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Computation of Independent Units in Number Fields by Dirichlet's Method*

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Abstract. Using the basis reduction algorithm of A. K. Lenstra, H. W. Lenstra, Jr. and L. Lovász [8] and an idea of Buchmann [4], we describe a method for computing maximal systems of independent units in arbitrary number fields. The tables in the supplements section display such systems for the fields $\mathbf{Q}(\sqrt[n]{D})$ where $6 \le n \le 11$.

1. Introduction. Let K be an algebraic number field of degree $n \geq 2$ over $\mathbf{Q}$, let R be an order in K and let E be the group of units of R. The structure of E was described in 1846 by Dirichlet [6]. He proved that if K has s real and $2t$ nonreal conjugate fields, then E is the direct product of the finite group of the roots of unity in E and $r = s+t-1$ infinite cyclic groups. In the sequel we assume $r \geq 1$.

Dirichlet's proof was based on his diophantine approximation theorem: Let $\alpha_1, \dots, \alpha_n \in R$, $n \geq 2$; then there exist for any $Q \in R$, $Q > 1$, integers $x_1, \dots, x_n$ which are not all zero such that

$$\begin{aligned} &|x_i| \le Q, \qquad i = 2, \dots, n, \\ &\left|\sum \alpha_i x_i\right| \le |\alpha_1| Q^{-(n-1)}. \end{aligned} \tag{1.1}$$

One can find this proof, for example, in Dedekind's classical book [5, §183]. If $n = 2$, then the convergents of the continued fraction expansion of α_1/α_2 solve (1.1).

Unfortunately, there exists for $n > 2$ no general practical method for the solution of the approximation problem (1.1).

The importance of the unit group inspired many mathematicians to find algorithms which produce systems of fundamental units, or at least systems of independent units. A system $\{\varepsilon_1, \dots, \varepsilon_u\} \subseteq E$ is called *independent* if $\varepsilon_1^{m_1} \cdots \varepsilon_u^{m_u} = 1$ implies $m_1 = \cdots = m_u = 0$ for every system of integers $\{m_1, \dots, m_u\}$. $\{\varepsilon_1, \dots, \varepsilon_r\} \subseteq E$ is called a system of *fundamental units* if it generates the maximal torsion free subgroup of E. Most of the former algorithms are based on generalizations of the continued fraction algorithm and are applicable only to special number fields. For a complete list of references we refer to Brentjes [2] and Buchmann [3].

The method of Pohst and Zassenhaus [10] has another foundation. It produces many integers of bounded norm by solving certain inequalities. This procedure

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yields units because there are only finitely many nonassociated elements of bounded norm in the order R. This method was improved by Fincke and Pohst [7].

The basis reduction algorithm of Lenstra, Lenstra and Lovász [8]—in the following LLL-algorithm—solves the following approximation problem very fast:

$$(1.2)\qquad \begin{aligned} &|x_i| \le 2^{n/4}Q, \qquad i = 2, \dots, n, \\ &\left|\sum x_i\alpha_i\right| \le |\alpha_1|Q^{-(n-1)}, \end{aligned}$$

which is slightly weaker than (1.1). Thus, the LLL-algorithm combined with Dirichlet's original idea yields theoretically a useful method for finding independent units. But in practical computation, this combination has the disadvantage that if Q increases, then $\sum x_i\alpha_i$ decreases very fast, and one must use multiprecision arithmetic. We were able to remove this disadvantage using an idea of Buchmann [4].

We do not vary Q but the α_i's. We apply the LLL-algorithm in each step two times. First we vary the α_i's in such a way that all their conjugates have always the same "small" order of magnitude, and then we solve (1.2) for the new α_i's and with the unchanged Q. In this way we compute independent units without handling too large or too small numbers. We are working with such numbers only if we want to calculate the coefficients of the units in the original basis of the order.

The comparison of our computational results in pure quintic fields with tables of fundamental units, computed by the method [3], showed that our method yields often fundamental units. If this is not the case, then one can compute such a system from a set of independent units, for example by the method of Fincke and Pohst [7].

In Section 2 we give an informal description of the basic steps of the algorithm. In Section 3 we study the connection between LLL-reduced bases of lattices, diophantine approximation and the algorithmization of Dirichlet's proof of the unit theorem. Section 4 contains the detailed description of the algorithm. To illustrate the efficiency of the method, we have computed maximal systems of independent units in number fields $\mathbf{Q}(\sqrt[n]{D})$, where $6 \le n \le 11$, which are presented in the tables of the supplements section at the end of this issue.

2. First Outline of the Method. Let K be an algebraic number field with s real conjugate fields $K^{(1)}, \dots, K^{(s)}$ and t pairs of complex conjugate fields $K^{(s+1)}, \overline{K^{(s+1)}}, \dots, K^{(s+t)}, \overline{K^{(s+t)}}$, and let R be an order of K. For every "conjugate direction" $i \in \{1, \dots, s+t\}$ we construct a sequence $(\gamma_k)_{k\in\mathbf{N}}$ of numbers of bounded norm in R with

$$(2.1)\qquad \begin{aligned} &|\gamma_k^{(i)}| < |\gamma_{k-1}^{(i)}| \quad \text{for } k \ge 2, \\ &|\gamma_k^{(j)}| > |\gamma_{k-1}^{(j)}| \quad \text{for } j \in \{1, \dots, s+t\}, j \ne i, k \ge 2. \end{aligned}$$

Obviously, these numbers have to be pairwise distinct, and after a finite number of steps two of these numbers are associated with a nontrivial unit ε_i satisfying

$$(2.2)\qquad |\varepsilon_i^{(i)}| < 1 \text{ and } |\varepsilon_i^{(j)}| > 1 \quad \text{for } j \ne i.$$

It is well known that every subsystem of cardinality $s+t-1$ in $\{\varepsilon_1, \dots, \varepsilon_{s+t}\}$ is a maximal system of independent units in R (cf. [9]).

