le 4$, where a term $x^n T_0 \cdot T_1$ is placed in S_i if $n \equiv i \pmod 5$, we find that $S_1 = S_4 = 0$, using (14). We also discover that

$$S_2 = x^{12} T_0(75, 50) T_1(50, -30) + x^{32} T_0(75, -50) T_1(50, 70) = 0,$$

because

$$\begin{aligned} x^{32} T_1(50, -70) &= x^{12} \sum_{-\infty}^{\infty} (-1)^n x^{50n^2 - 70n + 20} \\ &= -x^{12} \sum_{-\infty}^{\infty} (-1)^n x^{50n^2 + 30n} = -x^{12} T_1(50, 30) \qquad (n \to n + 1). \end{aligned} \tag{40}$$

Next,

$$S_3 = -x^8 T_0(75, -50) T_1(50, 10) + x^{33} T_0(75, -100) T_1(50, 10) = 0,$$

because

$$x^{33} T_0(75, -100) = x^8 \sum_{-\infty}^{\infty} x^{75n^2 - 100n + 25} = x^8 \sum_{-\infty}^{\infty} x^{75n^2 + 50n} = x^8 T_0(75, 50).$$

Finally,

$$\begin{aligned} S_0 = {} & T_0(75, -10) T_1(50, 10) - x^5 T_0(75, 40) T_1(50, 10) + x^{25} T_0(75, -80) T_1(50, 30) \\ & + x^{25} T_0(75, 20) T_1(50, -70). \end{aligned}$$

But

$$x^{25} T_0(75, -80) = x^{20} \sum_{-\infty}^{\infty} x^{75n^2 - 80n + 5} = x^{20} \sum_{-\infty}^{\infty} x^{75n^2 + 70n} = x^{20} T_0(75, 70),$$

and by (40)

$$x^{25} T_1(50, -70) = -x^5 T_1(50, 30),$$

so

$$\begin{aligned} S_0 &= T_1(50, 10)[T_0(75, 10) - x^5 T_0(75, 40)] \\ &\quad - x^5 T_1(50, 30)[T_0(75, 20) - x^{15} T_0(75, 70)] \\ &= T_1(50, 10) Q(50, 5) - x^5 T_1(50, 30) Q(50, 15), \end{aligned}$$

which is the desired right-hand side of (37). □

THEOREM 4. *There holds*

$$\prod_{\substack{n=1\\ n\equiv 0,\pm 3\\ (\mathrm{mod}\ 10)}}^{\infty}(1-x^n)\prod_{\substack{n=1\\ n\equiv \pm 1,\pm 2,\pm 4\\ (\mathrm{mod}\ 10)}}^{\infty}(1+x^n)=\sum_{n=0}^{\infty}(x^{n(n+1)}+x^{5n(n+1)+1}). \tag{41}$$

Proof. The result can also be written

$$2(0,\pm 3)_{10}[\pm 1,\pm 2,\pm 4]_{10}=\sum_{-\infty}^{\infty}(x^{n(n+1)}+x^{5n(n+1)+1}).$$

Replacing x by x^4 and multiplying by x gives

$$\sum_{-\infty}^{\infty}(x^{(2n+1)^2}+x^{5(2n+1)^2})=2x(0,\pm 12)_{40}[\pm 4,\pm 8,\pm 16]_{40},$$

the form in which we prove Theorem 4. Now (32) can be rewritten as

$$\begin{aligned}2(0,\pm 1)_{10}(\pm 12)_{40}[\pm 4]_{20}&=\sum_{-\infty}^{\infty}(-1)^n(x^{n^2}+x^{5n^2})\\ &=\sum_{-\infty}^{\infty}(x^{(2n)^2}+x^{5(2n)^2})-\sum_{-\infty}^{\infty}(x^{(2n+1)^2}+x^{5(2n+1)^2}).\end{aligned}\tag{42}$$

