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# The Construction of Preconditioners for Elliptic Problems by Substructuring, IV*

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**Abstract.** We consider the problem of solving the algebraic system of equations which result from the discretization of elliptic boundary value problems defined on three-dimensional Euclidean space. We develop preconditioners for such systems based on substructuring (also known as domain decomposition). The resulting algorithms are well suited to emerging parallel computing architectures. We describe two techniques for developing these preconditioners. A theory for the analysis of the condition number for the resulting preconditioned system is given and the results of supporting numerical experiments are presented.

**1. Introduction.** The aim of this series of papers is to propose and analyze methods for efficiently solving the equations resulting from finite element discretizations of second-order elliptic boundary value problems on general domains in $R^2$ and $R^3$. In particular, we shall be concerned with constructing computationally "effective" preconditioners for these discrete equations which can be used in a preconditioned iterative algorithm to define a rapid solution method. The methods developed are well suited to parallel computing architectures.

In Part I, [6], a flexible domain decomposition algorithm for the two-dimensional problems was developed and analyzed. This algorithm had the novel feature that it enabled subdivision into an arbitrary number of subdomains without the deterioration of the resulting iterative convergence rates. This property has important implications in parallel applications since for this type of algorithm, the number of subdomains is proportional to the number of parallel tasks.

In Parts II, [7] and III, [8], we extended the domain decomposition techniques along two directions. In Part II, we developed some simplified domain decomposition strategies for two- and three-dimensional problems, including a class of singularly perturbed systems which occur in parabolic timestepping applications. In Part III, we introduced a technique for two-dimensional problems which gave rise to domain decomposition strategies whose convergence rates stayed bounded independently of both the subdomain size $d$ and the mesh size $h$.

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In this part of the series, we will develop two domain decomposition algorithms for problems in three dimensions. We shall present a general theoretical approach for the analysis of such methods. These methods lead to preconditioned systems with condition number bounded by $c(1+\ln^2(d/h))$. In contrast, the simplified strategies of Part II give rise to a condition number bounded by $cd/h$.

Let $\Omega$ be a bounded domain in $R^3$ with boundary $\partial\Omega$. As a model problem for a second-order uniformly elliptic equation, we shall consider the Dirichlet problem

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where

$$Lv = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_j}\right)$$

with $\{a_{ij}\}$ uniformly positive definite, bounded and piecewise smooth on $\Omega$.

In this paper, we shall develop and analyze preconditioners for the matrices which result from finite element and finite difference discretization of (1.1). This is most naturally carried out from the finite element point of view. Accordingly, we shall proceed with the general finite element framework with a detailed formulation of both cases considered in Section 2.

The generalized Dirichlet form corresponding to (1.1) is given by

$$A(v,w) = \sum_{i,j=1}^{3} \int_\Omega a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j}\,dx, \tag{1.2}$$

which is defined for all $v$ and $w$ in the Sobolev space $H^1(\Omega)$ (the space of distributions with square-integrable first derivatives). The $L^2(\Omega)$ inner product is denoted

$$(v,w) = \int_\Omega vw\,dx.$$

The subspace $H_0^1(\Omega)$ is the completion of the smooth functions with support in $\Omega$, with respect to the norm in $H^1(\Omega)$. The weak formulation of the problem defined by (1.1) is: Find $u \in H_0^1(\Omega)$ such that

$$A(u,w) = (f,w) \quad \text{for all } w \in H_0^1(\Omega). \tag{1.3}$$

This leads immediately to the standard Galerkin approximation. Let $S_h(\Omega)$ be a finite-dimensional subspace of $H_0^1(\Omega)$. The Galerkin approximation is defined as the solution of the following problem: Find $U \in S_h(\Omega)$ such that

$$A(U,\Phi) = (f,\Phi) \quad \text{for all } \Phi \in S_h(\Omega). \tag{1.4}$$

The underlying method which we will consider is a preconditioned iterative method. As explained in Part I, the task of defining a preconditioner for the matrix problem corresponding to (1.4) is the same as that of defining another positive definite form $B(\cdot,\cdot)$ on $S_h(\Omega)\times S_h(\Omega)$. The importance of making a "good" choice for $B$ is well known. The form $B$ will define a good preconditioner provided it has two basic properties. First, the problem of finding the function $W \in S_h(\Omega)$ satisfying

$$B(W,\Phi) = G(\Phi) \quad \text{for all } \Phi \in S_h(\Omega), \tag{1.5}$$

for a given linear functional $G$, should be more economical to solve on a given computer architecture than (1.4). Secondly, $B$ should be spectrally close to $A$ in the sense that there are positive numbers $\beta_0$ and $\beta_1$ satisfying

$$\beta_0 B(V,V) \le A(V,V) \le \beta_1 B(V,V) \quad \text{for all } V \in S_h(\Omega), \tag{1.6}$$

where the ratio $\beta_1/\beta_0$ is not too large.

We will define the preconditioning form $B$ using domain decomposition and 'mapping' techniques. The domain $\bar{\Omega}$ is written as a union of subdomains $\bigcup \bar{\Omega}_i$. The mesh on each subdomain is assumed to be related to the mesh on the reference cube $\hat{\Omega}$ under a transformation $T_i$. The framework developed in this paper reduces the task of defining domain decomposition preconditioners on $\Omega$ to a problem of defining an appropriate form $Q$ acting on subspaces of functions defined on the boundary of the reference domain. A consequence of this approach is that the most significant part of the analysis need only be carried out on the reference domain in conjunction with a reference subspace.

The outline of the remainder of the paper is as follows. In Section 2, we describe the finite element and finite difference discretizations. We also give the assumptions on the subspaces on $\Omega$ and $\hat{\Omega}$. In Section 3, we show how the construction of the preconditioner $B$ can be reduced to the definition of an appropriate form $Q$. The preconditioner $B$ is described in terms of $Q$ in this section. In Section 4, we develop two forms $Q = Q_1$ and $Q = Q_2$ which lead to different domain decomposition algorithms. It is shown that (1.6) holds with

$$\beta_1/\beta_0 \le c(1 + \ln^2(d/h))$$

for the domain decomposition form $B$ resulting from either of these two forms. Here, $d$ is roughly the domain size and $h$ is the mesh size. The most computationally effective preconditioner results from the form $Q_1$. We describe the algorithm for the solution of (1.5) in Section 5. Finally, in Section 6, we give the results of numerical experiments for some three-dimensional problems.

For earlier papers dealing with domain decomposition techniques applied to the solution of the linear systems resulting from numerical approximation of boundary value problems see [2], [5], [9], [10]. The obvious generalizations of these methods lead to preconditioned systems whose condition number increases with the number of subdomains. Thus, these methods may not lead to effective algorithms on parallel computers. For some numerical results for domain decomposition methods on parallel computers see [11]. Additional papers and references for recent work on domain decomposition can be found in the proceedings to be published by SIAM of the 'First International Symposium on Domain Decomposition Methods for Partial Differential Equations' held in Paris 1987.

A domain decomposition technique which is well suited to applications with refinements is developed in [4]. The resulting algorithms are sometimes the same as those developed with the FAC approach of [15] which represent yet another technique for developing domain decomposition-like algorithms for refinement problems.

Before proceeding, we give some notation. In what follows, edges, faces, and subdomains will be open sets in $R^1$, $R^2$, and $R^3$, respectively. Let $\tilde{\Omega}$ be a generic domain in $R^j$ for $j = 1, 2, 3$. For nonnegative $s$, the Sobolev space of order $s$ on

$\tilde{\Omega}$ will be denoted $H^s(\tilde{\Omega})$ (cf. [14], [16]). The norm on $H^s(\tilde{\Omega})$ will be denoted by $\|\cdot\|_{s,\tilde{\Omega}}$ when $j=3$ and $|\cdot|_{s,\tilde{\Omega}}$ when $j=1,2$. The $L^2$ inner products and norms will be denoted

$$(u,v)_{\tilde{\Omega}} = \int_{\tilde{\Omega}} uv\,dx$$

and $\|u\|_{\tilde{\Omega}} = (u,u)_{\tilde{\Omega}}^{1/2}$ when $j=3$, and

$$\langle u,v\rangle_{\tilde{\Omega}} = \int_{\tilde{\Omega}} uv\,dx$$

and $|u|_{\tilde{\Omega}} = \langle u,u\rangle_{\tilde{\Omega}}^{1/2}$ when $j=1,2$.

Throughout this paper, $c$ and $C$, with or without subscripts, will denote positive constants which are independent of the subdivision, $d$ and $h$. These constants may take on different values in different places.

**2. Discretization of (1.1).** In this section, we shall formulate the finite element and finite difference methods to be considered. In order to do so, we shall first describe our assumptions on the domain $\Omega$ and its decomposition $\bar{\Omega} = \bigcup \bar{\Omega}_i$. We next use this decomposition to define the finite element and finite difference approximations. Finally, we describe some norms which will play an essential role in the analysis of the preconditioners to be developed in later sections.

An important aspect of this paper is to reduce the problem of defining domain decomposition algorithms on the union of subdomains to a problem on the unit cube $\hat{\Omega}$ with respect to a reference subspace. The faces of $\hat{\Omega}$ will be denoted $\hat{\Gamma}_i^f$ for $i=1,\dots,6$. In addition, the union of the closures of the edges of $\hat{\Omega}$ will be denoted by $\hat{\Gamma}^e$.

We make the following assumptions with respect to the domain $\Omega$:

(A.1) $\Omega$ can be subdivided into $m$ subdomains $\bar{\Omega} = \bigcup \bar{\Omega}_i$ with $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$.

(A.2) These subdomains are related to the unit cube in that for each $i$ there is an orientation-preserving trilinear mapping $T_i$ which takes $\hat{\Omega}$ onto $\Omega_i$. We assume that there exists a positive constant $d$ such that

$$d^{-1}\,|DT_i(x)| \le C \quad \text{for all } x \in \hat{\Omega} \tag{2.1}$$

and

$$d\,\left|DT_i^{-1}(x)\right| \le C \quad \text{for all } x \in \Omega_i. \tag{2.2}$$

Here, $DT_i(x)$ is the Jacobian matrix of $T_i$ at $x$. In (2.1) and (2.2), $|\cdot|$ denotes the matrix norm. Note that the subdomains are roughly of size $d$.

