MATHEMATICS OF COMPUTATION
VOLUME 53, NUMBER 187
JULY 1989, PAGES 327–333

A Table of Elliptic Integrals: Cubic Cases*

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Abstract. Forty-one integrands that are rational except for the square root of a cubic polynomial with known real zeros are integrated in terms of R-functions for which Fortran codes are available. In contrast to conventional tables the interval of integration is not required to begin or end at a singular point of the integrand. The table contains one elliptic integral of the first kind, 26 of the second kind, and 14 of the third kind. Only 10 of the integrals are treated in standard tables, which list a large number of special cases that are unified here.

1. Introduction. Two earlier installments [2], [3] of a new table of elliptic integrals deal primarily with "quartic cases" in which the integrand is rational except for the square root of a quartic polynomial with known real zeros. In this paper we consider "cubic cases" of the form

$$[p] = [p_1, \ldots, p_4] = \int_y^x \prod_{i=1}^4 (a_i + b_i t)^{p_i/2}\, dt, \tag{1.1}$$

where p_1, p_2, p_3 are odd integers, p_4 is an even integer (omitted if it is zero), and all quantities are real. Integrands containing complex conjugate factors will be considered in a later paper. Although a cubic case can usually be calculated numerically by choosing $a_i = 1$ and $b_i = 0$ for some value of i in a suitable quartic case, it is preferable to have an explicit formula, which is often tedious to obtain in a uniform notation from the quartic case.

The integral (1.1) is an elliptic integral of the third kind if p_4 is even and negative. Otherwise, it is second kind except for $[-1,-1,-1]$, which is first kind. Many integrals like

$$\int (\alpha \cos^2\theta + \beta \sin^2\theta)^{p_1/2}\, d\theta \quad \text{and} \quad \int (\alpha + \beta z^2)^{p_1/2}(\gamma + \delta z^2)^{p_2/2}\, dz$$

can be put in the form (1.1) by letting $t = \sin^2\theta$ or $t = z^2$.

The integrals in the table are expressed in terms of four R-functions:

$$R_F(x,y,z) = \frac{1}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}\, dt, \tag{1.2}$$

$$R_J(x,y,z,w) = \frac{3}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}(t+w)^{-1}\, dt, \tag{1.3}$$

Received July 8, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 33A25; Secondary 33A30.
*Part of this work was done at the University of Maryland, where the author was a visitor at the Institute for Physical Science and Technology, with the support of AROD contract DAAG 29-80-C-0032. The rest was done in the Ames Laboratory, which is operated for the U. S. Department of Energy by Iowa State University under contract no. W-7405-ENG-82. The work was supported by the Director of Energy Research, Office of Basic Energy Sciences.

and two special cases,

$$R_C(x,y) = R_F(x,y,y) \quad \text{and} \quad R_D(x,y,z) = R_J(x,y,z,z). \tag{1.4}$$

The functions R_F and R_J are symmetric in x, y, z; R_F and R_C are homogeneous of degree $-1/2$; R_J and R_D are homogeneous of degree $-3/2$; each of the four functions has the value unity when all its arguments are unity; and R_C and R_J are interpreted as Cauchy principal values when the last argument is negative. Fortran codes for numerical computation of all four functions, including Cauchy principal values, are listed in the Supplements to [2] and [3]. The functions R_F, R_D, and R_J respectively replace Legendre's integrals of the first, second, and third kinds, while R_C includes the inverse circular and inverse hyperbolic functions.

To select integrals that are relatively simple, we arbitrarily require $\sum |p_i| \le 7$ and $\sum p_i \le 3$. Apart from permutation of subscripts in (1.1), there are just 40 cases of this kind. The table in Section 2 contains all 40 as well as $[-3,-3,-3]$, while only 10 of the 41 are included in [4] and nine in [5]. Each of the formulas for $[1,-1,-1]$, $[1,1,-1]$, $[-1,-1,-3]$, and $[-1,-1,-5]$ unifies 18 formulas in [4], and that for $[1,-1,-3]$ unifies 36. Moreover, the table in Section 2 does not require the interval of integration to begin or end at a singular point of the integrand.