The sequence $(\gamma_k)_{k\in\mathbf{N}}$ is constructed as follows: To initialize the sequence, we set

$$(2.3)\qquad \gamma_1 = 1.$$

Now suppose that we know γ_k. Then we define

$$R_k = \frac{1}{\gamma_k} R, \qquad N_k = |N_{K|\mathbf{Q}}(\gamma_k)|, \tag{2.4}$$

and using techniques of diophantine approximation, we compute a number β_k in the module R_k satisfying

$$|\beta_k^{(i)}| < 1, f_1 > |\beta_k^{(j)}| > 1 \quad \text{for } j \in \{1, \dots, s+t\}, j \neq i, \tag{2.5}$$

and

$$|N_{K|\mathbf{Q}}(\beta_k)| \leq f_2 N_k^{-1},$$

where f_1, f_2 are constants depending only on the degree n of K and on the discriminant of the order R. Then we set

$$\gamma_{k+1} := \gamma_k \beta_k. \tag{2.6}$$

Obviously, the sequence $(\gamma_k)_{k \in \mathbf{N}}$ constructed like this satisfies the requirements of (2.1).

The advantage of our method is the following: All the conjugates of the numbers β_k and of the elements in the basis of R_k are—independent of k—of "small" size during the whole algorithm. Moreover, the question of whether two of the γ_k's, e.g., γ_{k_1} and γ_{k_2}, are associated can be answered in terms of the basis of the corresponding modules, since

$$\gamma_{k_1} \sim \gamma_{k_2} \Leftrightarrow R_{k_1} = R_{k_2}. \tag{2.7}$$

In fact, (2.7) follows directly from

$$\bigwedge_{\alpha \in K} (\alpha R = R \Leftrightarrow \alpha \text{ is a unit of } R). \tag{2.8}$$

Finally, if $\gamma_{k_1} \sim \gamma_{k_2}$ $(k_1 < k_2)$, then the corresponding unit can be computed by the formula

$$\varepsilon_i = \prod_{l=k_1}^{k_2-1} \beta_l. \tag{2.9}$$

So we do not have to know the γ_k's explicitly, and we can carry out all computations, except for the final computation of the unit ε_i, using only "small" numbers. For this reason, our method can be applied very efficiently to fields of high degrees and large discriminants.

3. Basis Reduction and Diophantine Approximation. First of all, let us briefly recall some definitions and results of the basis reduction theory of Lenstra, Lenstra and Lovász [8].

Let L be a complete lattice in $\mathbf{R}^n$, and let $d(L)$ be the volume of its fundamental parallelotope. For a basis $b_1, \dots, b_n$ of L the vectors b_i^* $(1 \leq i \leq n)$ and the real numbers μ_{ij} $(1 \leq j < i \leq n)$ are inductively defined by

$$b_i^* := b_i - \sum_{j=1}^{i-1} \mu_{ij} b_j^*,$$

$$\mu_{ij} := (b_i, b_j^*)/(b_j^*, b_j^*),$$

where (,) denotes the ordinary inner product on $\mathbf{R}^n$. The basis $b_1, \dots, b_n$ is called *LLL-reduced* if and only if

$$|\mu_{ij}| \le \tfrac{1}{2} \quad \text{for } 1 \le j < i \le n,$$

and

$$|b_i^* + \mu_{i,i-1} b_{i-1}^*|^2 \ge \tfrac{3}{4} |b_{i-1}^*|^2 \quad \text{for } 1 < i \le n.$$

(3.1) LEMMA. *Let $b_1, \dots, b_n$ be a reduced basis of L; then we have*

$$d(L) \le \prod_{i=1}^{n} |b_i| \le 2^{n(n-1)/4} d(L), \tag{3.2}$$

$$|b_1| \le 2^{(n-1)/4} d(L)^{1/n}. \tag{3.3}$$

The LLL-algorithm yields an LLL-reduced basis of any lattice.

In view of (2.4) we now discuss free $\mathbf{Z}$-modules of rank n in K of the form

$$M = \frac{1}{\gamma} R \tag{3.4}$$

with a number $\gamma \in R$.

We apply the LLL-algorithm in two different situations:

(a) Since we want to carry out computations in M, we need a convenient basis of M. From the geometry of numbers it is well known that the mapping

$$K \to \mathbf{R}^n$$
$$\alpha \to \underline{\alpha}: (\alpha^{(1)}, \dots, \alpha^{(s)}, \operatorname{Re} \alpha^{(s+1)}, \dots, \operatorname{Re} \alpha^{(s+t)}, \operatorname{Im} \alpha^{(s+1)}, \dots, \operatorname{Im} \alpha^{(s+t)})^T$$

is a monomorphism of K, and that the image $\underline{M}$ of the module M is a complete lattice in $\mathbf{R}^n$ (cf. [1, Chapter II, §3]). We call a $\mathbf{Z}$-module basis of M *LLL-reduced*, if the corresponding lattice basis has this property.

(3.5) LEMMA. *Let $\alpha_1, \dots, \alpha_n$ be an LLL-reduced basis of M. Then we have*

$$C_1^{-(n-1)} N^{-1/n} \le |\alpha_i^{(j)}| \le C_1 N^{-1/n} \quad \textit{for } 1 \le i \le n,\ 1 \le j \le s+t,$$

with $N := |N_{K|\mathbf{Q}}(\gamma)|$ and $C_1 = (2^{(n+2)/2} n^{-1})^{(n-1)/2} \Delta$, where Δ is the volume of the fundamental parallelotope of the lattice $\underline{R}$.

Proof. First of all, note that the volume of the fundamental parallelotope of $\underline{M}$ is given by the formula

$$d(\underline{M}) = N^{-1} \Delta. \tag{3.6}$$

Now it follows from (3.2) that

$$\prod_{i=1}^{n} |\underline{\alpha}_i| \le 2^{n(n-1)/4} N^{-1} \Delta, \tag{3.7}$$

where $|\underline{\alpha}_i|^2 = \sum_{j=1}^{s+t} |\alpha_i^{(j)}|^2$ for $1 \le i \le n$.

On the other hand, we have for every $0 \ne \alpha \in M$,

$$|\underline{\alpha}| \ge (n/2)^{1/2} N^{-1/n}. \tag{3.8}$$

In fact, if $\alpha \in M$, then there is a number $\tilde{\alpha} \in R$ with $\alpha = \tilde{\alpha}/\gamma$ and

$$2|\underline{\alpha}|^2 \geq \sum_{j=1}^{s} |\alpha^{(j)}|^2 + 2 \sum_{j=s+1}^{s+t} |\alpha^{(j)}|^2 \geq n|N_{K|\mathbf{Q}}(\alpha)|^{2/n}.$$

The second inequality of (3.5) follows from (3.7) and (3.8). In order to prove the first inequality, note that

$$N^{-1} \leq |N_{K|\mathbf{Q}}(\alpha_i)| \leq |\alpha_i^{(j)}| C_1^{(n-1)} N^{-(n-1)/n} \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq s+t. \quad \square$$

(b) In view of (2.2), the second application of the LLL-algorithm yields a number $\beta \in M$ satisfying

$$|\beta^{(i)}| < 1, |\beta^{(j)}| > 1 \quad \text{for } j \neq i \text{ and } |N_{K|\mathbf{Q}}(\beta)| \leq CN^{-1} \tag{3.9}$$

for every conjugate direction $i \in \{1, \ldots, s+t\}$. The constant C does not depend on M but only on R.