(A.3) The set of faces of $\Omega_i$ is denoted by $\{\Gamma_{ij}^f\}$, where $\Gamma_{ij}^f$ is defined to be the image of the $j$th face of $\partial\hat{\Omega}$ under $T_i$. Furthermore, we require that if two faces, $\Gamma_{ij}^f$ and $\Gamma_{kl}^f$, share a common point $x$, then $\Gamma_{ij}^f = \Gamma_{kl}^f$ and

$$T_i^{-1}(x) = T_{ik}^r \circ T_k^{-1}(x) \quad \text{for all } x \in \Gamma_{ij}^f,$$

where $T_{ik}^r$ is a rigid body rotation of $\hat{\Omega}$.

We next consider the definition of the approximation procedures. For simplicity of presentation, we shall consider particular finite element and finite difference

applications. Many generalizations are possible. For either procedure, we partition the unit cube into $m_1 \times m_2 \times m_3$ regular rectangular parallelepipeds and define $S_{\hat{h}}(\hat{\Omega})$ to be the functions which are continuous on $\hat{\Omega}$ and piecewise trilinear with respect to this partition. The reference mesh size $\hat{h}$ is defined to be

$$\hat{h} = \max(1/m_1, 1/m_2, 1/m_3).$$

We assume that

$$\min(1/m_1, 1/m_2, 1/m_3) \geq C\hat{h}.$$

We first consider the finite element case. Let $h = d\hat{h}$ and define

$$S_h(\Omega_i) = \{\phi = \psi \circ T_i^{-1} | \phi = 0 \text{ on } \partial\Omega \text{ and } \psi \in S_{\hat{h}}(\hat{\Omega})\}.$$

Define the map $I_i \colon S_h(\Omega_i) \mapsto S_{\hat{h}}(\hat{\Omega})$ by $I_i V = V \circ T_i$. Define the space $S_h(\Omega)$ to be the set of continuous functions on $\bar{\Omega}$ whose restrictions on each subdomain $\Omega_i$ are functions in $S_h(\Omega_i)$. We assume that

(A.4) $S_h(\Omega_i) = \{\phi|_{\Omega_i} \text{ for } \phi \in S_h(\Omega)\}$.

This implies that the boundary nodes of $S_h(\Omega_i)$ and $S_h(\Omega_j)$ coincide on common faces.

Note that we obviously have

$$A(V, W) = \sum_{i=1}^{m} A_i(V, W)$$

where

$$A_i(V, W) = \sum_{j,k=1}^{3} \int_{\Omega_i} a_{jk} \frac{\partial V}{\partial x_j} \frac{\partial W}{\partial x_k} \, dx,$$

and $m$ is defined in (A.1). The finite element approximation to the solution $u$ of (1.1) is the function $U \in S_h(\Omega)$ satisfying (1.4). We will derive preconditioners for this problem.

We next define the finite difference approximation. We only consider the case when $L$ is given by

$$Lv = -\nabla \cdot a \nabla v.$$

Assume that (A.1), (A.2) and (A.3) hold and furthermore that the mappings $T_i$ are simply dilatation and translation with respect to the coordinate axes. Also assume that we have a regular grid of nodal points $\{p_j\}_{j=1}^{\bar{N}}$ on a mesh of size $h$ defined on $\bar{\Omega}$. We label these nodes so that $\{p_j\}_{j=1}^{N}$ are the nodes in $\Omega$ and assume that the nodal points of $\bar{\Omega}_i$ coincide with the image of the nodal points (corresponding to the subspace $S_{\hat{h}}(\hat{\Omega})$ defined above) of $\hat{\Omega}$ under $T_i$. The space $S_h(\Omega)$ consists of $N$-dimensional vectors of nodal values at the nodes of $\Omega$. The subspace $S_h(\Omega_i)$ consists of the nodal values at nodes in $\Omega \cap \bar{\Omega}_i$. The map $I_i \colon S_h(\Omega_i) \mapsto S_{\hat{h}}(\hat{\Omega})$ is defined by interpolation, i.e., $I_i V$ is the function in $S_{\hat{h}}(\hat{\Omega})$ defined by

$$I_i V(p) = \begin{cases} V(T_i(p)) & \text{when } T_i(p) \text{ is a node of } \Omega \cap \bar{\Omega}_i, \\ 0 & \text{for the remaining nodes of } \hat{\Omega}. \end{cases}$$

Note that $S_{\hat{h}}(\hat{\Omega})$ is a finite element subspace of trilinear functions, even when we are using the finite difference approximation on $\Omega$.

Let $\mathcal{N}_i$ be the list of neighbors for $S_h(\Omega_i)$, i.e., $(k,l) \in \mathcal{N}_i$ if and only if $p_k, p_l$ are nodal points in $\bar{\Omega}_i$ which are a distance of $h$ apart. Let $p_{k,l}$ be the midpoint between $p_k$ and $p_l$ and set

$$A_i(V,W) = h^3 \sum_{(k,l)\in\mathcal{N}_i} w_{kl}\, a(p_{k,l}) \frac{(V(p_k)-V(p_l))(W(p_k)-W(p_l))}{h^2},$$

where we set $V(p_l) \equiv W(p_l) \equiv 0$ for nodes $p_l$ on $\partial\Omega$. Here, $w_{kl}$ is the weight function defined by

$$w_{kl} = \begin{cases} 1 & \text{if the line segment between } p_k \text{ and } p_l \text{ is in } \Omega_i, \\ 1/2 & \text{if the line segment between } p_k \text{ and } p_l \text{ is in some face of } \Omega_i, \\ 1/4 & \text{if the line segment between } p_k \text{ and } p_l \text{ is in an edge of } \partial\Omega_i. \end{cases}$$

For functions $V, W \in S_h(\Omega)$ define

$$A(V,W) = \sum_{i=1}^{m} A_i(V,W). \tag{2.3}$$

The finite difference approximation to the solution $u$ of (1.1) at the nodes is the function $U \in S_h(\Omega)$ satisfying

$$A(U,\Phi) = F \cdot \Phi \quad \text{for all } \Phi \in S_h(\Omega), \tag{2.4}$$

where $F$ is the vector $\{h^3 f(p_k)\}$ and $\cdot$ denotes the usual Euclidean inner product.

*Remark* 2.1. By summation by parts, it is not difficult to see that the form $A$ can be written

$$A(V,W) = (L_h V) \cdot W,$$

where $L_h$ is the usual second-order 7-point difference operator multiplied by $h^3$. Thus, the solution $U$ of (2.4) is the standard finite difference approximation to the solution of (1.1). We have taken the above approach for developing these equations because it naturally gives rise to the decomposition of the form given by (2.3).

*Remark* 2.2. For both finite element and finite difference discretizations, we allow for the case when each subdomain $S_h(\Omega_i)$ has a different number of nodes. Accordingly, the reference subspace may differ with $i$. We have suppressed this dependence in the notation for convenience.

We finish this section with some additional notation. Let $\Gamma = \bigcup \partial\Omega_i$ and $S_h(\Gamma)$ be the space of functions which are restrictions of those in $S_h(\Omega)$ to $\Gamma$. Let $S_h^0(\Omega_i)$ be the subspace of $S_h(\Omega_i)$ of functions which vanish on $\partial\Omega_i$. Finally, let $S_{\hat{h}}(\partial\hat{\Omega})$ denote the space of restrictions of the functions of $S_{\hat{h}}(\hat{\Omega})$ to $\partial\hat{\Omega}$ and $S_{\hat{h}}^0(\hat{\Omega})$ denote the space of functions in $S_{\hat{h}}(\hat{\Omega})$ which vanish on $\partial\hat{\Omega}$.

**3. A General Construction of $B(\cdot,\cdot)$.** We will define our domain decomposition form by replacing the terms $A_i(V,W)$ in (2.3). To do this, we decompose an arbitrary function $W \in S_h(\Omega_i)$ into $W = W_P + W_H$, where $W_P \in S_h^0(\Omega_i)$ and

$$A_i(W_H,\Phi) = 0 \quad \text{for all } \Phi \in S_h^0(\Omega_i). \tag{3.1}$$

$W_H$ is the unique function in $S_h(\Omega_i)$ which equals $W$ on $\partial\Omega_i$ and satisfies (3.1). Such a function will be called 'discrete $A_i$-harmonic.' A consequence of (3.1) is that

$$A_i(W,W) = A_i(W_P,W_P) + A_i(W_H,W_H). \tag{3.2}$$

To define our preconditioner $B$, we shall replace the term $A_i(W_H, W_H)$ above.

We note that assumptions (2.1) and (2.2) imply

$$cA_i(V,V) \le d\delta_i D(I_iV, I_iV) \le CA_i(V,V) \quad \text{for all } V \in S_h(\Omega_i), \tag{3.3}$$

in the finite element case. Here $D(\cdot,\cdot)$ denotes the Dirichlet integral and is defined by

$$D(v,w) \equiv \int_{\hat\Omega} \nabla v \cdot \nabla w \, dx.$$

The constant $\delta_i$ appearing in (3.3) is a scaling factor. One reasonable choice is to take $\delta_i = (\lambda_1^i + \lambda_0^i)/2$, where $\lambda_1^i$ and $\lambda_0^i$ are respectively the largest and smallest eigenvalue of the $3 \times 3$ matrix $\{a_{ij}(x_0)\}$ at some point $x_0 \in \Omega_i$. Then the values of $c$ and $C$ appearing in (3.3) only depend on the local variation of the coefficients $\{a_{ij}\}$ on the subregions. It is straightforward to show that (3.3) also holds in the case of finite differences.

The problem of defining a replacement for $A_i(W_H, W_H)$ is thus the same as that of finding one for $dD(I_iW_H, I_iW_H)$. Note that $I_iW_H$ depends only on its boundary values. Accordingly, the form $dD(I_iW_H, I_iW_H)$ can be replaced by a form which explicitly depends only on the boundary values of $I_iW_H$.