Derivation of the formulas is discussed in Section 3. All integral formulas have been checked by numerical integration, and some details of the checks are given in Section 4.

2. Table of Cubic Cases. We assume $x > y$ and $a_i + b_i t > 0$, $y < t < x$, for $i = 1, 2, 3$. Assumptions about $a_4 + b_4 t$ will be stated where necessary. We define

$$d_{ij} = a_i b_j - a_j b_i, \qquad r_{ij} = \frac{d_{ij}}{b_i b_j} = \frac{a_i}{b_i} - \frac{a_j}{b_j}; \tag{2.1}$$

$$X_i = (a_i + b_i x)^{1/2}, \qquad Y_i = (a_i + b_i y)^{1/2}; \tag{2.2}$$

$$U_i = (X_i Y_j Y_k + Y_i X_j X_k)/(x-y), \tag{2.3}$$

where i, j, k is any permutation of $1, 2, 3$;

$$W_2^2 = U_1^2 - b_4 d_{12} d_{13}/d_{14}; \tag{2.4}$$

$$Q_2^2 = (X_4 Y_4 W_2 / X_1 Y_1)^2, \qquad P_2^2 = Q_2^2 + b_4 d_{24} d_{34}/d_{14}; \tag{2.5}$$

$$A(p_1, \ldots, p_n) = X_1^{p_1} \cdots X_n^{p_n} - Y_1^{p_1} \cdots Y_n^{p_n}. \tag{2.6}$$

These definitions imply, if P_2 is chosen positive,

$$P_2 = (X_1^{-1} X_2 X_3 Y_4^2 + Y_1^{-1} Y_2 Y_3 X_4^2)/(x-y), \tag{2.7}$$

$$U_2^2 = U_1^2 - b_3 d_{12}, \qquad U_3^2 = U_1^2 - b_2 d_{13} = U_2^2 - b_1 d_{23}, \tag{2.8}$$

$$W_2^2 = U_2^2 + b_1 d_{12} d_{34}/d_{14} = U_3^2 + b_1 d_{13} d_{24}/d_{14}. \tag{2.9}$$

If one limit of integration is infinite, (2.3) simplifies to

$$\begin{aligned} U_i &= (b_j b_k)^{1/2} Y_i, \quad & x = +\infty, \\ U_i &= (b_j b_k)^{1/2} X_i, \quad & y = -\infty, \end{aligned} \tag{2.10}$$

where the square roots are nonnegative, while

$$\begin{aligned} Q_2^2 &= (b_4/b_1)(Y_4 W_2 / Y_1)^2, \quad & x = +\infty, \\ Q_2^2 &= (b_4/b_1)(X_4 W_2 / X_1)^2, \quad & y = -\infty. \end{aligned} \tag{2.11}$$

Cubic cases will be expressed in terms of the quantities

$$I_{1c} = 2R_F(U_3^2, U_2^2, U_1^2), \tag{2.12}$$

$$I_{2c} = \tfrac{2}{3} d_{12} d_{13} R_D(U_3^2, U_2^2, U_1^2) + 2X_1Y_1/U_1, \tag{2.13}$$

$$I_{3c} = \frac{-2b_1 d_{12} d_{13}}{3d_{14}} R_J(U_3^2, U_2^2, U_1^2, W_2^2) + 2R_C(P_2^2, Q_2^2), \tag{2.14}$$

which are integrals of the first, second, and third kinds, respectively. It will be seen from the tables that

$$I_{1c} = [-1,-1,-1], \quad I_{2c} = [1,-1,-1], \quad I_{3c} = [1,-1,-1,-2]. \tag{2.15}$$

Thus I_{3c} reduces to I_{2c} if $a_4 = 1$ and $b_4 = 0$. The extra subscript c stands for "cubic," and the quantities defined here are obtained from those used in [3] for quartic cases. Specifically, if we put $a_4 = 1$ and $b_4 = 0$ and subsequently replace the subscript 5 by 4, then (U_{12}, U_{13}, U_{14}) reduces to (U_3, U_2, U_1), (W, P, Q) to (W_2, P_2, Q_2), and (I_1, I_2, I_3, I_3') to $(I_{1c}, I_{2c}, I_{3c}, I_{2c})$.