Replacing x by $-x$ in (42) gives

$$\begin{aligned}2(0)_{10}(\pm 12)_{40}[\pm 1]_{10}[\pm 4]_{20}&=\sum_{-\infty}^{\infty}(x^{n^2}+x^{5n^2})\\ &=\sum_{-\infty}^{\infty}(x^{(2n)^2}+x^{5(2n)^2})+\sum_{-\infty}^{\infty}(x^{(2n+1)^2}+x^{5(2n+1)^2}).\end{aligned}\tag{43}$$

Subtracting (42) from (43) and using (8), (9), and (17) gives

$$\begin{aligned}&\sum_{-\infty}^{\infty}(x^{(2n+1)^2}+x^{5(2n+1)^2})\\ &\quad=(0)_{10}(\pm 12)_{40}[\pm 1]_{10}[\pm 4]_{20}-(0,\pm 1)_{10}(\pm 12)_{40}[\pm 4]_{20}\\ &\quad=(\pm 12)_{40}[\pm 4]_{20}\{(0)_{10}[\pm 1]_{10}-(0,\pm 1)_{10}\}\\ &\quad=(\pm 12)_{40}[\pm 4]_{20}\{T_0(5,4)-T_1(5,4)\}\\ &\quad=(\pm 12)_{40}[\pm 4]_{20}2xT_0(20,12)=2x(0,\pm 12)_{40}[\pm 4,\pm 8,\pm 16]_{40}.\quad\square\end{aligned}$$

Remark. Replacing x by $-x$ in Theorem 4 gives the formula

$$\prod_{\substack{n=1\\ n\equiv 0,\pm 1\\ (\mathrm{mod}\ 10)}}^{\infty}(1-x^n)\prod_{\substack{n=1\\ n\equiv \pm 2,\pm 3,\pm 4\\ (\mathrm{mod}\ 10)}}^{\infty}(1+x^n)=\sum_{n=0}^{\infty}(x^{n(n+1)}-x^{5n(n+1)+1}).$$

6. Two Nonexistence Results. Since Theorems 1(a) and 3(a) in [3], as well as (32) above, have the form

$$\prod_{n=1}^{\infty}(1+\gamma_n x^n)=1+\sum_{n=1}^{\infty}(-1)^n(x^{n^2}+x^{kn^2}) \tag{44}$$

for $k = 2, 3$, or 5 and certain $\gamma_n = 0, \pm 1$ for each $n \geq 1$, it is natural to ask whether there are any other values of $k \geq 1$ for which (44) holds. We will answer this question for the more general equation

$$\prod_{n=1}^{\infty}(1+\gamma_n x^n) = 1 + \sum_{n=1}^{\infty}(\delta_n x^{n^2} + \varepsilon_n x^{kn^2}), \tag{45}$$

where $\gamma_n = 0, \pm 1$ and $\delta_n, \varepsilon_n = \pm 1$ for each $n \geq 1$. Using a simple search algorithm, which runs quickly on an IBM PC, we discover that (45) can hold only when $k = 2, 3, 5$, or 9.

This search algorithm relies on the function *DeadendDegree*, which attempts to factor the series $1 + \sum_{n=1}^{L} \alpha_n x^n$ (stored as an array $(\alpha_1, \ldots, \alpha_L)$) into a product of the form on the left-hand side of (45) up to the specified degree L. If no factorization is possible, the program returns the degree at which a contradiction is first obtained. Otherwise, the value $L+1$ is returned. For example, the series

$$f(x) = 1 - x + x^4 - x^7 + x^9 + x^{16} + x^{25} - x^{28} + x^{36},$$

whose exponents are of the form n^2 and $7n^2$, factors uniquely up to degree 29 as

$$p(x) = (1-x)(1+x^4)(1+x^5)(1+x^6)(1+x^{12})(1+x^{13})(1+x^{14})(1+x^{16})$$
$$(1-x^{18})(1+x^{24})(1+x^{26})(1+x^{27})(1-x^{29}).$$

However, when this product is multiplied out, we find that

$$p(x) = 1 - x + x^4 - x^7 + x^9 + x^{16} + x^{25} - x^{28} + 2x^{30} + x^{31} + \cdots.$$