For the rest of this section we fix a conjugate direction $i \in \{1, \ldots, s+t\}$, and we assume that $\alpha_1, \ldots, \alpha_n$ is an LLL-reduced basis of M. Moreover, the numbers C_k, $k \in \mathbf{N}$, always denote effective constants depending only on the degree n of K and on the volume Δ of the fundamental parallelotope of R. Every number $\beta \in M$ has a representation

$$\beta = \sum_{l=1}^{n} x_l \alpha_l \quad \text{with } x_l \in \mathbf{Z} \text{ for } 1 \leq l \leq n.$$

We compute β of (3.9) solving the following approximation problem:

$$\begin{aligned} |\beta^{(i)}|^{e_i} &< C_2 \kappa^{-(n-e_i)} N^{-e_i/n}, \\ |x_l| &< C_3 \kappa \quad \text{for } 1 \leq l \leq n, \end{aligned} \tag{3.10}$$

with $\kappa \geq 1$ and

$$e_i = \begin{cases} 1 & \text{if } 1 \leq i \leq s, \\ 2 & \text{if } s+1 \leq i \leq s+t. \end{cases}$$

(3.11) LEMMA. *If β satisfies* (3.10), *then we have*

$$C_4 \kappa N^{-1/n} \leq |\beta^{(j)}| \leq C_5 \kappa N^{-1/n} \quad \textit{for } j \neq i,$$
$$|N_{K|\mathbf{Q}}(\beta)| \leq C_6 N^{-1}.$$

Proof. Applying (3.5) and (3.10), we find

$$|\beta^{(j)}| = \left| \sum_{l=1}^{n} x_l \alpha_l^{(j)} \right| \leq C_5 \kappa N^{-1/n}. \tag{3.12}$$

By virtue of the fact that

$$N^{-1} \leq |N_{K|\mathbf{Q}}(\beta)| = \prod_{l=1}^{s} |\beta^{(l)}| \prod_{l=s+1}^{s+t} |\beta^{(l)}|^2,$$

the first inequality follows from (3.10) and (3.12). $\square$

Since the bound for the norm of β does not depend on the constant κ, we choose κ such that (3.9) is satisfied. The approximation problem (3.10) is solved by means of the LLL-algorithm.

First of all, let us assume that i is a real direction, i.e., $1 \le i \le s$.

Consider the matrix

$$U := \begin{bmatrix} 0 & 0 & 0 & \cdots & \delta \\ 0 & 0 & 0 & \cdots & 0 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & \delta & \cdots & 0 \\ 0 & \delta & 0 & \cdots & 0 \\ \alpha_1^{(i)} & \alpha_2^{(i)} & \alpha_3^{(i)} & \cdots & \alpha_n^{(i)} \end{bmatrix}, \tag{3.13}$$

where $\alpha_1, \ldots, \alpha_n$ is a L^3-reduced basis of M and

$$\delta := 2^{-n/4} |\alpha_1^{(i)}| \kappa^{-n}. \tag{3.14}$$

We apply the LLL-algorithm to the columns of U. The result is a matrix $\tilde{U}$ which we get from U by multiplication by a unimodular transformation matrix $T = (t_{ij})_{1 \le i,j \le n} \in \mathbf{Z}^{(n,n)}$. If we define

$$x_l := t_{li} \quad \text{for } 1 \le l \le n,$$
$$\beta := \sum_{l=1}^{n} x_l \alpha_l,$$

then β solves (3.10).

In fact, since the fundamental parallelotope of the lattice spanned by the columns of U is of volume

$$d(U) = |\alpha_1^{(i)}| \delta^{n-1} = 2^{-n(n-1)/4} |\alpha_1^{(i)}|^n \kappa^{-n(n-1)},$$

it follows from (3.3) and (3.5) that

$$\begin{aligned} |\beta^{(i)}| &\le |\alpha_1^{(i)}| \kappa^{-(n-1)} \le C_1 \kappa^{-(n-1)} N^{-1/n}, \\ |x_l| &\le 2^{n/4} \kappa \quad \text{for } 2 \le l \le n, \end{aligned} \tag{3.15}$$

and we are ready if we prove an upper bound for x_1. We get this upper bound if we divide

$$\begin{aligned} |x_1 \alpha_1^{(i)}| &= \left| \sum_{l=1}^{n} x_l \alpha_l^{(i)} - \sum_{l=2}^{n} x_l \alpha_l^{(i)} \right| \\ &\le |\beta^{(i)}| + (n-1) 2^{n/4} C_1 \kappa N^{-1/n} \end{aligned}$$

by $|\alpha_1^{(i)}|$. In fact, this yields

$$|x_1| \le C_7 \kappa N^{-1/n} / |\alpha_1^{(i)}| \le C_3 \kappa. \tag{3.16}$$

Now assume that i is a "complex direction", i.e., $s < i \leq s+t$. This time we apply the LLL-algorithm to the columns of

$$U = \begin{bmatrix} 0 & 0 & 0 & \cdots & \delta \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \delta & \cdots & 0 \\ \operatorname{Re}\alpha_1^{(i)} & \operatorname{Re}\alpha_2^{(i)} & \operatorname{Re}\alpha_3^{(i)} & \cdots & \operatorname{Re}\alpha_n^{(i)} \\ \operatorname{Im}\alpha_1^{(i)} & \operatorname{Im}\alpha_2^{(i)} & \operatorname{Im}\alpha_3^{(i)} & \cdots & \operatorname{Im}\alpha_n^{(i)} \end{bmatrix}, \tag{3.17}$$

where

$$\delta = 2^{-n(n+1)/(4(n-2))} D^{1/2} \kappa^{-n/2}, \tag{3.18}$$

with

$$D = |\operatorname{Re}\alpha_1^{(i)} \operatorname{Im}\alpha_2^{(i)} - \operatorname{Re}\alpha_2^{(i)} \operatorname{Im}\alpha_1^{(i)}|.$$

Let $(t_{ij})_{1\leq i,j\leq n} \in \mathbf{Z}^{(n,n)}$ be the unimodular matrix which transforms U into the corresponding reduced matrix. Then we again fix

$$\begin{aligned} x_l &= t_{l1} \quad \text{for } 1 \leq l \leq n, \\ \beta &:= \sum_{l=1}^{n} x_l \alpha_l, \end{aligned} \tag{3.19}$$

and we prove that β satisfies (3.10).