To this end, we introduce a bilinear form $Q$ on $S_{\hat h}(\partial\hat\Omega) \times S_{\hat h}(\partial\hat\Omega)$ and define the form $B$ by

$$B(W,W) = \sum_{i=1}^{m} \{A_i(W_P, W_P) + d\delta_i \, Q(I_iW - \gamma_i(W), I_iW - \gamma_i(W))\}, \tag{3.4}$$

where $\gamma_i(W)$, for each $i$ and $W$, is the constant function on $\hat\Omega$ whose value is determined by

$$Q(I_iW - \gamma_i(W), 1) = 0. \tag{3.5}$$

Notice that the function $W_P$ depends upon $i$ in (3.4). For convenience we have suppressed this dependence in the notation.

Two constructions of $Q$ which lead to effective domain decomposition algorithms will be given in the next section. For the remainder of this section, we assume that such a form has been given which satisfies

$$\alpha_0(\hat h)Q(V,V) \le |V|^2_{1/2,\partial\hat\Omega} \le \alpha_1(\hat h)Q(V,V) \quad \text{for all } V \in S_{\hat h}(\partial\hat\Omega). \tag{3.6}$$

For the $Q$ to be defined, the constants $\alpha_0(\hat h)$ and $\alpha_1(\hat h)$ can be estimated in terms of $\hat h$ (see Proposition 2).

We then have the following proposition.

PROPOSITION 1. *Assume that* (3.6) *holds. There are constants* $\beta_2$ *and* $\beta_3$ *which do not depend on* $d$ *or* $h$ *satisfying*

$$\beta_2\alpha_0(d/h)\, B(W,W) \le A(W,W) \le \beta_3\alpha_1(d/h)\, B(W,W) \quad \textit{for all } W \in S_h(\Omega). \tag{3.7}$$

*Proof.* By (2.3) and (3.4), it suffices to consider a fixed subdomain $\Omega_i$. Let $W \in S_h(\Omega_i)$ be decomposed into $W = W_P + W_H$ as in (3.1). By the definition of $B$ and (3.2), it suffices to show

$$d\delta_i\alpha_0(d/h)\, Q(I_iW - \gamma_i(W), I_iW - \gamma_i(W)) \le cA_i(W,W) \tag{3.8}$$

and

$$A_i(W_H, W_H) \le Cd\delta_i\alpha_1(d/h)Q(I_iW - \gamma_i(W), I_iW - \gamma_i(W)), \tag{3.9}$$

where $\gamma_i(W)$ is the constant appearing in (3.5). Let $(I_iW)_H$ be the discrete harmonic function in $S_{\hat{h}}(\hat{\Omega})$ which equals $I_iW$ on $\partial\hat{\Omega}$, i.e., $(I_iW)_H$ is the unique function in $S_{\hat{h}}(\hat{\Omega})$ which equals $I_iW$ on $\partial\hat{\Omega}$ and satisfies the homogeneous equation

$$D((I_iW)_H, \Phi) = 0 \quad \text{for all } \Phi \in S^0_{\hat{h}}(\hat{\Omega}). \tag{3.10}$$

For any constant $\gamma$ on $\hat{\Omega}$, (3.5) implies

$$Q(I_iW - \gamma_i(W), I_iW - \gamma_i(W)) \le Q(I_iW - \gamma, I_iW - \gamma).$$

In particular, taking $\gamma$ to be the average value of $(I_iW)_H$ on $\hat{\Omega}$, it follows from (3.6), the trace theorem and Poincaré's inequality that

$$\begin{aligned} Q(I_iW - \gamma_i(W), I_iW - \gamma_i(W)) &\le c\alpha_0^{-1}(\hat{h})D((I_iW)_H, (I_iW)_H) \\ &\le c\alpha_0^{-1}(\hat{h})D(I_iW, I_iW). \end{aligned}$$

Inequality (3.8) then follows from (3.3).

We next prove (3.9). We first show that for discrete harmonic functions $V$,

$$\|V\|^2_{H^1(\hat{\Omega})} \le C_1|V|^2_{1/2,\partial\hat{\Omega}}. \tag{3.11}$$

By the Poincaré inequality and the minimization property of discrete harmonic functions, (3.11) will follow if we can construct a function $\tilde{W} \in S_{\hat{h}}(\hat{\Omega})$ with the same boundary values as $V$ satisfying

$$\|\tilde{W}\|^2_{H^1(\hat{\Omega})} \le C_1|V|^2_{1/2,\partial\hat{\Omega}}. \tag{3.12}$$

To do this, we use a variation of an argument given in [1]. Let $v$ be the harmonic function on $\hat{\Omega}$ which is equal to $V$ on $\partial\hat{\Omega}$. There exists a function $W \in S_{\hat{h}}(\hat{\Omega})$ (which may differ from $V$ on $\partial\hat{\Omega}$) satisfying

$$\|v - W\|^2_{\hat{\Omega}} + \hat{h}^2\|v - W\|^2_{H^1(\hat{\Omega})} \le C\|v\|^2_{H^1(\hat{\Omega})}. \tag{3.13}$$

The function $W$ in (3.13) can be taken to be, for example, the $L^2(\hat{\Omega})$ projection of $v$. We define $\tilde{W}$ to be the function in $S_{\hat{h}}(\hat{\Omega})$ which is equal to $V$ on the nodes of $S_{\hat{h}}(\hat{\Omega})$ which lie on $\partial\hat{\Omega}$ and is equal to $W$ on the nodes of $S_{\hat{h}}(\hat{\Omega})$ which are in the interior of $\hat{\Omega}$. Clearly,

$$\left\|\tilde{W}\right\|_{H^1(\hat{\Omega})} \le \left\|\tilde{W} - W\right\|_{H^1(\hat{\Omega})} + \|W\|_{H^1(\hat{\Omega})} \le c\hat{h}^{-1/2}|v - W|_{\partial\hat{\Omega}} + \|W\|_{H^1(\hat{\Omega})}.$$

By a well-known trace inequality,

$$|v - W|^2_{\partial\hat{\Omega}} \le C(\hat{h}^{-1}\|v - W\|^2_{\hat{\Omega}} + \hat{h}\|v - W\|^2_{H^1(\hat{\Omega})}).$$

Combining the above inequalities with the well-known inequality for harmonic functions,

$$\|v\|_{H^1(\hat{\Omega})} \le C|V|_{1/2,\partial\hat{\Omega}},$$

completes the proof of (3.12).

Let $X$ be the function in $S_h(\Omega_i)$ satisfying $I_iX = (I_iW)_H$. Note that

$$X = W \quad \text{on } \partial\Omega_i,$$

and hence by (3.1),(3.3),

$$\begin{aligned} A_i(W_H, W_H) &\le A_i(X, X) \le Cd\delta_i D((I_iW)_H, (I_iW)_H) \\ &= Cd\delta_i D((I_iW)_H - \gamma_i(W), (I_iW)_H - \gamma_i(W)). \end{aligned}$$

Applying (3.11) gives

$$A_i(W_H, W_H) \le cd\delta_i |I_iW - \gamma_i(W)|^2_{1/2,\partial\hat{\Omega}}.$$

Inequality (3.9) now follows from (3.6), which completes the proof of Proposition 1.

**4. The Construction and Analysis of $Q$.** In this section, we construct and analyze two forms $Q_j$ which gives rise to effective domain decomposition preconditioners for three-dimensional problems. It will be shown that for each of these forms, (3.6) holds with $\alpha_1(\hat{h}) \le C$ and $\alpha_0(\hat{h}) \ge c/(1 + \ln^2(\hat{h}^{-1}))$. Thus, by Proposition 1, the preconditioner $B$ defined by (3.4) using these $Q_j$ will give rise to preconditioned systems with condition number growth bounded by $C_0(1 + \ln^2(d/h))$. As will be demonstrated in Section 5, the first form $Q_1$ gives rise to a more efficient computational strategy and is hence the preferred method. We include the second form since it is, in some sense, the natural extension of the method of Part I to three dimensions.

We want to derive replacement forms for the norm $|\cdot|_{1/2,\partial\hat{\Omega}}$ on $S_{\hat{h}}(\partial\hat{\Omega})$. As in [14], [16], this norm is given by

$$|w|_{1/2,\partial\hat{\Omega}} = \left(\int_{\partial\hat{\Omega}}\int_{\partial\hat{\Omega}} \frac{(w(x) - w(y))^2}{|x-y|^3}\, ds(x)\, ds(y) + |w|^2_{\partial\hat{\Omega}}\right)^{1/2}, \tag{4.1}$$

where $s$ denotes area on $\partial\hat{\Omega}$. Let $\hat{\Gamma}_i^f$ be a face of $\hat{\Omega}$. The space $\mathring{\mathring{H}}^{1/2}(\hat{\Gamma}_i^f)$ is defined to be the completion of the smooth functions defined on $\partial\hat{\Omega}$ with support in $\hat{\Gamma}_i^f$ with respect to the norm given by (4.1).

*Remark* 4.1. It is well known that the space $\mathring{\mathring{H}}^{1/2}(\hat{\Gamma}_i^f)$ is the interpolation space which is halfway between $H_0^1(\hat{\Gamma}_i^f)$ and $L^2(\hat{\Gamma}_i^f)$ [14], [16]. For smooth functions with support in $\hat{\Gamma}_i^f$, $\langle -\Delta u, u\rangle^{1/2}_{\hat{\Gamma}_i^f}$ is equivalent to the norm on $H_0^1(\hat{\Gamma}_i^f)$ (here, $\Delta$ denotes the two-dimensional Laplacian on the face). Consequently, the completion of the norm given by

$$\left(\left\langle (-\Delta)^{1/2} w, w\right\rangle_{\hat{\Gamma}_i^f}\right)^{1/2} \quad \text{for } w \in H_0^1(\hat{\Gamma}_i^f) \tag{4.2}$$

is equivalent to the norm on $\mathring{\mathring{H}}^{1/2}(\hat{\Gamma}_i^f)$.