The discrepancy of 2 between the coefficients of x^{30} in $p(x)$ and $f(x)$ guarantees that the factorization of the original series $f(x)$ into the form on the left-hand side of (45) has reached a dead end at degree 30. *DeadendDegree* uses an auxiliary array $\beta(x) = 1 + \sum_{n=1}^{L} \beta_n x^n$:

```
function DeadEndDegree(L, 1 + Σ_{n=1}^{L} α_n x^n)
    begin
        set β(x) = 1
        for n = 1 to L do
        begin
            δ_n = α_n − β_n
            if |δ_n| ≥ 2 then return (n)
            β(x) = β(x)(1 + δ_n x^n)
        end
        return (L + 1)
    end.
```

Given L and a sequence $s = \{s_1, \ldots, s_t\}$, $1 \leq s_i \leq L$, the intermediate function *LargestDegree* returns the value

$$\max_{\substack{\varepsilon_i = 1 \text{ or } -1 \\ 1 \leq i \leq t}} \textit{DeadendDegree}\left(L, 1 + \sum_{i=1}^{t} \varepsilon_i x^{s_i}\right).$$

The search algorithm can now be described:

program *Search*
begin
input (L)
$S = \{n^2 : 1 \le n^2 \le L\}$
$MaxK = LargestDegree(L, S)$
if $MaxK \le L$ then
for $k = 2$ to $MaxK$ do
begin
$T = \{kn^2 : 1 \le kn^2 \le L\}$
if $LargestDegree(L, S \cup T) \le L$ then
print ("No Identity for $k =$ ", k)
else
output parameters for those series which factor to degree L
end
end.

On execution, with $L = 30$, *Search* finds that $MaxK = 15$. This means that all series of the form $1 + \sum_{n=1}^{\infty} \delta_n x^{n^2}$, for any choice of sign $\delta_n = \pm 1$, fail to factor into the form on the left-hand side of (45) by degree 15. Thus, (45) is impossible for $k \ge 16$. Of the values of k between 2 and 15 which fail to factor to degree 30 (all but $k = 2, 3, 5$, and 9), the "longest survivor" is the series $f(x)$ displayed above.

For $k = 2, 3$, and 5, the values of γ_n, δ_n, and ε_n determined by the search program agree up to degree 144 with the coefficients of the known identities or the trivial variation of them obtained by replacing x by $-x$. This suggests that the known identities are the only solutions in this case. To prove this, however, would require a much more elaborate investigation of the kind used in [1, Chapter 4]. Note that Theorems 1(a), 3(a) in [3], and (32) above, along with their companions with $x \to -x$, are the only solutions to (44).

For $k = 9$, the values determined by the program for γ_n, δ_n, and ε_n, $1 \le n \le 144$, indicated four basic identities, whose proofs rely on the four functions $Q(6,1)$, $Q_1(6,1)$, $Q_2(6,1)$, and $Q_3(6,1)$. The first two give the identities

$$\begin{aligned}
Q(6,1) &= (0, \pm 1, \pm 4, \pm 5, 6)_{12} \\
&= \sum_{-\infty}^{\infty} x^{3(3n^2+n)}(x^{-3n} - x^{3n+1}) = \sum_{-\infty}^{\infty}(x^{(3n)^2} - x^{(3n+1)^2}) \\
&= \sum_{n=-\infty}^{-1} (x^{(3n)^2} - x^{(3n+1)^2}) + 1 - x + \sum_{n=1}^{\infty}(x^{(3n)^2} - x^{(3n+1)^2}) \\
&= 1 + \sum_{n=1}^{\infty}(\delta_n x^{n^2} + x^{9n^2}), \quad \text{where } \delta_n = \begin{cases} 1, & \text{if } 3 \mid n, \\ -1, & \text{if } 3 \nmid n \end{cases}
\end{aligned}$$