Obviously, it follows from (3.5) that

$$D \leq C_8 N^{-2/n}. \tag{3.20}$$

But we also need a lower bound for D, and this is given in

(3.21) Lemma. *For $l_1, l_2 \in \{1, \ldots, n\}$ set*

$$D_{l_1 l_2} := |\operatorname{Re}\alpha_{l_1}^{(i)} \operatorname{Im}\alpha_{l_2}^{(i)} - \operatorname{Re}\alpha_{l_2}^{(i)} \operatorname{Im}\alpha_{l_1}^{(i)}|.$$

Then there are numbers $l_1, l_2 \in \{1, \ldots, n\}$ such that

$$D_{l_1 l_2} \geq C_9 N^{-2/n}. \tag{3.22}$$

Proof. If $D^*_{l_1 l_2}$ for $l_1, l_2 \in \{1, \ldots, n\}$ denotes the absolute value of the adjoint determinant of $D_{l_1 l_2}$ in the matrix $(\underline{\alpha}_1, \ldots, \underline{\alpha}_n)$, then we get from (3.5), (3.6) and Laplace's formula

$$\begin{aligned} N^{-1}\Delta &\leq \sum_{1\leq l_1<l_2\leq n} D_{l_1 l_2} D^*_{l_1 l_2} \\ &\leq C_{10} N^{-(n-2)/n} \max_{1\leq l_1<l_2\leq n} D_{l_1 l_2}, \end{aligned}$$

and this proves the lemma. □

Without loss of generality we assume that (3.22) is true for $l_1 = 1$ and $l_2 = 2$. Then we have

$$D \geq C_{11} N^{-2/n}. \tag{3.23}$$

Notice that we have to renumber the basis at this point in order to get $D \neq 0$.

Now we are able to prove that β, defined in (3.19), satisfies (3.10). This time the fundamental parallelotope of the lattice spanned by the columns of U is of volume

$$d(U) = D \cdot \delta^{n-2} = 2^{-n(n+1)/4} \cdot D^{n/2} \cdot \kappa^{-n(n-2)/2},$$

and thus (3.3) and (3.20) yield

$$\begin{aligned} |\beta^{(i)}|^2 &\leq D \cdot \kappa^{-(n-2)} \leq C_2 \kappa^{-(n-2)} N^{-2/n}, \\ |x_l| &\leq 2^{(n^2+1)/(4(n-2))} \kappa \quad \text{for } 3 \leq l \leq n. \end{aligned} \tag{3.24}$$

An upper bound for x_1 and x_2 follows from

$$\begin{aligned} |x_1 \operatorname{Re} \alpha_1^{(i)} + x_2 \operatorname{Re} \alpha_2^{(i)}| &= \left| \operatorname{Re} \beta^{(i)} - \sum_{l=3}^{n} x_l \operatorname{Re} \alpha_l^{(i)} \right|, \\ |x_1 \operatorname{Im} \alpha_1^{(i)} + x_2 \operatorname{Im} \alpha_2^{(i)}| &= \left| \operatorname{Im} \beta^{(i)} - \sum_{l=3}^{n} x_l \operatorname{Im} \alpha_l^{(i)} \right|. \end{aligned}$$

Applying Cramer's rule, we get in view of (3.5), (3.23) and (3.24),

$$|x_l| \leq C_{12} \kappa N^{-2/n} D^{-1} \leq C_3 \kappa \quad \text{for } l = 1, 2. \quad \square \tag{3.25}$$

4. Computational Aspects of the Algorithm. Let $i \in \{1, \ldots, s+t\}$ be again a fixed conjugate direction. Before we give a detailed description of the algorithm, we give some preparatory explanations.

Assume that we know for a $k \in \mathbf{N}$ the number $N_k = |N_{K|\mathbf{Q}}(\gamma_k)|$ and an LLL-reduced basis $\alpha_1(k), \ldots, \alpha_n(k)$ of the module $R_k = R/\gamma_k$.

In order to compute the number β_k satisfying (2.5), we have to proceed as follows:

- choose κ,
- set δ according to (3.14) or (3.18),
- set U according to (3.13) or (3.17),
- apply the LLL-algorithm to the columns of U resulting in $\tilde{U} = U \cdot T$, with $T = (t_{l,j})_{1 \leq l,j \leq n} \in \mathbf{Z}^{(n,n)}$,
- set $x_l(k) \leftarrow t_{l1}$ for $1 \leq l \leq n$ and set $\beta_k \leftarrow \sum_{l=1}^{n} x_l \alpha_l(k)$.

But how to choose κ? To make sure that the algorithm yields a maximal system of independent units, we have to choose κ such that β_k satisfies (3.9). Since we know by (3.11) and (2.6) that

$$N_k \leq C_6, \tag{4.1}$$

this means

$$\kappa = \max\{C_4^{-1} C_6^{1/n}, C_2^{e_i/(n-e_i)}\} + \varepsilon \tag{4.2}$$

with an arbitrary small constant ε.

Now in almost all our examples it has turned out to be enough to choose κ such that only

$$|\beta_k^{(i)}| < 1, \tag{4.3}$$

in order to compute maximal systems of independent units. This condition is necessary to avoid trivial units. Recall that we have by (3.15) and (3.16), (3.24) and (3.25),

$$\begin{aligned} &|\beta_k^{(i)}|^{e_i} \le \lambda_i \kappa^{-(n-e_i)}, \\ &|x_l| \le C_{13}\kappa\lambda_i^{-1} \quad \text{for } 1 \le l \le e_i, \\ &|x_l| \le C_{14}\kappa \qquad \text{for } e_i < l \le n, \end{aligned} \tag{4.4}$$

with

$$\lambda_i = \begin{cases} |\alpha_1^{(i)}| & \text{for } i \le s, \\ |\operatorname{Re}\alpha_1^{(i)} \operatorname{Im}\alpha_2^{(i)} - \operatorname{Re}\alpha_2^{(i)} \operatorname{Im}\alpha_1^{(i)}| & \text{for } i > s. \end{cases} \tag{4.5}$$

Now on the one hand, we want to satisfy (4.3); on the other hand, we want to make the $|x_l|$ small in order to get units with small coefficients. Hence we have to choose κ such that

$$\lambda_i \kappa^{-(n-e_i)} = 1 - \varepsilon \tag{4.6}$$

with a small number ε, and this means that the bound for x_l, $1 \le l \le e_i$, increases if κ decreases, whereas for the bounds of x_l, $e_i < l \le n$, the contrary is true.

