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Comment on Calderón's Paper: "On an Inverse Boundary Value Problem"*

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Dedicated to Professor Eugene Isaacson on the occasion of his 70th birthday

Abstract. Calderón determined a method to approximate the conductivity σ of a conducting body in R^n (for $n \geq 2$) based on measurements of boundary data. The approximation is good in the L_∞ norm provided that the conductivity is a small perturbation from a constant. We calculate the approximation exactly for the case of homogeneous concentric conducting disks in R^2 with different conductivities. Here, the difference in the conductivities is the perturbation. We show that the approximation yields precise information about the spatial variation of σ, even when the perturbation is large. This ability to distinguish spatial regions with different conductivities is important for clinical monitoring applications.

1. Introduction. In an elegant short paper [2], Calderón discussed the problem of determining an approximation to the electrical conductivity σ inside a bounded domain B in R^n, for $n \geq 2$, from electrical measurements made on the surface S of B.

Calderón gave an explicit method for finding an approximation to σ from data measured on S when $\sigma = 1 + \eta$ and $\|\eta\|_\infty$ is small. He proved that the method yields an approximation to a smoothed version of η whose error is $O(\|\eta\|_\infty^\alpha)$ for some α with $1 < \alpha < 2$.

We show for a specific example that Calderón's method can yield precise spatial information about σ when $\|\eta\|_\infty$ is not small, even though the method does not yield a good approximation to σ in any L_p norm.

This specific example is of interest in the design and evaluation of electrical impedance imaging systems [1], [9]. These systems have made qualitative images of conductivity changes inside living bodies using algorithms, some of which [11] are in theory, if not in practice, closely related to Calderón's.

In clinical monitoring applications it is not always necessary to display reconstructions that are accurate in L_p norms. One may want only to distinguish two tissues clearly, or one might be interested only in how much blood or gas passes through some local region in a given time. For such applications, it is sufficient to provide the observer with rapid reconstructions that distinguish regions whose properties, such as conductivity, may differ greatly. A simple linearization, such as Calderón's, is fast and, as seen in this example, sometimes may satisfy the requirement above.

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The reader interested in learning more about the theory and algorithms that have been applied to this problem can profitably consult [3], [6], [7], [8], [10], [12], [13], [14].

In Section 2 we derive Calderón's approximation formula. The approximation formula can be evaluated explicitly for the case of homogeneous concentric disks in R^2 with different conductivities. We exhibit this formula in Section 3 and show that the approximation gives the exact spatial variation of the conductivity. The details of the calculations are presented in Section 4.

2. Calderón's Method. We present a slightly different formulation of Calderón's method for determining an approximation to the conductivity σ inside a body B from measurements on its surface S. Here, σ is a small perturbation from a constant conductivity on B: $\sigma = 1 + \eta$, where η is small. We assume also that η is zero on a neighborhood of S.

Let $u(p)$ denote the electrical potential at a point p inside B and $j(p)$ the current density applied at the point p on the surface S. Then

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0 \quad \text{in } B, \\ \frac{\partial u}{\partial \nu} &= j \quad \text{on } S, \end{aligned}$$

where ν denotes the outward unit normal to S. The current density j must satisfy the compatibility condition

$$\int_S j(p)\, dS = 0.$$

Also, since u is defined only up to an additive constant, we assume that it is normalized by

$$\int_S u(p)\, dS = 0.$$

The inverse boundary value problem is then to determine σ from the boundary mapping R that takes applied current densities j into measured voltages $u = u|_S$. Physically, the inverse problem can be stated as follows. Apply all possible currents to S and measure all resulting voltages on S. From this boundary data, determine σ in B.