We shall use a discrete operator $l_0^{1/2}$ which approximates $(-\Delta)^{1/2}$ in the definitions of the computational forms $Q_1$ and $Q_2$. Let

$$S_{\hat{h}}^0(\hat{\Gamma}_i^f) \equiv \{\phi|_{\hat{\Gamma}_i^f} \text{ such that } \phi \in S_{\hat{h}}(\partial\hat{\Omega}) \text{ and } \phi = 0 \text{ on the edges of } \hat{\Omega}\}.$$

The discrete operator $l_0\colon S_{\hat{h}}^0(\hat{\Gamma}_i^f) \mapsto S_{\hat{h}}^0(\hat{\Gamma}_i^f)$ is defined by

$$\langle l_0\Psi, \Phi\rangle_{\hat{\Gamma}_i^f} = \int_{\hat{\Gamma}_i^f} \nabla\Psi \cdot \nabla\Phi\, ds \quad \text{for all } \Phi \in S_{\hat{h}}^0(\hat{\Gamma}_i^f). \tag{4.3}$$

Here, $\nabla$ denotes the two-dimensional gradient on $\hat{\Gamma}_i^f$. The operator $l_0$ is symmetric positive definite on $S_{\hat{h}}^0(\hat{\Gamma}_i^f)$ and $l_0^{1/2}$ is defined to be its positive square root.

*Remark* 4.2. Note that the discrete operator $l_0$ is a finite-dimensional approximation to $-\Delta$. It can be shown by interpolation [13, Theorem 9.1] and the inverse assumptions on $S_{\hat{h}}^0(\hat{\Gamma}_i^f)$ that

$$c\,|V|^2_{H^{1/2}(\hat{\Gamma}_i^f)} \le \left\langle l_0^{1/2}V, V\right\rangle_{\hat{\Gamma}_i^f} \le C\,|V|^2_{H^{1/2}(\hat{\Gamma}_i^f)} \quad \text{for all } V \in S_{\hat{h}}^0(\hat{\Gamma}_i^f). \tag{4.4}$$

The constants $c$ and $C$ in (4.4) can be chosen to be independent of $\hat{h}$.

We now construct the first form $Q_1$.

*Method* 1. We decompose functions $V \in S_{\hat{h}}(\partial\hat{\Omega})$ by $V = V_{e,1} + V_{f,1}$, where $V_{e,1}$ and $V_{f,1}$ satisfy:

(1) $V_{f,1} = 0$ on $\hat{\Gamma}^e \equiv \bigcup_{i=1}^6 \partial\hat{\Gamma}_i^f$.
(2) $V_{e,1} = 0$ on all nodes on the faces of $\hat{\Omega}$.

Define

$$Q_1(V,V) \equiv \hat{h} \sum_{x_i \in \hat{\Gamma}^e} V(x_i)^2 + \sum_{i=1}^6 \left\langle l_0^{1/2}V_{f,1}, V_{f,1}\right\rangle_{\hat{\Gamma}_i^f}. \tag{4.5}$$

The first sum in (4.5) is over the nodes $x_i$ on $\hat{\Gamma}^e$.

*Remark* 4.3. The quasi-uniformity of the mesh defined on $\hat{\Gamma}^e$ implies that

$$c\,\langle V,V\rangle_{\hat{\Gamma}^e} \le \hat{h} \sum_{x_i \in \hat{\Gamma}^e} V(x_i)^2 \le C\,\langle V,V\rangle_{\hat{\Gamma}^e}.$$

The construction of the second form $Q_2$ differs from the first only in the way that $V$ is decomposed.

*Method* 2. We define $V_{e,2}$ on $\partial\hat{\Omega}$ to be the function which equals $V$ on $\hat{\Gamma}^e$ and is discrete harmonic in the faces, i.e., for each face $\hat{\Gamma}_i^f$,

$$\int_{\hat{\Gamma}_i^f} \nabla V_{e,2} \cdot \nabla\Phi\, ds = 0 \quad \text{for all } \Phi \in S_{\hat{h}}^0(\hat{\Gamma}_i^f). \tag{4.6}$$

Again, we set $V = V_{e,2} + V_{f,2}$ and define

$$Q_2(V,V) \equiv \hat{h} \sum_{x_i \in \hat{\Gamma}^e} V(x_i)^2 + \sum_{i=1}^6 \left\langle l_0^{1/2}V_{f,2}, V_{f,2}\right\rangle_{\hat{\Gamma}_i^f}. \tag{4.7}$$

Note that the definitions (4.5) and (4.7) only differ in their respective use of $V_{f,1}$ and $V_{f,2}$. These constructions lead to completely different quadratic forms.

The following proposition provides estimates for $\alpha_0$ and $\alpha_1$ for the forms $Q_1$ and $Q_2$. Its proof will be given later in this section.

PROPOSITION 2. *For $j = 1, 2$, there are positive constants $c$ and $C$ which are independent of $\hat{h}$ and satisfy*

$$c(1+\ln^2(\hat{h}^{-1}))^{-1}Q_j(V,V) \le |V|^2_{1/2,\partial\hat{\Omega}} \le CQ_j(V,V) \quad \textit{for all } V \in S_{\hat{h}}(\partial\hat{\Omega}). \tag{4.8}$$

Combining Propositions 1 and 2 gives the following theorem.

THEOREM. *Let $B$ be defined by* (3.4) *with $Q = Q_1$ or $Q = Q_2$. Then there are constants $c$ and $C$ which are independent of $d$ and $h$ satisfying*

$$c(1+\ln^2(d/h))^{-1}B(W,W) \le A(W,W) \le CB(W,W) \quad \textit{for all } W \in S_h(\Omega).$$

*Remark* 4.4. A construction analogous to that used in Method 1 can be carried out in the two-dimensional case. This leads to a preconditioned system with condition number on the order of $\ln^2(d/h)$. Instead of the corner problem of the preconditioner of [6], this method requires the solution of a sparse system with the number of variables equal to the number of subdomains.

We next give a proof of Proposition 2. We shall start by stating some lemmas which are used in the proof. Two of these lemmas (Lemmas 4.2 and 4.3) represent a fundamental part of the analysis. We prove the proposition assuming the lemmas and then devote the remainder of the section to the proof of the lemmas.

LEMMA 4.1. *Let $V \in S_{\hat{h}}(\partial\hat{\Omega})$ and $V_e = V_{e,1}$ or $V_e = V_{e,2}$; then*

$$|V_e|^2_{1/2,\partial\hat{\Omega}} \le C\,\langle V,V\rangle_{\hat{\Gamma}^e}.$$

LEMMA 4.2. *Let $V \in S_{\hat{h}}(\partial\hat{\Omega})$; then*

$$\langle V,V\rangle_{\hat{\Gamma}^e} \le C(1+\ln(\hat{h}^{-1}))|V|^2_{1/2,\partial\hat{\Omega}}.$$

LEMMA 4.3. *Let $V \in S_{\hat{h}}(\partial\hat{\Omega})$ and $V_f = V_{f,1}$ or $V_f = V_{f,2}$. Then*

$$\left\langle l_0^{1/2}V_f, V_f\right\rangle_{\hat{\Gamma}^f_i} \le C(1+\ln^2(\hat{h}^{-1}))|V|^2_{1/2,\partial\hat{\Omega}}$$

*holds for every face $\hat{\Gamma}^f_i$ of $\partial\hat{\Omega}$.*

Assuming Lemmas 4.1–4.3, we can prove Proposition 2.

*Proof of Proposition* 2. Let $V$ in $S_{\hat{h}}(\partial\hat{\Omega})$ be decomposed into $V = V_e + V_f$. Then

$$|V|^2_{1/2,\partial\hat{\Omega}} \le 7\left(|V_e|^2_{1/2,\partial\hat{\Omega}} + \sum_{i=1}^{6}|V_f^i|^2_{1/2,\partial\hat{\Omega}}\right), \tag{4.9}$$

where $V_f^i$ is the function defined on $\partial\hat{\Omega}$ which equals $V_f$ on $\hat{\Gamma}^f_i$ and is zero on $\partial\hat{\Omega}/\hat{\Gamma}^f_i$. The second inequality of the proposition follows from (4.9), Lemma 4.1, Remarks 4.3, 4.2 and the definition of $\mathring{\mathring{H}}^{1/2}(\hat{\Gamma}^f_i)$. The first inequality of the proposition follows from Remark 4.3 and Lemmas 4.2 and 4.3.

We now proceed with the proof of the lemmas. Some of the details for the proofs in the case of Method 2 are somewhat technical. So as not to disturb the flow of the domain decomposition analysis, these details will be given in the Appendix.

*Proof of Lemma* 4.1 *for Method* 1. By convexity,

$$|V_{e,1}|^2_{1/2,\partial\hat{\Omega}} \le c\,|V_{e,1}|_{\partial\hat{\Omega}}\,|V_{e,1}|_{1,\partial\hat{\Omega}}. \tag{4.10}$$

Using the fact that $V_{e,1}$ vanishes on the nodes of $\partial\hat{\Omega}$ which are not on $\hat{\Gamma}^e$, a straightforward computation gives

$$|V_{e,1}|^2_{\partial\hat{\Omega}} \le C\hat{h}^2 \sum_{x_i\in\hat{\Gamma}^e} V_{e,1}(x_i)^2$$

and

$$|V_{e,1}|^2_{1,\partial\hat{\Omega}} \le C \sum_{x_i \in \hat{\Gamma}^e} V_{e,1}(x_i)^2.$$

The lemma for Method 1 then follows from Remark 4.3.

The proof of Lemma 4.1 in the case of Method 2 involves some technical estimates for functions which are discrete harmonic on the faces of $\partial\hat{\Omega}$ and will be given in the Appendix.