So the best thing to do is to renumber $\alpha_1(k), \dots, \alpha_n(k)$ such that $|\lambda_i - 1|$ is as small as possible and $D_{12} \ne 0$ if $i > s$, and then to fix

$$\kappa = \lambda_i^{1/(n-e_i)} + \varepsilon. \tag{4.7}$$

The next question we are going to discuss is the representation of the reduced basis $\alpha_1(k), \dots, \alpha_n(k)$ and of the number β_k.

Note that all the basis elements have a representation

$$\alpha_j(k) = \frac{1}{N_k} \sum_{l=1}^{n} a_{lj}(k)\alpha_l(1) \quad \text{for } 1 \le j \le n, \tag{4.8}$$

with

$$A_k := (a_{lj}(k))_{1 \le l,j \le n} \in \mathbf{Z}^{(n,n)}. \tag{4.9}$$

Since by (3.5) the conjugates of the $\alpha_j(k)$'s are—independent of k—all of the same small size, the same is true for the elements of A_k. Similarly, the number β_k is representable as

$$\beta_k = \frac{1}{N_k} \sum_{l=1}^{n} b_l(k)\alpha_l(1), \qquad b_l(k) \in \mathbf{Z}, \tag{4.10}$$

and because of (3.11) also the $b_l(k)$'s are small.

Finally, we explain how to decide whether the algorithm terminates, i.e., whether $\gamma_{k+1} = \gamma_k\beta_k$ is associated with a γ_Z, $Z \le k$.

By (2.7) we know that we have to check whether the corresponding modules are equal. A necessary condition is of course $N_{k+1} = N_Z$. If this condition is satisfied, then we have to test whether $A_Z \cdot A_{k+1}^{-1} \in \mathrm{GL}(n, \mathbf{Z})$. So we get:

(4.11) ALGORITHM.

Input: The conjugate direction $i \in \{1, \dots, s+t\}$.

Rational approximations to the conjugates of the elements of an LLL-reduced basis $\alpha_1, \dots, \alpha_n$ of the order R. A constant $\varepsilon > 0$.**

Output: The unit ε_i.

1. *Initialization.* $a_l(1) \leftarrow \alpha_l$ for $1 \le l \le n$,
$N_1 \leftarrow 1$,
$k \leftarrow 1$,

2. *Repeat.*

   a) Renumber $\alpha_1(k), \dots, \alpha_n(k)$ such that $|\lambda_i - 1|$ is minimal and $D_{12} \neq 0$ if $i > s$, cf. (4.6).
   b) $\kappa \leftarrow \lambda_i^{1/(n-e_i)} + \varepsilon$.
   c) Set δ according to (3.14) or (3.18) and U according to (3.13) or (3.17).
   d) Apply the LLL-algorithm to the columns of U. The corresponding unimodular transformation is $T = (t_{lj})_{1 \le l,j < n}$.
   e) Set $\beta_k \leftarrow \sum_{l=1}^{n} t_l \alpha_l(k)$, $N_{k+1} \leftarrow N_k |N_{K|\mathbf{Q}}(\beta_k)|$; compute the coefficients $b_l(k)$, $1 \le l \le n$ (cf. (4.10)).
   f) Compute an LLL-reduced basis $\alpha_1(k+1), \dots, \alpha_n(k+1)$ of the module $R_{k+1} = (1/\beta_k) R_k$, applying the LLL-algorithm to $\{\alpha_1(k)/\beta_k, \dots, \alpha_n(k)/\beta_k\}$. Compute the corresponding representation matrix A_k (cf. (4.8)).
   g) For $Z = 1$ until k:
   If $N_Z = N_{k+1}$ then
   if $A_Z \cdot A_{k+1}^{-1} \in \mathrm{GL}(n, \mathbf{Z})$ then set $\varepsilon_i \leftarrow \prod_{l=Z}^{k} \beta_l$.
   Return.
   h) $k \leftarrow k + 1$.

After we have applied this algorithm to every coordinate direction, we know a set $\{\varepsilon_1, \dots, \varepsilon_{s+t}\}$ of nontrivial units. If this set does not contain a subset of $s+t-1$ independent units, we apply our algorithm again, but with a bigger κ in 2b).

Tables (see the supplements section at the end of this issue). By the method described, we have computed maximal systems of independent units in the order $\mathbf{Z}[\rho]$ of the field $\mathbf{Q}(\rho)$, where $\rho = \sqrt[n]{D}$ for $6 \le n \le 11$. For $n \le 5$ and $n = 6$, $D < 0$, there are efficient methods (cf. [3]) for computing fundamental units; therefore we have omitted these cases. In the tables we use D and n in the above sense. Moreover, we denote by

P : $\max_{i \in \{1, \dots, s+t\}} \{\text{number of iterations in direction } i\}$,
R : regulator of the system,
$x_1, \dots, x_n$: the coefficients of the units in the basis $1, \rho, \dots, \rho^{n-1}$.

**It is possible to determine the necessary precision of approximation theoretically, but this theoretical value could hardly be realized. In our computation, double-precision floating-point arithmetic (26 decimal digits) was always sufficient.

For the sake of readability of the tables we decided not to list the coefficients to more than 8 decimal digits. All the computations were carried out on the CYBER 76 of the University of Cologne. The computation of the units of each field took at most a few CPU-seconds.