and the more elaborate

$$\begin{aligned} Q_1(6,1) &= (0,\pm 1)_{12}[\pm 4, \pm 5, 6]_{12} \\ &= \sum_{-\infty}^{\infty}(-1)^{(3n^2+n)/2}(x^{(3n)^2} - x^{(3n+1)^2}) \\ &= 1 + \sum_{n=1}^{\infty}(\delta_n x^{n^2} + x^{9n^2}), \end{aligned}$$

$$\text{where } \delta_n = \begin{cases} (-1)^{\frac{3n^2-n}{2}-1} & \text{if } n \equiv 0 \pmod 3, \\ (-1)^{(3n^2-n)/2} & \text{if } n \equiv 1 \pmod 3, \\ (-1)^{\frac{3n^2-n}{2}+1} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Note in the latter equation that the terms x^{9n^2} cancel when n is odd and double when n is even. Since we can switch the coefficients of $x^{(3n)^2}$ and x^{9n^2} when these terms cancel, there are infinitely many trivial variations of this identity which satisfy (45). The other two basic identities for $k = 9$ are obtained from the two above by replacing x by $-x$, or equivalently, by using $Q_2(6,1)$ and $Q_3(6,1)$.

In a similar way we have searched for identities of the form

$$\prod_{n=1}^{\infty}(1 + \gamma_n x^n) = 1 + \sum_{n=1}^{\infty}(\delta_n x^{n^2-1} + \varepsilon_n x^{kn^2-1}), \tag{46}$$

where $\gamma_n = 0, \pm 1$ and $\delta_n, \varepsilon_n = \pm 1$. (This generalizes Theorems 1(b) and 3(b) in [3] and (33) above.) The program in this case is the same as *Search*, but with $n^2 - 1$ and $kn^2 - 1$ being used in the definitions of S and T. This time we find that $MaxK = 25$ and that all series dead-end by degree 44 except when $k = 2, 3, 4, 5$, and 9. As before, for $k = 2, 3$, and 5, the values of γ_n, δ_n, and ε_n suggest the known identities are the only ones. For $k = 4$, there are four basic identities, stemming from the four functions $T_0(16,8)$, $T_1(16,18)$, $T_2(16,8)$, and $T_3(16,8)$. The first is

$$\begin{aligned} T_0(16,8) = (0)_{32}[\pm 8]_{32} &= \sum_{-\infty}^{\infty} x^{(4n+1)^2-1} \\ &= \sum_{n=1}^{\infty} x^{(2n-1)^2-1} = \sum_{n=1}^{\infty}(x^{n^2-1} - x^{4n^2-1}). \end{aligned}$$

For the other 3 identities, we use (9), (10), and (11) to obtain

$$T_1(16,8) = (0,\pm 8)_{32} = 1 + \sum_{n=1}^{\infty}(-1)^{n(n+1)/2}x^{(2n-1)^2-1},$$

$$T_2(16,8) = (0,\pm 24)_{64}[\pm 8, 32]_{64} = 1 + \sum_{n=1}^{\infty}(-1)^{(n-1)n(n+1)(n+2)/8}x^{(2n-1)^2-1},$$

and

$$T_3(16,8) = (0,\pm 8)_{64}[\pm 24, 32]_{64} = 1 + \sum_{n=1}^{\infty}(-1)^{(n-2)n(n+1)(n+3)/8}x^{(2n-1)^2-1}.$$

To put these into the form of (46), we need only make sure that the terms $x^{(2n)^2-1}$ and x^{4n^2-1} cancel.

For $k = 9$, there are four basic identities, derived from the four functions $T_0(9,6)$, $T_1(9,6)$, $T_2(9,6)$ and $T_3(9,6)$. We give only the first here:

$$T_0(9,6) = (0)_{182}[\pm 3]_{18} = \sum_{-\infty}^{\infty} x^{(3n+1)^2-1}$$
$$= 1 + \sum_{n=1}^{\infty} (x^{(3n+1)^2-1} + x^{(3n-1)^2-1}) = \sum_{n=1}^{\infty} (x^{n^2-1} - x^{9n^2-1}).$$

The other three identities are obtained in a straightforward manner, similar to that for $k = 4$.

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