In preparation for the proofs of the remaining two lemmas, we state a certain type of two-dimensional discrete Sobolev inequality whose proof can be found in [3], [6]. Let $\tilde{S}_h$ be a subspace of approximating functions with mesh size $\tilde{h}$ defined on a two-dimensional domain $\tilde{\Omega}$ (in our applications, $\tilde{S}_h$ will be $S_{\hat{h}}(\hat{\Omega})$ restricted to some two-dimensional slice of $\hat{\Omega}$). We assume that $\tilde{\Omega}$ satisfies a cone condition of size $d$ and angle $\alpha$ bounded away from zero and that $\tilde{S}_h$ satisfies the following inverse inequality:

$$|\nabla V|_{L^\infty(\tilde{\Omega})} \le C_1 \tilde{h}^{-1} |V|_{L^\infty(\tilde{\Omega})} \quad \text{for all } V \in \tilde{S}_h. \tag{4.11}$$

Then there exists a positive constant $C$ independent of $\tilde{h}$ such that

$$|V|^2_{L^\infty(\tilde{\Omega})} \le C(1+\ln(\tilde{h}^{-1}))\, |V|^2_{1,\tilde{\Omega}} \quad \text{for all } V \in \tilde{S}_h. \tag{4.12}$$

*Proof of Lemma* 4.2. Let $V \in S_{\hat{h}}(\partial\hat{\Omega})$. Define $\tilde{V}$ to be the discrete harmonic extension of $V$ (into the interior of $\hat{\Omega}$). By (3.11), it suffices to show that

$$\langle V, V\rangle_{\hat{\Gamma}^e} \le c(1+\ln(\hat{h}^{-1}))\|\tilde{V}\|^2_{1,\hat{\Omega}}.$$

Without loss of generality, we consider the integral over that part of the edge which corresponds to $x = z = 0$. Then by (4.12),

$$\int_0^1 V^2(0,y,0)\,dy \le c(1+\ln(\hat{h}^{-1})) \int_0^1 \left|\tilde{V}(\cdot,y,\cdot)\right|^2_{H^1} dy. \tag{4.13}$$

The $H^1$ norm in the last integral of (4.13) is over the intersection of $\hat{\Omega}$ with the plane at the given $y$-value. This integral is clearly bounded by $\|\tilde{V}\|^2_{1,\hat{\Omega}}$, and hence the proof is complete.

*Proof of Lemma* 4.3. Much of the proof of the lemma is the same whether we are considering Method 1 or Method 2. Accordingly, let $V \in S_{\hat{h}}(\partial\hat{\Omega})$ be decomposed into $V = V_e + V_f$, where $V_e$ and $V_f$ are given by either Method 1 or Method 2. By Remark 4.2, it suffices to prove

$$|V_f|^2_{\mathring{H}^{1/2}(\hat{\Gamma}^f_i)} \le C(1+\ln^2(\hat{h}^{-1}))|V|^2_{1/2,\partial\hat{\Omega}}. \tag{4.14}$$

Let $w$ be the function defined on $\partial\hat{\Omega}$ which equals $V_f$ on $\hat{\Gamma}^f_i$ and is zero on $\partial\hat{\Omega}/\hat{\Gamma}^f_i$. Then the $\mathring{H}^{1/2}(\hat{\Gamma}^f_i)$ norm of $V_f$ is given by (4.1). The corresponding integral term in (4.1) reduces to

$$\int_{\hat{\Gamma}^f_i}\int_{\hat{\Gamma}^f_i} \frac{(V_f(x)-V_f(y))^2}{|x-y|^3}\,ds(x)\,ds(y) + 2\int_{\hat{\Gamma}^f_i}\int_{\partial\hat{\Omega}/\hat{\Gamma}^f_i} \frac{V_f(x)^2}{|x-y|^3}\,ds(x)\,ds(y).$$

Let the four edges of $\hat{\Gamma}^f_i$ be denoted $\Gamma^{f,e}_{i,j}$ for $j = 1,2,3,4$. A straightforward computation gives that

$$c\int_{\partial\hat{\Omega}/\hat{\Gamma}^f_i} |x-y|^{-3}\,ds(x) \le \sum_{j=1}^4 \mathrm{Dist}(y,\Gamma^{f,e}_{i,j})^{-1} \le C\int_{\partial\hat{\Omega}/\hat{\Gamma}^f_i} |x-y|^{-3}\,ds(x),$$

where $\mathrm{Dist}(y, \Gamma_{i,j}^{f,e})$ denotes the distance from $y$ to $\Gamma_{i,j}^{f,e}$. Thus, the quantity $|V_f|^2_{H^{1/2}(\hat{\Gamma}_i^f)}$ is equivalent to

$$\int_{\hat{\Gamma}_i^f}\int_{\hat{\Gamma}_i^f} \frac{(V_f(x)-V_f(y))^2}{|x-y|^3}\,ds(x)\,ds(y) + \sum_{j=1}^{4}\int_{\hat{\Gamma}_i^f} \frac{V_f(y)^2}{\mathrm{Dist}(y,\Gamma_{i,j}^{f,e})}\,ds(y). \tag{4.15}$$

To bound the double integral term in (4.15), it suffices to bound the square of the $H^{1/2}(\partial\hat{\Omega})$ norm of $V_f$. Lemmas 4.1 and 4.2 give

$$|V_f|^2_{1/2,\partial\hat{\Omega}} \le 2(|V|^2_{1/2,\partial\hat{\Omega}} + |V_e|^2_{1/2,\partial\hat{\Omega}}) \le C(1+\ln(\hat{h}^{-1}))|V|^2_{1/2,\partial\hat{\Omega}}. \tag{4.16}$$

Thus, to complete the proof of the theorem, we need only bound the single integral terms in (4.15).

Without loss of generality, it suffices to consider the face $\hat{\Gamma}_i^f$ in the plane $z=0$ and a typical term, for example

$$\int_0^1\int_0^1 \frac{V_f(x,y,0)^2}{x}\,dx\,dy. \tag{4.17}$$

Thus it suffices to prove

$$\begin{aligned}\int_0^1\int_0^{\hat{h}} \frac{V_f(x,y,0)^2}{x}\,dx\,dy + \int_0^1\int_{\hat{h}}^1 \frac{V_f(x,y,0)^2}{x}\,dx\,dy \\ \le c(1+\ln^2(\hat{h}^{-1}))|V|^2_{1/2,\partial\hat{\Omega}}.\end{aligned} \tag{4.18}$$

For the first term in (4.18), we have

$$\int_0^1\int_0^{\hat{h}} \frac{V_f(x,y,0)^2}{x}\,dx\,dy \le c\hat{h}^2\int_0^1 \left|\frac{\partial V_f(\cdot,y,0)}{\partial x}\right|^2_{L^\infty} dy.$$

Let $\tilde{V}_f$ be the discrete harmonic extension of $V_f$ into $\hat{\Omega}$. By an inverse property of the subspace $S_{\hat{h}}(\hat{\Omega})$ restricted to the plane $y=$ constant and (4.12),

$$\int_0^1\int_0^{\hat{h}} \frac{V_f(x,y,0)^2}{x}\,dx\,dy \le c(1+\ln(\hat{h}^{-1}))\int_0^1 \left|\tilde{V}_f(\cdot,y,\cdot)\right|^2_{H^1} dy. \tag{4.19}$$

The integral term in the right-hand side of (4.19) is clearly bounded by $\|\tilde{V}_f\|^2_{1,\hat{\Omega}}$. But $\tilde{V}_f$ is discrete harmonic and hence by (3.11) and (4.16),

$$\begin{aligned}\int_0^1\int_0^{\hat{h}} \frac{V_f(x,y,0)^2}{x}\,dx\,dy &\le c(1+\ln(\hat{h}^{-1}))|V_f|^2_{1/2,\partial\hat{\Omega}} \\ &\le C(1+\ln^2(\hat{h}^{-1}))|V|^2_{1/2,\partial\hat{\Omega}}.\end{aligned} \tag{4.20}$$

We next consider the second term of (4.18). For the case of Method 1, we have

$$\begin{aligned}&\int_0^1\int_{\hat{h}}^1 \frac{V_{f,1}(x,y,0)^2}{x}\,dx\,dy \\ &\quad\le \ln(\hat{h}^{-1})\int_0^1 |V_{f,1}(\cdot,y,0)|^2_{L^\infty}\,dy \\ &\quad\le 2\ln(\hat{h}^{-1})\left(\int_0^1 |V(\cdot,y,0)|^2_{L^\infty}\,dy + \int_0^1 |V_{e,1}(\cdot,y,0)|^2_{L^\infty}\,dy\right).\end{aligned} \tag{4.21}$$

Let $\tilde{V}$ denote the discrete harmonic extension of $V$ into $\hat{\Omega}$. By (4.12) and (3.11),

$$\int_0^1 |V(\cdot,y,0)|_{L^\infty}^2\, dy \le c(1+\ln(\hat{h}^{-1}))\|\tilde{V}\|_{1,\hat{\Omega}}^2 \le c(1+\ln(\hat{h}^{-1}))|V|_{1/2,\partial\hat{\Omega}}^2. \tag{4.22}$$

Since $V_{e,1}$ vanishes on the interior nodes of the faces,

$$|V_{e,1}(\cdot,y,0)|_{L^\infty}^2 \le V_{e,1}(0,y,0)^2 + V_{e,1}(1,y,0)^2.$$

Hence by Lemma 4.2,

$$\begin{aligned}\int_0^1 |V_{e,1}(\cdot,y,0)|_{L^\infty}^2\, dy &\le C\,\langle V_{e,1},V_{e,1}\rangle_{\hat{\Gamma}^e}\\ &\le c(1+\ln(\hat{h}^{-1}))|V|_{1/2,\partial\hat{\Omega}}^2.\end{aligned} \tag{4.23}$$

Inequality (4.18) follows from (4.20), (4.21), (4.22), and (4.23). This completes the proof of the lemma in the case of Method 1.

To complete the proof of the lemma, we have only to bound the second integral term in (4.18) in the case of Method 2. Applying the arithmetic geometric mean inequality, and changing the order of integration, gives

$$\begin{aligned}&\int_0^1\int_{\hat{h}}^1 \frac{V_{f,2}(x,y,0)^2}{x}\,dx\,dy\\ &\le 2\int_0^1\int_{\hat{h}}^1 \frac{V(x,y,0)^2}{x}\,dx\,dy + 2\int_{\hat{h}}^1\int_0^1 \frac{V_{e,2}(x,y,0)^2}{x}\,dy\,dx\\ &\le 2\ln(\hat{h}^{-1})\left(\int_0^1 |V(\cdot,y,0)|_{L^\infty}^2\,dy + \sup_{x\in[0,1]}\int_0^1 V_{e,2}(x,y,0)^2\,dy\right).\end{aligned} \tag{4.24}$$

In the Appendix, we shall show that

$$\sup_{x\in[0,1]}\int_0^1 V_{e,2}(x,y,0)^2\,dy \le c\,\langle V_{e,2},V_{e,2}\rangle_{\hat{\Gamma}^e}. \tag{4.25}$$

Applying (4.22) to the first term in (4.24) and (4.25) and Lemma 4.2 to the second gives the desired bound for the second term of (4.18). This completes the proof of the lemma.