If η is small, then $u(p) = u_0(p) + O(\eta)$, where u_0 satisfies

$$\begin{aligned} \nabla^2 u_0 &= 0 \quad \text{in } B, \\ \frac{\partial u_0}{\partial \nu} &= j \quad \text{on } S. \end{aligned}$$

For any $\xi \in R^n$, introduce the auxiliary potential $v(p) = \exp(-i\bar{z} \cdot p)$, where $p \in R^n$ and $z = \frac{1}{2}(\xi + i\xi^\perp)$, $\bar{z} = \frac{1}{2}(\xi - i\xi^\perp)$. Here, $\xi^\perp$ denotes any vector in R^n that is perpendicular to ξ and has the same length; that is,

$$\xi \cdot \xi = |\xi|^2 = \xi^\perp \cdot \xi^\perp \quad \text{and} \quad \xi \cdot \xi^\perp = 0.$$

If we take $j(p) = -iz \cdot \nu \exp(-iz \cdot p)$, then

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0, \\ \nabla^2 v &= 0 \end{aligned}$$

in B, and

$$\frac{\partial u}{\partial \nu} = -iz \cdot \nu \exp(-iz \cdot p), \qquad \frac{\partial v}{\partial \nu} = -i\bar{z} \cdot \nu \exp(-i\bar{z} \cdot p)$$

on S. Integration by parts and the divergence theorem yield

$$\int_B \eta \nabla u \cdot \nabla v \, dp = \int_S \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) dS.$$

The right side of this equation can be determined from surface measurements alone; we denote it by $C(\xi; \eta)$. Then

$$C(\xi; \eta) \equiv \int_S \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) dS = \int_B \eta \nabla u \cdot \nabla v \, dp. \tag{2.1}$$

If η is small, then $u(p) = \exp(-iz \cdot p) + O(\eta)$, so that

$$C(\xi; \eta) = \frac{1}{2} \int_B \eta(p)(-|\xi|^2) \exp(-i\xi \cdot p) \, dp + O(\eta^2).$$

Calderón's method is the following. Neglect the $O(\eta^2)$ term to obtain the approximation

$$\int_B \eta(p) \exp(-i\xi \cdot p) \, dp \approx -\frac{2}{|\xi|^2} C(\xi; \eta).$$

Since η is assumed to be zero in a neighborhood of S, it can be extended smoothly to be zero everywhere outside of B. This leads to an approximate formula for the Fourier transform of η:

$$\begin{aligned} \hat{\eta}(\xi) &\equiv \int_{R^n} \eta(p) \exp(-i\xi \cdot p) \, dp \\ &= \int_B \eta(p) \exp(-i\xi \cdot p) \, dp \approx -\frac{2}{|\xi|^2} C(\xi; \eta). \end{aligned}$$

Finally, we obtain the reconstruction formula for the approximation γ to η as an inverse Fourier transform:

$$\begin{aligned} \gamma(p) &\equiv \frac{1}{(2\pi)^n} \int_{R^n} \left(-\frac{2}{|\xi|^2} \right) C(\xi; \eta) \exp(i\xi \cdot p) \, d\xi \\ &\approx \frac{1}{(2\pi)^n} \int_{R^n} \hat{\eta}(\xi) \exp(i\xi \cdot p) \, d\xi = \eta(p). \end{aligned}$$

There is no reason to believe that this formula yields a "good" approximation to η when $\|\eta\|_\infty$ is large. In the next section we show exactly what it yields for simple examples, including those for which $\eta \equiv \infty$, i.e., a perfect conductor.

3. A Simple Example. Let D denote the unit disk in R^2, and let S denote its boundary, the unit circle. For $a \geq -1$ and $0 < \Gamma < 1$, define the perturbation η by

$$\eta(p) = \begin{cases} a, & \text{for } |p| \leq \Gamma, \\ 0, & \text{for } \Gamma < |p| < 1. \end{cases}$$

The body then consists of two homogeneous concentric disks: the inner disk of radius Γ has conductivity $1 + a$, and outside this disk the conductivity is 1.

We compute explicitly both $C(\xi; \eta)$ and the Calderón approximation γ to η defined by

$$\gamma(p) = \frac{1}{(2\pi)^2} \int_{R^2} \left(-\frac{2}{|\xi|^2} \right) C(\xi; \eta) \exp(i\xi \cdot p) \, d\xi.$$

In particular, we show that

$$C(\xi;\eta) = 4\pi\mu \sum_{n=1}^{\infty} \frac{n}{1+\mu\Gamma^{2n}} \frac{(-\frac{1}{4}\Gamma^2|\xi|^2)^n}{(n!)^2} \tag{3.1}$$

and

$$\gamma(p) = 2\mu \sum_{m=0}^{\infty} (-\mu)^m \chi_{\Gamma^{m+1}}(p) \tag{3.2}$$

where

$$\mu = \frac{a}{2+a}$$

and χ_b denotes the characteristic function of the disk of radius b. We note that $-1 \le a \le \infty$ implies that $-1 \le \mu \le 1$.