It is convenient to define also the quantities

$$\begin{aligned} J_{1c} &= d_{12} d_{13} I_{1c} - 2b_1 A(1,1,1) \\ &= 2d_{12} d_{13} R_F(U_3^2, U_2^2, U_1^2) - 2b_1 A(1,1,1), \end{aligned} \tag{2.16}$$

$$\begin{aligned} J_{2c} &= b_2 I_{2c} - 2A(1,1,-1) \\ &= \frac{2}{3} b_2 d_{12} d_{13} R_D(U_3^2, U_2^2, U_1^2) + \frac{2d_{13}X_2Y_2}{X_3Y_3U_1}. \end{aligned} \tag{2.17}$$

The first of these appears in the formula for $[1,1,-1]$ and is transmitted by recurrence relations to a dozen others; likewise, J_{2c} is transmitted from $[1,-1,-3]$. The second line of (2.17) follows from the first with the help of the identity

$$\begin{aligned} (x-y)U_1X_3Y_3A(1,1,-1) &= X_2Y_2(X_1^2Y_3^2 - X_3^2Y_1^2) + X_1X_3Y_1Y_3(X_2^2 - Y_2^2) \\ &= (x-y)(d_{31}X_2Y_2 + b_2X_1X_3Y_1Y_3). \end{aligned} \tag{2.18}$$

It is important to use I_{1c} and J_{2c}, not J_{1c} or I_{2c}, to evaluate integrals with $\sum p_i < -2$, which converge when $x = +\infty$ or $y = -\infty$. Both J_{1c} and I_{2c} then become infinite while I_{1c} and J_{2c} are finite. The second term in the second line of (2.17) becomes

$$\begin{aligned} 2d_{13}X_2Y_2/X_3Y_3U_1 &= 2d_{13}Y_2/b_3Y_1Y_3, & x &= +\infty, \\ 2d_{13}X_2Y_2/X_3Y_3U_1 &= -2d_{13}X_2/b_3X_1X_3, & y &= -\infty. \end{aligned} \tag{2.19}$$

If one limit of integration is a branch point of the integrand, then X_i or Y_i is 0 for some value of $i \le 3$, and one of the two terms on the right-hand side of (2.3) vanishes. If $X_1Y_1 = 0$ then P_2 and Q_2 are infinite, and the R_C function in (2.14) vanishes by homogeneity. If both limits of integration are branch points, the elliptic integral is called complete, and $U_1U_2U_3 = 0$. It is not assumed that $b_i \neq 0$ nor that $d_{ij} \neq 0$ unless one of these quantities occurs in a denominator. The relation $d_{ij} = 0$ is equivalent to proportionality of $a_i + b_it$ and $a_j + b_jt$.

We shall now list 41 cases of

$$[p_1, \ldots, p_4] = \int_y^x (a_1 + b_1t)^{p_1/2} \cdots (a_4 + b_4t)^{p_4/2}\, dt, \tag{2.20}$$

18 with $p_4 = 0$, nine with $p_4 = 2$ or 4, and 14 with $p_4 = -2$ or -4. Only the last 14 are integrals of the third kind involving I_{3c}. We omit $p_4 = 0$ in the first 18:

$$[-1,-1,-1] = I_{1c}. \tag{2.21}$$

$$[1,-1,-1] = I_{2c}. \tag{2.22}$$

$$[1,1,-1] = [(b_1 d_{23} + b_2 d_{13}) I_{2c} - J_{1c}]/3b_1 b_3. \tag{2.23}$$

$$[1,1,1] = [-2b_1 b_2 b_3 (r_{12} r_{13} + r_{23}^2) I_{2c} + (r_{12} + r_{13}) J_{1c} + 6A(3,1,1)]/15b_1. \tag{2.24}$$