**5. The Solution of the Preconditioning Problem (1.5).** In this section, we give an efficient algorithm for solving (1.5) in the case of Method 1. A similar algorithm can be developed for Method 2. Because of the discrete harmonic extensions on the faces, the algorithm for Method 2 is somewhat less efficient than that corresponding to Method 1 (see Remark 5.2).

In general, when $B$ is of the form (3.4), we solve first for $W_P$ on each subdomain, then for the values of $W_H$ on $\Gamma$, and finally extend $W_H$ to all of $\Omega$. Most of the ideas described in this section have appeared in our earlier papers. However, the application of these techniques is not transparent and hence we include a discussion here.

The solution of (1.5) involves a three-step procedure. As already mentioned, the problem of finding the solution $W$ to (1.5) reduces to that of computing $W_P$ and

$W_H$ on each subdomain. The first step is to compute $W_P$. By taking $\Phi \in S_h^0(\Omega_i)$ in (1.5) and using (3.4), it follows that

$$A_i(W_P, \Phi) = G(\Phi) \quad \text{for all } \Phi \in S_h^0(\Omega_i). \tag{5.1}$$

Thus, $W_P$ can be determined by solving independent discrete Dirichlet problems on the subregions. The second step involves the computation of the values of $W_H$ on $\Gamma$. These values are determined as the solution of the problem

$$\begin{aligned} d\sum_{i=1}^{m} \delta_i Q(I_i W - \gamma_i(W), \theta) \\ = G(\tilde{\theta}) - A(W_P, \tilde{\theta}) \equiv F(\theta) \quad \text{for all } \theta \in S_h(\Gamma). \end{aligned} \tag{5.2}$$

Here, $\tilde{\theta}$ denotes any extension of $\theta$ in $S_h(\Omega)$, and $\{\gamma_i(W)\}$ are the constants defined by (3.5). Note that by (5.1), the right-hand side of (5.2) is independent of the particular extension $\tilde{\theta}$. The development of an algorithm for solving (5.2) is an important part of this section and will be considered shortly. Assuming the values of $W_H$ on $\Gamma$ have been computed, the third step is to compute the 'discrete $A_i$-harmonic' extension into the interior of the subdomains. This is done separately on each subdomain as follows: Let $\tilde{W}_H$ be any extension of the boundary values of $W_H$ in $S_h(\Omega_i)$, e.g., the extension which is zero at all of the nodes not on $\partial\Omega_i$. Then on $\Omega_i$, $W_H = Y + \tilde{W}_H$, where $Y \in S_h^0(\Omega_i)$ is the solution of

$$A_i(Y, \Phi) = -A_i(\tilde{W}_H, \Phi) \quad \text{for all } \Phi \in S_h^0(\Omega_i). \tag{5.3}$$

Thus, the computation of $W_H$ (once its values on $\Gamma$ are known) reduces to the solution of independent discrete Dirichlet problems on the subdomains.

To complete the description of the algorithm, we provide an efficient way to compute the function $W_H$ on $\Gamma$, i.e., the solution of (5.2). This involves a two-step procedure. The first step requires the computation of the average values $\{\gamma_i(W)\}$ appearing in (5.2). We shall use a technique described in [7] to derive a sparse matrix problem for these values. We will only consider the case when all of the $\delta_i$'s are equal to one; the more general case is similar (cf. [7]). This matrix is derived using a special choice of test functions, $\phi_i \in S_h(\Gamma)$, for $i = 1, \ldots, m$. Consider a fixed $i$. We define the function $\phi_i$ to vanish on the nodes which are not on $\partial\Omega_i$. Its values on the nodes of $\partial\Omega_i$ are to be determined.

Let $X$ and $Y$ be in $S_h(\Omega)$ and $\Gamma_{ij}^f = \Gamma_{kl}^f$; then by (A.3) and (A.4),

$$\left\langle l_0^{1/2}(I_i X)_f, (I_i Y)_f \right\rangle_{\hat{\Gamma}_j^f} = \left\langle l_0^{1/2}(I_k X)_f, (I_k Y)_f \right\rangle_{\hat{\Gamma}_l^f}. \tag{5.4}$$

Here, $(\cdot)_f$ denotes the face component in the decomposition of Method 1. Let $\mathcal{N}(x_j)$ be the indices of the subregions which share a boundary node $x_j$ and $\mathcal{N}(\Gamma_{ij}^f)$ be the indices of the two subregions which share a boundary face $\Gamma_{ij}^f$. The number of indices in $\mathcal{N}(x_j)$ will be denoted $|\mathcal{N}(x_j)|$. Then (5.4) and the properties of $\phi_i$

imply

$$
\begin{aligned}
d\sum_{j=1}^{m} Q(I_jW-\gamma_j(W),\phi_i) &= d\hat{h}\sum_{x_j\in\Gamma_i^e}|\mathcal{N}(x_j)|\,W(x_j)\phi_i(x_j)\\
&\quad + 2d\sum_{j=1}^{6}\left\langle l_0^{1/2}(I_iW)_f,(I_i\phi_i)_f\right\rangle_{\hat{\Gamma}_j^f}\\
&\quad - d\hat{h}\sum_{x_j\in\Gamma_i^e}\Bigg(\sum_{k\in\mathcal{N}(x_j)}\gamma_k(W)\Bigg)\phi_i(x_j)\\
&\quad - d\sum_{j=1}^{6}\Bigg(\sum_{l\in\mathcal{N}(\Gamma_{ij}^f)}\gamma_l(W)\Bigg)\left\langle l_0^{1/2}(1)_f,(I_i\phi_i)_f\right\rangle_{\hat{\Gamma}_j^f}\\
&\equiv S_1+S_2+S_3+S_4,
\end{aligned}
\tag{5.5}
$$

where $\{x_j\}$ are the nodal values on $\Gamma_i^e=\bigcup_{j=1}^6\partial\Gamma_{ij}^f$.

We define $\phi_i$ at the nodal values on $\partial\Omega_i$ by

$$
\phi_i(x_j)=\begin{cases}1/|\mathcal{N}(x_j)| & \text{when } x_j\in\Gamma_i^e,\\ 1/2 & \text{when } x_j\in\Gamma_{ik}^f.\end{cases}
\tag{5.6}
$$

Then, by (3.5), the first two sums of (5.5) can be written

$$
S_1+S_2=dQ(I_iW,1)=\gamma_i(W)dQ(1,1).
\tag{5.7}
$$

Combining (5.5) and (5.7) gives

$$
dQ(1,1)\gamma_i(W)-M_{ik}\gamma_k(W)=F(\phi_i),
\tag{5.8}
$$

where

$$
M_{ik}=d\hat{h}\sum_{x_j\in\Gamma_i^e\cap\Gamma_k^e}\frac{1}{|\mathcal{N}(x_j)|}+d\sum_{\substack{j=1,\dots,6\\ \Gamma_{ij}^f\cap\partial\Omega_k\neq\varnothing}}\frac{\left\langle l_0^{1/2}(1)_f,(1)_f\right\rangle_{\hat{\Gamma}_j^f}}{2}.
$$

It is straightforward to check that $M$ is symmetric with nonnegative entries. Furthermore, the row sum of $M$ for any row is less than or equal to $dQ(1,1)$, with strict inequality when the row corresponds to a domain $\Omega_i$ with $\partial\Omega_i\cap\partial\Omega\neq\varnothing$. This means that $dQ(1,1)I-M$ (where $I$ denotes the $m\times m$ identity matrix) is an $M$-matrix [17] and resembles matrices arising in standard finite difference methods. We compute the values of $\{\gamma_i(W)\}$ by solving this $m\times m$ system.

*Remark* 5.1. In the case of many subdomains, the matrix $dQ(1,1)I-M$ is sparse since the $i$th equation only involves the values of $\gamma_k(W)$ for subdomains $\Omega_k$ with $\partial\Omega_i\cap\partial\Omega_j\neq\varnothing$.

Once the values of $\{\gamma_k(W)\}$ are known, we compute the values of $W_H$ on $\Gamma$ as follows. We are left to solve

$$
d\sum_{i=1}^{m}Q(I_iW,\theta)=F(\theta)+d\sum_{i=1}^{m}Q(\gamma_i(W),\theta)\equiv\tilde{F}(\theta)\quad\text{for all }\theta\in S_h(\Gamma).
\tag{5.9}
$$

By (5.4), we have

$$
\begin{aligned}
d\sum_{i=1}^{m} Q(I_iW,\theta) &= d\hat{h}\sum_{x_j\in\Gamma^e} |\mathcal{N}(x_j)|\,W(x_j)\theta(x_j) \\
&\quad + d\sum_{k,l}\left\langle l_0^{1/2}(I_kW)_f,(I_k\theta)_f\right\rangle_{\hat{\Gamma}_l^f} \qquad \text{for all } \theta\in S_h(\Gamma).
\end{aligned}
\tag{5.10}
$$

Here, $\Gamma^e$ is the union of the closures of the edges of the subdomains. For functions $\theta$ with support contained on the $j$th face of the $i$th subregion, (5.9)–(5.10) reduces to

$$
2d\left\langle l_0^{1/2}(I_iW)_f,(I_i\theta)_f\right\rangle_{\hat{\Gamma}_j^f} = \tilde{F}(\theta). \tag{5.11}
$$

Equation (5.11) completely determines the values of $W$ on the nodes of $\Gamma_{ij}^f$. For functions $\theta$ which vanish on the face nodes, (5.9)–(5.10) reduces to

$$
d\hat{h}\sum_{x_j\in\Gamma^e} |\mathcal{N}(x_j)|\,W(x_j)\theta(x_j) = \tilde{F}(\theta). \tag{5.12}
$$

The nodal values of $W$ on $\Gamma^e$ are trivially computed from (5.12) using $\theta$ corresponding to nodal basis functions.