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Supplement to Computation of Independent Units in Number Fields by Dirichlet's Method

By Johannes Buchmann and Attila Pethő

TABLE 1

n = 6

D	P	R	X_1	X_2	X_3	X_4	X_5	X_6
2	2	8.07	1	1	0	0	0	0
			1	1	1	1	1	1
			-1	0	0	1	0	0
3	3	77.99	-2	0	0	1	0	0
			-1	-1	1	-1	0	1
			1	-1	-1	-1	0	1
5	3	105.92	2	0	0	1	0	0
			-2	1	0	-1	1	0
			-2	-1	0	1	1	0
6	9	666.30	1	0	-3	0	0	1
			1	0	-3	0	0	-1
			-53	-36	0	18	12	-2
7	10	1140.68	44	0	23	0	12	0
			-3	3	11	11	6	1
			-4593	-3331	-2405	-1736	-1259	-909
10	17	2037.05	-181	0	-84	0	-39	0
			-101683	69276	-47197	32155	-21907	14925
			181	82	-21	-50	-21	10
11	10	4269.68	-1475	988	-662	444	-298	200
			-175	0	52	0	12	0
			353330	-237006	158800	-106533	71460	-47880
12	22	5314.04	7	0	0	2	0	0
			-78961825	-52185932	-34489736	-22794312	-15064792	-9956338
			-38135	-38758	41756	-10528	-10963	11993

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TABLE 1 (*continued*)

n = 6

D	P	R	X_1	X_2	X_3	X_4	X_5	X_6
13	4	747.04	34	22	14	9	6	4
			-4	0	-3	0	2	0
			22	-25	29	-6	7	-8
14	34	3445.52	-29	0	-12	0	-5	0
			-2199	1424	-919	590	-378	243
			-403201	0	0	107760	0	0
15	18	11794.33	31	0	0	-8	0	0
			-290524	19422	105372	-74971	5040	27206
			-290524	19422	105372	74971	5040	-27206
17	18	9939.50	-33	0	0	-6	0	0
						COEFFICIENTS > 1.E8		
			-465121	-461148	468540	-112824	-111870	113628
18	3	275.76	-1	-1	1	0	0	0
			-1	1	1	0	0	0
			17	0	0	4	0	0
19	16	13960.50	-12304	0	-4611	0	-1728	0
			-13216	8116	-4965	3028	-1848	1132
			-19651490	0	0	4508361	0	0
20	12	1880.76	-361	0	-133	0	-49	0
			-45741	-27763	-16851	-10228	-6208	-3768
			-11	-53	-21	6	14	6
21	29	8285.91	-1705	0	-618	0	-224	0
			5149	3096	1851	1108	668	405
			-665335	0	0	145188	0	0
22	23	14442.96	-793	0	-283	0	-101	0
						COEFFICIENTS > 1.E8		
			30580901	0	0	-6519870	0	0
23	29	8233.13	-24	0	0	5	0	0
			14332267	8498816	5039686	2988480	1772138	1050856
			-373912	106947	66785	-77827	22749	14143

TABLE 2

n = 7

D P	R	X_1	X_2	X_3	X_4	X_5	X_6	X_7
2 5	53.57	1	1	1	1	1	1	1
		-1	-2	1	1	-1	0	1
		-1	0	-2	-1	0	-1	-1
3 7	116.79	1	1	0	0	1	0	0
		10	9	8	7	6	5	4
		7	5	5	4	3	3	2
4 12	214.27	-13	-11	-9	-7	-6	-5	-4
		-1	-1	2	0	-2	0	1
		-3	7	-1	-4	2	2	-2
5 20	591.07	1	1	2	1	-1	-1	0
		4	5	-3	-1	-3	0	1
		11	-8	-3	5	1	-4	1
6 19	881.17	-11	22	-1	-13	5	6	-5
		-19	-11	-10	-9	-6	-4	-4
		-1	-1	-3	-5	-5	-3	-1
7 41	21045.34	-587	490	28	-375	308	-110	49
		1469	-329	35	250	-504	347	-289
		-2738021	-3554159	-1786064	351695	1356453	1079259	241250
8 18	857.09	-25	-19	-13	-11	-7	-6	-4
		1689	1255	932	692	514	382	284
		57	45	47	29	16	17	14
9 72	6306.77	-19	11	-13	3	0	0	3
		712	-428	184	-14	-80	113	-106
		-1873	2041	2859	1512	-147	-940	-779
10 32	1748.37	-9	5	-2	1	1	-1	1
		61	46	37	29	21	14	8
		-1063861	-765644	-551025	-396563	-285402	-205399	-147623
11 51	5989.02	-16644	-4470	9722	-507	-4424	1776	1618
		10	-27	4	12	-6	-4	4
				COEFFICIENTS > 1.E8				
12 39	7406.40	-2327	135	1290	1045	271	-266	-365
		-12239	138039	-74750	-75562	118056	-57373	4362
		-8735	5149	-2165	202	786	-1111	1013

TABLE 2 (*continued*)

13 24	1370.92	1	0	1	2	1	0	0
		12	-1	-6	-6	-1	1	2
		157	595	-259	-206	188	41	-103
14 90	378243.8029			COEFFICIENTS	> 1.E8			
				COEFFICIENTS	> 1.E8			
				COEFFICIENTS	> 1.E8			
15 95	16480.76	-61	-5	-8	29	2	5	-5
		-629	-1412	722	439	-463	-63	231
		-97379	-71262	-7890	31313	33639	16409	-15
16 89	13713.42	-15	70	-3	-28	7	7	-7
		-4143	3310	-2103	1036	-315	-77	238
		-63617	-45262	-28977	-18634	-13545	-9156	-5755

TABLE 3

n = 8

D	P	R	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
-2	5	300.06	-1	0	-2	-2	0	0	0	1
			1	0	2	-2	0	0	0	1
			1	0	2	0	0	0	0	1
-3	5	385.26	-4	-1	2	1	-1	-1	0	1
			4	-1	-2	1	1	-1	0	1
			-2	1	2	1	-1	-1	0	1
-5	12	4522.72	2	2	-2	0	1	-2	0	0
			-16298	30522	27664	-3440	-18297	-10968	5410	11572
			-2	2	2	0	-1	-2	0	0
-6	28	16952.36	-575	572	-294	-266	312	-114	-220	203
			-95	-76	-70	-30	-8	2	12	21
			95	-76	70	-30	8	2	-12	21
-7	18	1217.82	-6	-3	2	2	0	-1	-1	0
			6	-3	-2	2	0	-1	1	0
			34	-1	26	-3	9	-3	-2	-1
-8	9	2400.45	1	0	-2	-4	-2	1	1	0
			1	0	-2	4	-2	-1	1	0
			-1	4	-2	-4	2	1	-3	2
-9	27	16014.99	435	0	198	0	0	0	-66	0
			-53	30	9	-20	12	0	-1	-4
			2000539	-32060362	43857521	-43065604	35142668	-24476132	14074695	-5629476
-10	18	8330.27	1	16	-6	8	-4	-2	0	-3
			-173	-216	192	26	-127	56	45	-64
			1	-16	-6	-8	-4	2	0	3
-11	46	46376.61	2078	-24603	23604	-3645	-19719	9693	7854	-6591
			1	201	156	-15	72	33	-24	-9
			1	-201	156	15	72	-33	-24	9
-12	26	15649.07	25	-10	30	-44	36	-15	-1	4
			1585	-266	-1176	-424	456	377	12	-178
			-25	-10	-30	-44	-36	-15	1	4
-13	52	32271.41	1221	316	-830	290	276	-304	26	148
			-1221	316	830	290	-276	-304	-26	148
			473650	421212	317434	202828	103251	31172	-11896	-31842