$$[1,-1,-3] = (J_{2c} - d_{12} I_{1c})/d_{23}. \tag{2.25}$$

$$[-1,-1,-3] = (b_3 J_{2c} - b_2 d_{13} I_{1c})/d_{13} d_{23}. \tag{2.26}$$

$$[1,1,-3] = [2b_2 I_{2c} - d_{12} I_{1c} - 2A(1,1,-1)]/b_3. \tag{2.27}$$

$$[3,-1,-1] = [2(b_2 d_{13} + b_3 d_{12}) I_{2c} - J_{1c}]/3b_2 b_3. \tag{2.28}$$

$$\begin{aligned}[3,1,-1] = [&b_1 b_2 b_3 (3r_{13}^2 + 7r_{13} r_{23} - 2r_{23}^2) I_{2c} \\ &-(3r_{13} + r_{23}) J_{1c} + 6A(3,1,1)]/15b_3.\end{aligned} \tag{2.29}$$

$$[3,-1,-3] = [(b_1 d_{23} + b_2 d_{13}) I_{2c} - d_{12} d_{13} I_{1c} - 2d_{13} A(1,1,-1)]/b_3 d_{23}. \tag{2.30}$$

$$\begin{aligned}[3,1,-3] = [&(b_1 d_{23} + 7b_2 d_{13}) I_{2c} - 4d_{12} d_{13} I_{1c} \\ &+2b_1 A(1,1,1) - 6d_{13} A(1,1,-1)]/3b_3^2.\end{aligned} \tag{2.31}$$

$$[1,-3,-3] = [2b_3 J_{2c} - (b_2 d_{13} + b_3 d_{12}) I_{1c} + 2d_{23} A(1,-1,-1)]/d_{23}^2. \tag{2.32}$$

$$\begin{aligned}[-1,-3,-3] = [&b_3 (b_2 d_{13} + b_3 d_{12}) J_{2c} - 2b_2 b_3 d_{12} d_{13} I_{1c} \\ &+2b_2 d_{13} d_{23} A(1,-1,-1)]/d_{12} d_{13} d_{23}^2.\end{aligned} \tag{2.33}$$

$$\begin{aligned}&[-3,-3,-3] \\ &\quad = (b_3/d_{12} d_{13} d_{23}) \{ (2/r_{12} r_{13} r_{23})(r_{12} r_{13} + r_{23}^2) J_{2c} \\ &\qquad -(b_1 b_2/r_{23})(r_{12} + r_{13}) I_{1c} \\ &\qquad +(2/r_{12})[b_1 r_{23} A(-1,1,-1) + b_2 r_{13} A(1,-1,-1)] \}.\end{aligned} \tag{2.34}$$

$$[1,-1,-5] = [-b_2 (1 + r_{12}/r_{13}) J_{2c} + 2b_2 d_{12} I_{1c} - 2d_{23} A(1,1,-3)]/3d_{23}^2. \tag{2.35}$$

$$[1,1,-5] = [(r_{13}^{-1} + r_{23}^{-1}) J_{2c} - d_{12} r_{23}^{-1} I_{1c} - 2b_3 A(1,1,-3)]/3b_3^2. \tag{2.36}$$

$$\begin{aligned}[-1,-1,-5] = [&-2(r_{13}^{-1} + r_{23}^{-1}) J_{2c} + b_1 b_2 (1 + 2r_{13}/r_{23}) I_{1c} \\ &-2b_3 A(1,1,-3)]/3d_{13} d_{23}.\end{aligned} \tag{2.37}$$

$$\begin{aligned}[5,-1,-1] = [&b_1^2 b_2 b_3 (8r_{12}^2 + 8r_{13}^2 + 7r_{12} r_{13}) I_{2c} \\ &-4b_1 (r_{12} + r_{13}) J_{1c} + 6b_1 A(3,1,1)]/15b_2 b_3.\end{aligned} \tag{2.38}$$

The next nine integrals have $p_4 = 2$ or 4. No restriction is placed on a_4 or b_4.