*Remark* 5.2. An algorithm similar to that described above could be developed for the solution of the preconditioning form $B$ defined using $Q_2$ (i.e., Method 2). In fact, if $I_i\phi_i$ is discrete harmonic on the faces, (5.5) gets replaced by

$$
\begin{aligned}
d\sum_{i=1}^{m} Q(I_iW-\gamma_i(W),\phi_i) &= d\hat{h}\sum_{x_j\in\Gamma_i^e} |\mathcal{N}(x_j)|\,W(x_j)\phi_i(x_j) \\
&\quad - d\hat{h}\sum_{x_j\in\Gamma_i^e}\Bigg(\sum_{k\in\mathcal{N}(x_j)}\gamma_k(W)\Bigg)\phi_i(x_j).
\end{aligned}
$$

For this case, $\phi_i(x_j)$ is defined by (5.6) for $x_j\in\Gamma_i^e$ and extended discrete harmonically into the faces. An equation for the values of $\gamma_i(W)$ analogous to (5.8) can then be derived. As above, once the values of $\gamma_i(W)$ have been computed, we are left to solve (5.9). (5.10) gets replaced by

$$
\begin{aligned}
d\sum_{i=1}^{m} Q(I_iW,\theta) &= d\hat{h}\sum_{x_j\in\Gamma^e} |\mathcal{N}(x_j)|\,W(x_j)\theta(x_j) \\
&\quad + d\sum_{k,l}\left\langle l_0^{1/2}(I_kW)_{f,2},(I_k\theta)_{f,2}\right\rangle_{\hat{\Gamma}_l^f} \qquad \text{for all } \theta\in S_h(\Gamma).
\end{aligned}
\tag{5.13}
$$

The values of $(I_kW)_{f,2}$ can then be computed on the faces using equations similar to (5.11). In the case of Method 2, (5.12) is only valid for functions $\theta$ for which $I_i\theta$ is discrete harmonic on all faces of $\partial\hat{\Omega}$ for all $i$. Thus the discrete harmonic extension into the faces must be computed for each edge nodal function (i.e., a function which is one on one of the edge nodes and vanishes on the remaining edge nodes). Even if these extensions are preprocessed, one must compute $\tilde{F}$ applied to each of these for each inversion of $B$. This results in a work increase of $O(N)$ operations and substantially complicates the coding.

We conclude this section with a review of the procedure developed here for solving (1.5) when $B$ is given by (3.4) and $Q$ is given by Method 1.

**Algorithm for Solving (1.5).**

(1) Compute $W_P$ by solving (5.1). This involves the solution of Dirichlet problems on the subdomains, which can be done independently and in parallel.
(2) Compute the values of $W_H$ on $\Gamma$ solving (5.2). First, we compute the values $\{\gamma_i(W)\}$ by solving the matrix problem (5.8). The values of $W_H$ on $\Gamma$ are then computed by (5.11) and (5.12).
(3) Extend the boundary values of $W_H$ by solving (5.3). As in Step 1, this involves the solution of Dirichlet problems on the subdomains, which can be done independently and in parallel.
(4) Set $W = W_P + W_H$.

**6. Numerical Experiments.** In this section, we present the results of numerical experiments using the preconditioners developed earlier. We shall only report results for the more computationally effective algorithms resulting from Method 1. We have made no attempt to develop a general code, and consequently our results will be for model applications. These computations are designed to illustrate the theory developed in the earlier sections.

The domain $\Omega$ will be the unit cube partitioned into $m = m_0 \times m_0 \times m_0$ subdomains which are subcubes of side length $1/m_0$. We will use a finite difference approximation on a grid of size $k \times k \times k$. Let $h = 1/(k+1)$ and $J = (j_1, j_2, j_3)$ be a multi-integer. Then the nodes of the grid are the points $x_J = (j_1 h, j_2 h, j_3 h)$ for $1 \le j_1, j_2, j_3 \le k$.

*Example* 1. For the first example, we consider the model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Here, $\Delta$ denotes the Laplace operator. The finite difference approximation to $u$ is the nodal function $U$ which satisfies (2.4). In this case

$$A(V, W) = (L_h V) \cdot W,$$

where $L_h$ is the seven-point difference operator given by

$$\begin{aligned} (L_h V)_J = 6V_{j_1,j_2,j_3} &- V_{j_1+1,j_2,j_3} - V_{j_1-1,j_2,j_3} - V_{j_1,j_2+1,j_3} \\ &- V_{j_1,j_2-1,j_3} - V_{j_1,j_2,j_3+1} - V_{j_1,j_2,j_3-1}. \end{aligned} \tag{6.1}$$

We define $V_K = 0$ for indices $K$ appearing on the right-hand side of (6.1) which are not in $\Omega$.

For this example, the nodes on the faces of the subdomains are regularly spaced. We note that the definition of Method 1 and Method 2 only requires the computation of $l_0^{1/2}$ on the reference element with respect to the reference subspace. Because of the uniformly spaced grid on the faces, $l_0^{1/2}$ can be economically computed by use of the discrete Fourier transform. In addition, it is possible to replace $l_0^{1/2}$ on this subspace by any uniformly spectrally equivalent operator. For example, $l_0^{1/2}$ could be replaced by $\tilde{l}_0^{1/2}$, where $\tilde{l}_0^{1/2}$ is $\hat{h}$ times the square root of the five-point operator on the face. We use $\tilde{l}_0^{1/2}$ in the numerical examples of this section.

Table 6.1 gives computational results for Example 1. In this case, the cube was broken up into eight subcubes ($m$=2). We report the condition number $K$ for the preconditioned system as a function of $h$. For comparison, we provide the function

$$f(d/h) = 10.9 + .76 \log_2^2(d/h).$$

The close correlation between $K$ and $f(1/2h)$ suggests that the growth of the condition number of the preconditioned system is in agreement with the theorem in Section 4. We have also included the number of nodes, $N$, and the number of iterations, $N_i$, of preconditioned conjugate gradient required to reduce the $A$-norm error of a typical example by a factor of .001.

TABLE 6.1
*Iterative convergence for Example* 1.

| $h$ | $K$ | $f(1/2h)$ | $N_i$ | $N$ |
|---|---|---|---|---|
| 1/4 | 10.5 | 11.7 | 7 | 27 |
| 1/8 | 13.9 | 13.9 | 8 | 343 |
| 1/16 | 17.7 | 17.7 | 8 | 3375 |
| 1/32 | 23 | 23 | 7 | 29791 |

*Example* 2. In this example, we consider a variable coefficient problem which has large jumps in the coefficients across the subdomain boundaries. Specifically, we consider the problem

$$-\nabla \cdot \mu \nabla u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

For this example, we consider the unit cube broken down into twenty-seven subdomains. The function $\mu$ is piecewise constant on the subregions with values given by Figure 6.1. Table 6.2 gives the results of computational experiments for this example. Note that the results for the condition number $K$ of the preconditioned system

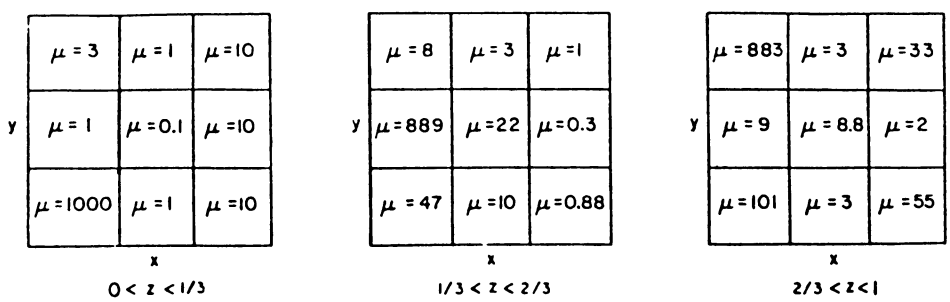


FIGURE 6.1
*Coefficients for Example* 2.

are of the same magnitude as those of Table 6.1. This suggests that the method gives rise to convergence rates which are independent of jumps in coefficients across the subregions. This is in agreement with the analysis since, with an appropriate choice of $\{\delta_i\}$, the constants $c$ and $C$ in (3.3) can be chosen independent of such jumps.

TABLE 6.2
*Iterative convergence for Example 2.*

| $h$ | $K$ | $f(1/3h)$ | $N_i$ | $N$ |
|---|---|---|---|---|
| 1/6 | 11.6 | 11.7 | 11 | 125 |
| 1/12 | 14.1 | 13.9 | 10 | 1331 |
| 1/24 | 18.3 | 17.7 | 10 | 12167 |

**7. Appendix.** We first prove Lemma 4.1 in the case of Method 2. To this end, we prove auxiliary lemmas involving harmonic functions. Then the proof of Lemma 4.1 will follow from approximation.

We will use the integral representation given in (4.1) for estimating the $H^{1/2}(\partial\hat{\Omega})$ norm. Let $\theta_1$ and $\theta_2$ be two-dimensional domains and $u$ be defined on $\theta_1 \cup \theta_2$. We define

$$I(\theta_1, \theta_2, u) = \int_{\theta_1}\int_{\theta_2} \frac{(u(x)-u(y))^2}{|x-y|^3}\, dx\, dy. \tag{7.1}$$

The first auxiliary lemma will involve the domain $\tilde{\Omega} = [-1,1] \times [0,1]$. Let $\tilde{\Omega}_1 = [-1,0] \times [0,1]$, $\tilde{\Omega}_2 = [0,1] \times [0,1]$ and $\tilde{\Gamma} = \partial\tilde{\Omega}_1 \cup \partial\tilde{\Omega}_2$.