TABLE 3 (*continued*)

-14	32	47055.93	7143361	8494380	981576	-3851200	-2626784	545434	1658132	630504
			-7143361	8494380	-981576	-3851200	2626784	545434	-1658132	630504
			-36770047	-29676652	-20418088	-11784224	-5100448	-684204	1727888	2649312
-15	19	1591.59	4	3	-5	5	-4	3	-2	1
			4	-3	-5	-5	-4	-3	-2	-1
			-46	9	12	-21	21	-17	12	-7
-16	23	40006.37	-7361	4760	-2540	952	0	-476	635	-595
			7361	1904	-2540	-2380	0	1190	635	-238
					COEFFICIENTS > 1.E8					
-17	30	40341.69	13072	0	-8861	0	3001	0	59	0
			-931	-516	-251	-24	101	131	124	93
			817240	7758303	3764591	-1798801	-2820137	-628853	1051095	874251

TABLE 4

n = 8

D	P	R	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
2	2	49.17	-1	1	0	0	0	0	0	0
			1	1	0	0	0	0	0	0
			1	0	1	0	0	1	0	0
			1	0	1	0	0	0	0	0
3	8	6455.46	2	-14	19	-6	-9	10	-5	2
			-2	-14	-19	-6	9	10	5	2
			4	-4	1	1	-1	1	-1	0
			1	0	-1	0	0	0	1	0
5	8	10515.62	-6	-7	16	-5	-5	5	-2	1
			126	22	28	66	-3	-82	-54	14
			-126	22	-28	66	3	-82	54	14
			3	0	2	0	0	0	0	0
6	22	26423.50	5	-32	16	8	-6	-3	-4	7
			-721	4	330	4	-120	96	130	-64
			721	4	-330	4	120	96	-130	-64
			293	0	260	0	-110	0	-100	0
7	20	117945.70	-3032	-790	1448	1832	1147	298	-548	-692
			-3032	790	1448	-1832	1147	-298	-548	692
			-27	8	12	-10	1	0	1	2
			-1	0	1	0	1	0	0	0
8	9	3147.02	-3	0	0	0	1	0	0	0
			-65	52	-40	30	-24	18	-14	11
			1	-2	0	1	0	0	1	1
			-1	-2	0	1	0	0	-1	1
10	60	371303.57	COEFFICIENTS > 1.E8							
			-319843	84510	201568	-15722	-91831	13779	44776	-17805
			319843	84510	-201568	-15722	91831	13779	-44776	-17805
			-1	0	-14	0	-4	0	2	0
11	27	136464.85	10	-3	-8	2	5	-2	-2	1
			-10	-3	8	2	-5	-2	2	1
			80884	65099	-45965	-33870	24127	18734	-12779	-10853
			98	0	-81	0	-3	0	39	0
12	6	25368.50	-1	-2	-4	0	0	1	1	0
			-1	2	-4	0	0	-1	1	0
			1	-8	8	-4	0	2	-2	1
			-7	0	-8	0	-6	0	-2	0
13	87	3845635.63	1119	490	-206	-382	-312	-136	58	106
			1119	-490	-206	382	-312	136	58	-106
			COEFFICIENTS > 1.E8							
			298	0	306	0	-81	0	-84	0
14	7	100723.38	-15	-12	0	14	0	-8	2	2
			15	32	64	80	68	40	16	4
			15	-32	64	-80	68	-40	16	-4
			15	0	16	0	12	0	4	0

TABLE 5

n = 9

D	P	R	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
2	5	165.95	-1	1	0	0	0	0	0	0	0
			-1	0	0	1	0	0	0	0	0
			1	-1	-1	1	1	0	-1	0	1
			1	0	0	1	1	0	-1	-1	0
3	11	1056.70	2	0	1	-1	0	-1	0	0	0
			-1	0	-1	-2	-1	0	0	-1	-1
			4	3	1	-1	-2	-2	-1	0	1
			-13	-6	8	7	-4	-7	1	6	1
4	11	1327.61	-17	-15	-13	-11	-9	-8	-7	-6	-5
			1	0	-1	1	0	0	0	0	0
			5	0	0	3	0	0	2	0	0
			-7	3	1	-3	5	-5	5	-4	3
5	27	182467.79	-1	0	0	4	0	0	-2	0	0
			-51	17	-4	-21	24	-27	28	-20	17
			-9	63	-56	34	55	-34	5	13	-32
			199	-884	686	-670	865	-641	285	-233	192
6	38	154638.34	1	0	0	-6	0	0	3	0	0
			-5	9	3	-6	-2	3	2	0	-2
			-1	-24	-15	-12	6	3	9	7	-1
			-13238993	18838326	32542473	28206786	13560254	-1917096	-11513318	-13167747	-8800581

7	16	10421.08	-2	0	0	1	0	0	0	0	0
			5	-3	-1	-1	-3	0	1	0	1
			-26	0	19	7	-9	-6	5	6	-1
			18	36	6	-38	3	11	12	-12	-5
9	12	9511.00	-2	0	0	1	0	0	0	0	0
			-28	24	-18	12	-7	3	0	-2	3
			28	39	30	12	-4	-12	-12	-7	-1
			1	24	-15	0	9	-9	0	5	-4
10	56	322546.87	-1	5	-7	0	2	-1	0	2	-1
			1	0	0	6	0	0	-3	0	0
			3529	-3791	3407	-2686	1859	-1097	488	-51	-222
			10500761	-12251295	-9592912	4759591	7028411	-967272	-4474559	-621305	2516329
11	67	1606649.22	-1	0	0	-4	0	0	2	0	0
			-21	-34	27	5	-16	10	-2	-5	7
					COEFFICIENTS >	1.E8					
					COEFFICIENTS >	1.E8					
12	58	229134.44	-323	-105	66	141	126	63	0	-35	-40
			61	123	-138	18	55	-57	13	27	-24
			107	0	0	-33	0	0	-6	0	0
			2963	1452	-1326	-1182	449	801	-49	-473	-97