$$[-1,-1,-1,2] = (b_4 I_{2c} - d_{14} I_{1c})/b_1. \tag{2.39}$$

$$[1,-1,-1,2] = (b_4/3)[(r_{13} - r_{34} - 2r_{24}) I_{2c} - J_{1c}/b_1 b_2 b_3]. \tag{2.40}$$

$$\begin{aligned}[1,1,-1,2] = (b_4/15b_1 b_3) \{ &-b_1 b_2 b_3 [r_{12}^2 + r_{13}^2 + r_{23}^2 + 5r_{34}(r_{13} + r_{23})] I_{2c} \\ &+(r_{12} + r_{14} + 4r_{34}) J_{1c} + 6A(3,1,1) \}.\end{aligned} \tag{2.41}$$

$$[-1,-1,-3,2] = (-d_{34} J_{2c} + d_{13} d_{24} I_{1c})/d_{13} d_{23}. \tag{2.42}$$

$$[1,-1,-3,2] = [(b_4d_{23} - b_2d_{34})I_{2c} + d_{12}d_{34}I_{1c} + 2d_{34}A(1,1,-1)]/b_3d_{23}. \tag{2.43}$$

$$\begin{aligned}[1,1,-3,2] = (b_4/3b_3)[b_2(r_{13} + r_{23} - 6r_{34})I_{2c} + d_{12}(3r_{34} - r_{13})I_{1c} \\ +(2/b_3)A(1,1,1) + 6r_{34}A(1,1,-1)].\end{aligned} \tag{2.44}$$

$$\begin{aligned}[3,-1,-1,2] = (b_4/15b_2b_3)\{b_1b_2b_3[8r_{12}^2 + 8r_{13}^2 + 7r_{12}r_{13} \\ -10r_{14}(r_{12} + r_{13})]I_{2c} \\ -(4r_{12} + 4r_{13} - 5r_{14})J_{1c} + 6A(3,1,1)\}.\end{aligned} \tag{2.45}$$

$$\begin{aligned}[-1,-1,-1,4] = (b_4^2/3b_1)[-2(r_{14} + r_{24} + r_{34})I_{2c} \\ +3b_1r_{14}^2I_{1c} - J_{1c}/b_1b_2b_3].\end{aligned} \tag{2.46}$$

$$\begin{aligned}[1,-1,-1,4] = (b_4^2/15b_1b_2b_3)\{b_1b_2b_3(8r_{23}^2 + 3r_{12}r_{13} - 5r_{14}^2 + 20r_{24}r_{34})I_{2c} \\ +2(r_{14} + 2r_{24} + 2r_{34})J_{1c} + 6A(3,1,1)\}.\end{aligned} \tag{2.47}$$

The final 14 integrals have $p_4 = -2$ or -4 and are integrals of the third kind. We assume either that $a_4 + b_4t$ is positive on the closed interval of integration or else, if $p_4 = -2$, that it changes sign in the open interval of integration. In the latter case the integral is interpreted as a Cauchy principal value (see [3, Section 6]).

$$[1,-1,-1,-2] = I_{3c}. \tag{2.48}$$

$$[-1,-1,-1,-2] = (b_4I_{3c} - b_1I_{1c})/d_{14}. \tag{2.49}$$

$$[1,1,-1,-2] = (d_{24}I_{3c} + b_2I_{2c})/b_4. \tag{2.50}$$

$$[1,1,1,-2] = [3b_2r_{24}d_{34}I_{3c} + b_2b_3(r_{14} + r_{24} + r_{34})I_{2c} - J_{1c}/b_1]/3b_4. \tag{2.51}$$

$$[1,1,-3,-2] = (d_{24}I_{3c} - J_{2c} + d_{12}I_{1c})/d_{34}. \tag{2.52}$$

$$[1,-1,-3,-2] = (b_4d_{23}I_{3c} - b_3J_{2c} + b_3d_{12}I_{1c})/d_{23}d_{34}. \tag{2.53}$$

$$[-1,-1,-3,-2] = [(b_4^2/d_{14})I_{3c} - (b_3^2/d_{13}d_{23})J_{2c} + (r_{23}^{-1} - r_{14}^{-1})I_{1c}]/d_{34}. \tag{2.54}$$