The expression (3.2) for γ is easy to interpret: for any nonzero p, γ is a finite sum of characteristic functions of disks whose radii decrease geometrically. The first term is (a multiple of) the characteristic function of the inner disk $|p| \le \Gamma$ on which η is nonzero. Consequently, γ vanishes where η does, and the spatial variation of η is captured exactly by γ.

For $k \ge 0$ and $\Gamma^{k+2} < |p| < \Gamma^{k+1}$, we have

$$\begin{aligned} \gamma(p) &= 2\mu \sum_{m=0}^{k} (-\mu)^m = 2\mu \frac{1-(-\mu)^{k+1}}{1+\mu} \\ &= \frac{a}{1+a}\left(1 - \left(\frac{-a}{2+a}\right)^{k+1}\right). \end{aligned}$$

Therefore, $\gamma = \eta + O(a^2)$ when $|a|$ is small.

On the other hand, when a is large, say $a = \infty$ (so that $\mu = 1$), then

$$\gamma(p) = 2\chi_{\Gamma}(p) - 2\chi_{\Gamma^2}(p) + - \cdots .$$

Thus, γ alternates between the values 0 and 2 in the rings $\Gamma^{k+1} < |p| < \Gamma^k$ for $k \ge 0$. We leave to the reader to decide whether a graphic display of a smoothed approximation to this γ could be used to distinguish a highly conductive region from its background.

4. An Explicit Calculation. Here we outline the calculation yielding Eqs. (3.1) and (3.2).

Using polar coordinates $p = r(\cos\theta, \sin\theta)$ on R^2, we can obtain Fourier series expansions for the normal derivatives on the boundary of the unit disk D:

$$\begin{aligned} \frac{\partial u}{\partial \nu} &= -iz\cdot\nu \exp(-iz\cdot p) = \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta), \\ \frac{\partial v}{\partial \nu} &= -i\bar{z}\cdot\nu \exp(-i\bar{z}\cdot p) = \sum_{n=1}^{\infty}(a_n' \cos n\theta + b_n' \sin n\theta). \end{aligned}$$

By solving the corresponding Neumann problems for

$$\begin{aligned} \nabla\cdot(\sigma\nabla u) &= 0, \\ \nabla^2 v &= 0, \end{aligned}$$

we obtain

$$u(1,\theta) = \sum_{n=1}^{\infty} \rho_n (a_n \cos n\theta + b_n \sin n\theta),$$
$$v(1,\theta) = \sum_{n=1}^{\infty} \rho_n^0 (a_n' \cos n\theta + b_n' \sin n\theta),$$

where

$$\rho_n = \frac{1}{n}\left(\frac{1-\mu\Gamma^{2n}}{1+\mu\Gamma^{2n}}\right), \qquad \rho_n^0 = \frac{1}{n}.$$

Hence, for $n = 1, 2, \ldots,$

$$\begin{aligned} a_n &= \frac{1}{\rho_n^0 \pi}\int_{-\pi}^{+\pi} \exp(-iz\cdot p)\cos n\theta\, d\theta, \\ a_n' &= \frac{1}{\rho_n^0 \pi}\int_{-\pi}^{+\pi} \exp(-i\bar{z}\cdot p)\cos n\theta\, d\theta, \end{aligned} \tag{4.1}$$

and replacing $\cos n\theta$ by $\sin n\theta$ yields the formulas for b_n and b_n'.