TABLE 6

n = 10

D	P	R	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
-2	7	10268.18	-1	-5	-5	1	5	2	-1	-2	-3	-2
			1	0	1	0	0	0	0	0	0	0
			1	-5	5	1	-5	2	1	-2	3	-2
			-5	6	9	2	-7	6	8	-4	-4	2
-3	14	744293.19	-180532	-159775	83070	150133	-7182	-60725	15651	3786	-57494	-21169
			1	0	-2	0	-1	0	1	0	0	0
			-2	144	-84	39	38	-57	47	-30	-27	11
			2	144	84	39	-38	-57	-47	-30	27	11
-4	9	45232.36	-1	6	-1	-2	2	0	0	0	-1	1
			1	6	1	-2	-2	0	0	0	1	1
			-231	-19	-11	-39	97	60	43	107	48	36
			231	-19	11	-39	-97	60	-43	107	-48	36
-5	33	351452.45	1	0	-5	-10	-5	0	-2	-2	0	0
			1	0	0	0	0	0	-1	0	-1	0
			1	0	-5	10	-5	0	-2	2	0	0
			-5863401	74626500	3580705	-55000445	-1549720	38869305	633629	-27697493	-1245674	19974611

-6 34 10397094.11	-9926569	-6761011	-185929	3912968	3528759	783692	-1468612	-1942930	-789117	870067
	1	0	-60	0	20	0	15	0	-20	0
	26191	15569	4313	-979	-8398	-9185	-10885	-10190	-8369	-6743
	-26191	15569	-4313	-979	8398	-9185	10885	-10190	8369	-6743
-7 52 16096856.94	-28167	-14695	7640	5742	9190	3856	-7425	-4780	-2112	-470
	1	0	-8	0	-3	0	-1	0	-2	0
	-28167	14695	7640	-5742	9190	-3856	-7425	4780	-2112	470
			COEFFICIENTS > 1.E8							
-8 21 2409599.67	1	0	-5	0	-3	0	2	0	1	0
	-22244921	14926038	14714445	-9869620	-9688762	6505490	6391101	-4290501	-4223545	2838260
	-425	162	-41	-164	29	0	54	34	-32	49
	-425	-162	-41	164	29	0	54	-34	-32	-49

TABLE 7

n = 10

D	P	R	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_8	X_{10}
2	5	6359.86	-1	1	0	0	0	0	0	0	0	0
			1	1	0	0	0	0	0	0	0	0
			3	-1	-2	1	1	-2	1	1	-2	0
			-1	-1	5	1	3	2	2	0	-1	1
			-3	-1	2	1	-1	-2	-1	1	2	0
3	8	564471.86	-1	1	-1	-2	4	0	0	-1	-1	2
			-1	-2	2	-1	-1	0	-1	1	-1	-1
			-478	683	-628	245	57	-83	234	-433	313	-9
			-478	-683	-628	-245	57	83	234	433	313	9
			1	-2	-2	-1	1	0	1	1	1	-1
5	17	46674845.69	-9	24	-37	32	-4	-22	23	-10	2	0
			9	24	37	32	4	-22	-23	-10	-2	0
			-2643	903	-719	899	2287	111	617	-971	-642	51
			185488	255685	14400	-187240	-102590	84098	105850	10080	-79494	-50750
			-2643	-903	-719	-899	2287	-111	617	971	-642	-51

6 22 286112723.68	-13	20	33	-3	-16	-5	8	10	-2	-5
			COEFFICIENTS > 1.E8							
	7279	-4959	-2883	4381	4926	-5781	-2178	2956	1485	-2162
			COEFFICIENTS > 1.E8							
	-7279	-4959	2883	4381	-4926	-5781	2178	-2956	-1485	-2162
7 54 315740274.75	13	-6	-5	7	1	-2	-1	3	-1	-2
	-13	-6	5	7	-1	-2	1	3	1	-2
	-65038	-23485	65130	-49179	46686	-22339	-9646	22072	-21456	15550
			COEFFICIENTS > 1.E8							
	-65038	23485	65130	49179	46686	22339	-9646	-22072	-21456	-15550
8 27 9117494.77	-17	4	19	8	-9	-10	-2	5	3	-2
	-11	18	11	-6	-13	-3	4	7	-2	-2
	18113939	-3412416	-6814162	4944916	-1051381	-3694959	4249980	1514683	-3821396	-164516
	-11	-18	11	6	-13	3	4	-7	-2	2
	-18113939	-3412416	6814162	4944916	1051381	-3694959	-4249980	1514683	3821396	-164516

TABLE 8

n = 11

D	P	R	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}
2	9	6602.07	1	1	0	0	0	0	0	0	0	0	0
			-1	1	1	-1	0	1	0	0	0	0	0
			-3	-4	5	1	5	0	-1	2	-2	2	0
			1	2	0	-1	-1	-2	0	1	1	1	0
			-1	0	0	-1	0	0	-1	0	1	0	0
3	14	48549.91	1	6	-4	-3	8	9	-1	-6	1	9	7
			-130	-112	131	115	-122	-70	63	53	-11	-52	-16
			-1	1	0	0	0	0	1	0	0	0	0
			4	3	0	0	-2	-3	-3	-4	-3	-3	-2
			1	-2	-1	0	-2	-1	1	0	0	1	0
4	39	422532.74	-11	22	-9	-10	15	-4	-8	8	1	-7	4
			-5	3	1	2	-1	-3	2	3	-3	-2	2
			-5	0	0	3	2	0	-1	-3	-1	0	1
			135	-913	545	404	-825	384	304	-518	169	222	-246
			5	-2	-1	1	1	2	0	-2	1	2	-1
5	39	948073.98	-4	5	1	-1	-1	0	2	-1	-1	0	1
			-4	2	-3	0	0	-1	1	-1	1	0	0
			4	-76	69	14	-53	27	10	-30	9	20	-17
			-569	-306	-96	131	313	378	367	357	345	291	198
			-1114	2036	1862	-102	1402	1510	-126	470	769	-350	-170