$$[3,-1,-1,-2] = (d_{14}I_{3c} + b_1I_{2c})/b_4. \tag{2.55}$$

$$\begin{aligned}[3,1,-1,-2] = b_1b_2r_{14}r_{24}I_{3c} + (b_1b_2/3b_4)(2r_{13} + 2r_{14} + r_{24})I_{2c} \\ -J_{1c}/3b_3b_4.\end{aligned} \tag{2.56}$$

$$\begin{aligned}[3,1,1,-2] = (d_{14}d_{24}d_{34}/b_4^3)I_{3c} \\ +\{b_1b_2b_3[5r_{14}(r_{14} + r_{24} + r_{34}) - r_{12}^2 - r_{13}^2 - r_{23}^2]I_{2c} \\ -(3r_{14} + r_{24} + r_{34})J_{1c} + 6A(3,1,1)\}/15b_4.\end{aligned} \tag{2.57}$$

$$\begin{aligned}[1,1,1,-4] = [b_2b_3(r_{24} + r_{34} + r_{24}r_{34}/r_{14})I_{3c} + (3b_2b_3/b_4)I_{2c} \\ -(d_{12}d_{13}/d_{14})I_{1c} - 2A(1,1,1,-2)]/2b_4.\end{aligned} \tag{2.58}$$

In the next three formulas we use the abbreviation

$$K_{2c} = b_2b_3I_{2c} - 2b_4A(1,1,1,-2) = b_3J_{2c} - 2d_{34}A(1,1,-1,-2). \tag{2.59}$$

The second equality, showing that K_{2c} is finite if $x = +\infty$ or $y = -\infty$, follows from (2.17) and [3, (4.8)].

$$[1,1,-1,-4] = [b_1 b_2 b_3 b_4 (r_{34}^2 - r_{13} r_{23}) I_{3c} + b_1 r_{14} K_{2c} - d_{12} d_{13} I_{1c}]/2 d_{14} d_{34}. \tag{2.60}$$

$$[1,-1,-1,-4] = (1/2b_4)(r_{14}^{-1} - r_{24}^{-1} - r_{34}^{-1}) I_{3c} + [K_{2c} - (b_4 d_{12} d_{13}/d_{14}) I_{1c}]/2 d_{24} d_{34}. \tag{2.61}$$

$$\begin{aligned}[-1,-1,-1,-4] = &-(1/2d_{14})(r_{14}^{-1} + r_{24}^{-1} + r_{34}^{-1}) I_{3c} \\ &+ (b_4/2 d_{14} d_{24} d_{34}) K_{2c} \\ &+ (b_1/d_{14})^2 (1 - r_{12} r_{13}/2 r_{24} r_{34}) I_{1c}.\end{aligned} \tag{2.62}$$

3. Derivation of the Formulas. Six of the 41 formulas are obtained by putting $a_4 = 1$ and $b_4 = 0$ (see the remarks following (2.15)) in suitable quartic cases in [3]. Thus $[-1,-1,-1]$, $[1,-1,-1]$, $[1,1,-1]$, $[1,-1,-3]$, $[-1,-1,-3]$, and $[1,1,-3]$ come respectively from $[-1,-1,-1,-1]$, $[1,-1,-1,-3]$, $[1,1,-1,-5]$, $[1,-1,-3,-3]$, $[-1,-1,-3,-3]$, and $[1,1,-3,-3]$. Seven more are obtained by putting $a_4 = 1$ and $b_4 = 0$ and then replacing a_5 by a_4 and b_5 by b_4. Thus $[-1,-1,-1,2]$, $[1,-1,-1,-2]$, $[3,-1,-1,-2]$, $[1,1,1,-4]$, $[1,1,-1,-4]$, $[1,-1,-1,-4]$, and $[-1,-1,-1,-4]$ come respectively from $[-1,-1,-1,-1,2]$, $[1,-1,-1,-1,-2]$, $[3,-1,-1,-1,-2]$, $[1,1,1,-1,-4]$, $[1,1,-1,-1,-4]$, $[1,-1,-1,-1,-4]$, and $[-1,-1,-1,-1,-4]$. The formulas are often simplified by using identities such as [3, (4.6) to (4.9)].