The orthogonality of the trigonometric functions yields

$$C(\xi;\eta) = \pi \sum_{n=1}^{\infty} \delta_n (a_n a_n' + b_n b_n'),$$

where

$$\delta_n \equiv \rho_n^0 - \rho_n = \frac{2}{n}\left(\frac{\mu\Gamma^{2n}}{1+\mu\Gamma^{2n}}\right).$$

It follows from Eq. (4.1) that if $\xi = |\xi|(\cos\alpha, \sin\alpha)$, then

$$\begin{aligned} a_n &= \frac{1}{\rho_n^0}\exp(-in\alpha)\frac{(-\frac{1}{2}i|\xi|)^n}{n!}, \\ b_n &= ia_n, \\ a_n' &= \frac{1}{\rho_n^0}\exp(+in\alpha)\frac{(-\frac{1}{2}i|\xi|)^n}{n!}, \\ b_n' &= -ia_n'. \end{aligned}$$

Thus,

$$C(\xi;\eta) = 4\pi\mu \sum_{n=1}^{\infty} \frac{n}{1+\mu\Gamma^{2n}} \frac{(-\frac{1}{4}\Gamma^2|\xi|^2)^n}{(n!)^2}.$$

Since $0 < \Gamma < 1$ and $|\mu| \le 1$, we can expand the factor $(1+\mu\Gamma^{2n})^{-1}$ in a geometric series and interchange the order of summation to obtain

$$C(\xi;\eta) = 4\pi\mu \sum_{m=0}^{\infty} (-\mu)^m \sum_{n=1}^{\infty} \frac{n}{(n!)^2}\left(-\frac{\Gamma^{2m+2}|\xi|^2}{4}\right)^n. \tag{4.2}$$

Before evaluating the inverse Fourier transform, we observe that

$$\begin{aligned}
\hat{\chi}_b(\xi) &\equiv \int_{R^2} \chi_b(p)\exp(-i\xi\cdot p)\,dp \\
&= \int_0^b \int_{-\pi}^{+\pi} \exp(-i|\xi|r\cos(\theta-\alpha))r\,d\theta\,dr \\
&= \int_0^b \int_{-\pi}^{+\pi} \exp(-i|\xi|r\cos\theta)r\,d\theta\,dr \\
&= \sum_{k=0}^{\infty} \frac{(-i|\xi|)^k}{k!} \int_0^b r^{k+1}\,dr \int_{-\pi}^{+\pi} \cos^k\theta\,d\theta \\
&= \sum_{k=0}^{\infty} \frac{(-i|\xi|)^k}{k!} \frac{b^{k+2}}{k+2} \begin{cases} 0, & \text{if } k \text{ is odd,} \\ 2\pi \binom{k}{k/2} 2^{-k}, & \text{if } k \text{ is even,} \end{cases} \\
&= 2\pi \sum_{m=0}^{\infty} \left(-\frac{b^2|\xi|^2}{4}\right)^m \frac{b^2}{2} \frac{m+1}{((m+1)!)^2} \\
&= -\frac{4\pi}{|\xi|^2} \sum_{n=1}^{\infty} \left(-\frac{b^2|\xi|^2}{4}\right)^n \frac{n}{(n!)^2}.
\end{aligned}$$

Then, from Eq. (4.2), we obtain

$$C(\xi;\eta) = 2\mu \sum_{m=0}^{\infty} (-\mu)^m \left(-\frac{|\xi|^2}{2}\right) \hat{\chi}_{\Gamma^{m+1}}(\xi).$$

Finally, since

$$\gamma(p) = \frac{1}{(2\pi)^2} \int_{R^2} \left(-\frac{2}{|\xi|^2}\right) C(\xi;\eta)\exp(i\xi\cdot p)\,d\xi,$$

and

$$\chi_b(p) = \frac{1}{(2\pi)^2} \int_{R^2} \hat{\chi}_b(\xi)\exp(i\xi\cdot p)\,dp,$$

we obtain

$$\gamma(p) = 2\mu \sum_{m=0}^{\infty} (-\mu)^m \chi_{\Gamma^{m+1}}(p).$$

5. Conclusion. We have shown that if infinitely many precise measurements could be made, then a centered circular inhomogeneity inside a larger circular body could by "imaged" and precise knowledge of its radius obtained by Calderón's method, even if its conductivity differed greatly from the background. The image would in general not be close in L_p to the actual conductivity distribution.

We do not study here the more complicated behavior arising from a conductivity distribution with several discontinuities, nor do we consider the numerically important question of the stability of this method [4].

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