The remaining cases are then obtained by recurrence relations. Let e_i denote an n-tuple with 1 in the ith place and 0's elsewhere (for example, $[p + 2e_1] = [p_1 + 2, p_2, \ldots, p_n]$). From [4, Section 4] we have

$$(p_1 + \cdots + p_n + 2) b_i [p] = \sum_{j \neq i} p_j d_{ji} [p - 2e_j] + 2A(p + 2e_i), \tag{Ai}$$

$$d_{ij}[p] = b_j [p + 2e_i] - b_i [p + 2e_j], \tag{Bij}$$

$$b_j [p] = b_i [p - 2e_i + 2e_j] + d_{ij} [p - 2e_i]. \tag{Cij}$$

To get $[1,1,1]$ we use (A1) and evaluate $[1,-1,1]$ by interchanging the subscripts 2 and 3 in $[1,1,-1]$. Equations (2.28) to (2.31) then follow in order from (C12), (C13), (C13), and (C13). To get $[\pm 1,-3,-3]$ we use (B23) and evaluate $[\pm 1,-3,-1]$ by interchanging the subscripts 2 and 3 in $[\pm 1,-1,-3]$. Then $[-3,-3,-3]$ follows from putting $[p] = [-1,-3,-3]$ and $i = 1$ in [2, (5.5)] and using [3, (4.8)]. After $[1,-1,-5]$ has been obtained from (A2) with $[p] = [1,-1,-3]$, Eqs. (2.36) and (2.37) follow respectively from (C23) and (B13). Equation (2.38) comes from (C12).

Equations (2.40) to (2.47) follow in order from (C42), (C43), (C43), (C43), (C43), (C42), (C41), and (C42). To get (2.49), (2.50), and (2.51), we use (B14), (C24), and (C34), respectively. Equations (2.52), (2.53), (2.54), (2.56), and (2.57) follow in order from (B34), (B34), (B34), (C24), and (C14).

4. Numerical Checks. The 41 formulas in Section 2 were checked numerically when $x = 2.0$, $y = 0.5$, $a_i = 0.1 + 0.2i$, $b_i = 0.5 - 0.2i$, $1 \leq i \leq 4$. In each formula the integral on the left side, defined by (2.20), was integrated numerically by the SLATEC code QNG. On the right-hand side, I_{1c}, I_{2c}, and I_{3c} were calculated from

(2.12) to (2.14) by using the codes for R-functions in the Supplements to [2] and [3]. The remaining calculations, including those of J_{1c}, J_{2c}, and K_{2c} by (2.16), (2.17), and (2.59), were done with a hand calculator. For each of the 41 formulas the values obtained for the two sides agreed to better than one part in a million.

Some of the intermediate values in these calculations are listed here:

$$\begin{array}{ll}
U_1^2 = 0.41309998, & W_2^2 = 0.38909998, \\
U_2^2 = 0.40109998, & P_2^2 = 0.24016665, \\
U_3^2 = 0.43709998, & Q_2^2 = 0.21616665, \\
 & \\
R_C(P_2^2, Q_2^2) = 2.1128946, & I_{1c} = 3.0973715, \\
R_F(U_3^2, U_2^2, U_1^2) = 1.5486858, & I_{2c} = 2.0520132, \\
R_D(U_3^2, U_2^2, U_1^2) = 3.7353179, & I_{3c} = 4.2877248, \\
R_J(U_3^2, U_2^2, U_1^2, W_2^2) = 3.8709720, & J_{1c} = -0.00688951, \\
 & J_{2c} = -0.80566308, \\
 & \\
A(1,1,1) = 0.16015635, & K_{2c} = 0.78110328, \\
A(1,1,-1) = 0.50543220, & \\
A(1,-1,-1) = 0.48163106, & A(3,1,1) = 0.32463223, \\
A(-1,1,-1) = -0.12403646, & A(1,1,1,-2) = 1.3360390, \\
A(1,1,-3) = 1.2956636, & A(1,1,-1,-2) = 2.9189040.
\end{array}$$

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