LEMMA 7.1. *Let $u \in H^{1/2}(\partial\Omega)$ be harmonic in $\tilde{\Omega}_i$ for $i = 1, 2$. Then*

$$|u|^2_{1/2,\tilde{\Omega}} \le c\, |u|^2_{\tilde{\Gamma}}\,. \tag{7.2}$$

*Proof.* We define

$$u_e(x,y) = (u(x,y) + u(-x,y))/2$$

and

$$u_o(x,y) = (u(x,y) - u(-x,y))/2.$$

Now $u = u_e + u_o$ gives an orthogonal decomposition of $u$ in the $L^2(\tilde{\Gamma})$-inner product. Consequently, it suffices to prove (7.2) for $u = u_e$ and $u = u_o$. By the Schwarz reflection principle, $u_o$ is harmonic in $\tilde{\Omega}$ and hence

$$|u_o|^2_{1/2,\tilde{\Omega}} \le c\, |u_o|^2_{\partial\tilde{\Omega}} \le c\, |u_o|^2_{\tilde{\Gamma}}\,.$$

By a representation analogous to (4.1),

$$\begin{aligned} |u_e|^2_{1/2,\tilde{\Omega}} &= I(\tilde{\Omega}, \tilde{\Omega}, u_e) + |u_e|^2_{\tilde{\Omega}} \\ &= 2I(\tilde{\Omega}_1, \tilde{\Omega}_1, u_e) + 2I(\tilde{\Omega}_1, \tilde{\Omega}_2, u_e) + |u_e|^2_{\tilde{\Omega}}\,. \end{aligned}$$

Since $|x-y| \ge |(-x_1, x_2) - y|$ holds when $x \in \tilde{\Omega}_1$ and $y \in \tilde{\Omega}_2$,

$$I(\tilde{\Omega}_1, \tilde{\Omega}_2, u_e) \le I(\tilde{\Omega}_1, \tilde{\Omega}_1, u_e).$$

Hence,

$$|u_e|^2_{1/2,\tilde{\Omega}} \le c\, |u_e|^2_{1/2,\tilde{\Omega}_1} \le c\, |u_e|^2_{\partial\tilde{\Omega}_1} \le c\, |u_e|^2_{\tilde{\Gamma}}\,.$$

This completes the proof of the lemma.

The following lemma gives the result corresponding to Lemma 4.1 for functions which are harmonic on the faces of $\partial\hat{\Omega}$.

LEMMA 7.2. *Let $w \in H^{1/2}(\partial\Omega)$ be a function which is harmonic on each face of $\partial\hat{\Omega}$. Then*

$$|w|^2_{1/2,\partial\hat{\Omega}} \le c\,\langle w, w\rangle_{\hat{\Gamma}^e}\,.$$

*Proof.* We again use (4.1) to bound the $H^{1/2}(\partial\hat{\Omega})$ norm. The integral term of (4.1) is given by

$$I(\partial\hat{\Omega}, \partial\hat{\Omega}, w) = \sum_{i,j=1}^{6} I(\hat{\Gamma}^f_i, \hat{\Gamma}^f_j, w).$$

Let $\omega_i$ be the union of the two faces adjacent to the $i$th edge. If two faces $\hat{\Gamma}^f_i$ and $\hat{\Gamma}^f_j$ do not share an edge, then

$$I(\hat{\Gamma}^f_i, \hat{\Gamma}^f_j, w) \le c\,|w|^2_{\partial\hat{\Omega}}$$

and hence

$$|w|^2_{1/2,\partial\hat{\Omega}} \le c\left(\sum_{i=1}^{12} I(\omega_i, \omega_i, w) + |w|^2_{\partial\hat{\Omega}}\right). \tag{7.3}$$

The lemma follows from (7.3) and Lemma 7.1.

*Proof of Lemma* 4.1 *for Method* 2. Let $V$ be a function which is discrete harmonic on the faces of $\partial\hat{\Omega}$. Let $v$ be the function which equals $V$ on $\hat{\Gamma}^e$ and is harmonic on the faces of $\partial\hat{\Omega}$. By Lemma 7.2, it obviously suffices to show that

$$|V - v|^2_{1/2,\partial\hat{\Omega}} \le c\,|V|^2_{\hat{\Gamma}^e}\,.$$

By convexity,

$$|V - v|^2_{1/2,\partial\hat{\Omega}} \le c\,|V - v|_{\partial\hat{\Omega}}\,|V - v|_{1,\partial\hat{\Omega}}\,. \tag{7.4}$$

Applying well-known finite element techniques to estimate $|V - v|_{\partial\hat{\Omega}}$ and the Poincaré inequality gives

$$|V - v|^2_{1/2,\partial\hat{\Omega}} \le c\hat{h}D_{\partial\hat{\Omega}}(V - v, V - v) \le c\hat{h}D_{\partial\hat{\Omega}}(\tilde{V} - v, \tilde{V} - v), \tag{7.5}$$

where $\tilde{V}$ is the function in $S_{\hat{h}}(\hat{\Omega})$ which equals $V$ on $\hat{\Gamma}^e$ and vanishes on the face nodes and $D_{\partial\hat{\Omega}}(\cdot,\cdot)$ denotes the Dirichlet inner product on $\partial\hat{\Omega}$. By the arithmetic geometric mean inequality, an inequality similar to (3.11), inverse assumptions and an obvious computation,

$$|V - v|^2_{1/2,\partial\hat{\Omega}} \le c\hat{h}\sum_{i=1}^{6}\{|V|^2_{1/2,\partial\hat{\Gamma}^f_i} + D_{\hat{\Gamma}^f_i}(\tilde{V}, \tilde{V})\} \le C\,|V|^2_{\hat{\Gamma}^e}\,.$$

This completes the proof of the lemma.

We next prove (4.25). To do this, we first prove the analogous result for harmonic functions. The discrete result will then be derived by approximation.

LEMMA 7.3. *Let $u$ be harmonic on the face $\hat{\Gamma}_i^f$ in the plane $z = 0$. Then*

$$\sup_{x\in[0,1]} \int_0^1 u^2(x,y,0)\,dy \le C \int_{\partial\hat{\Gamma}_i^f} u^2\,ds. \tag{7.6}$$

*Proof.* Let $u = \sum_{j=1}^4 u_j$, where $u_j$ is the harmonic function which equals $u$ on $\Gamma_{i,j}^{f,e}$ and vanishes on the remaining three edges. By the arithmetic geometric mean inequality and obvious properties of the integral, it suffices to prove that

$$\sup_{x\in[0,1]} \int_0^1 u_j^2(x,y,0)\,dy \le C \int_{\Gamma_{i,j}^{f,e}} u_j^2\,ds$$

holds for $j = 1,2,3,4$. By obvious symmetries involving the $\{u_j\}$, it suffices to show that

$$\sup_{x\in[0,1]} \int_0^1 u_j^2(x,y,0)\,dy + \sup_{y\in[0,1]} \int_0^1 u_j^2(x,y,0)\,dx \le C \int_{\Gamma_{i,j}^{f,e}} u_j^2\,ds \tag{7.7}$$

holds for any $j$. Without loss of generality, we consider $u_1$, the function which is nonzero on the line $x = 1$. Expanding $u_1$ in a sine series gives

$$u_1(x,y,0) = \sum_{k=1}^{\infty} \alpha_k \sin(\pi k y) \left( \frac{e^{\pi k x} - e^{-\pi k x}}{e^{\pi k} - e^{-\pi k}} \right). \tag{7.8}$$

We consider the two terms of (7.7) separately. For the first, we use (7.8) and the Plancherel Theorem to get

$$\begin{aligned} \sup_{x\in[0,1]} \int_0^1 u_1^2(x,y,0)\,dy &= 1/2 \sup_{x\in[0,1]} \sum_{k=1}^{\infty} \alpha_k^2 \left( \frac{e^{\pi k x} - e^{-\pi k x}}{e^{\pi k} - e^{-\pi k}} \right)^2 \\ &\le 1/2 \sum_{k=1}^{\infty} \alpha_k^2 = \int_0^1 u_1(1,y,0)^2\,dy. \end{aligned}$$

For the second term of (7.7), using (7.8) and changing the order of summation and integration gives

$$\begin{aligned} &\sup_{y\in[0,1]} \int_0^1 u_1^2(x,y,0)\,dx \\ &\quad \le \sum_{k,l=1}^{\infty} |\alpha_k|\,|\alpha_l| \int_0^1 \left( \frac{e^{\pi k x} - e^{-\pi k x}}{e^{\pi k} - e^{-\pi k}} \right) \left( \frac{e^{\pi l x} - e^{-\pi l x}}{e^{\pi l} - e^{-\pi l}} \right) dx. \end{aligned} \tag{7.9}$$

We clearly have that

$$\left( \frac{e^{\pi k x} - e^{-\pi k x}}{e^{\pi k} - e^{-\pi k}} \right) \le \frac{2}{1 - e^{-2\pi}} e^{\pi k (x-1)}$$

and hence

$$\int_0^1 \left( \frac{e^{\pi k x} - e^{-\pi k x}}{e^{\pi k} - e^{-\pi k}} \right) \left( \frac{e^{\pi l x} - e^{-\pi l x}}{e^{\pi l} - e^{-\pi l}} \right) dx \le \frac{C}{k+l}. \tag{7.10}$$

Combining the above inequalities gives

$$\sup_{y\in[0,1]} \int_0^1 u_1^2(x,y,0)\,dx \le c \sum_{k,l=1}^{\infty} \frac{|\alpha_k|\,|\alpha_l|}{k+l}.$$

Applying Hilbert's Double Series Theorem (cf. [12]) finally gives

$$\sup_{y\in[0,1]} \int_0^1 u_1^2(x,y,0)\,dx \le C \sum_{k=1}^{\infty} \alpha_k^2 \le C \int_0^1 u_1(1,y,0)^2\,dy. \tag{7.11}$$

This completes the proof of the lemma.

*Proof of* (4.25). We must prove that (7.6) holds for functions $U \in S_{\hat{h}}(\partial\hat{\Omega})$ which are discrete harmonic on the face $\hat{\Gamma}_i^f$. Let $u$ be the function which equals $U$ on $\partial\hat{\Gamma}_i^f$ and is harmonic on $\hat{\Gamma}_i^f$. By the arithmetic geometric mean inequality and Lemma 7.3, it suffices to show that

$$\int_0^1 (u(x,y,0) - U(x,y,0))^2\,dy \le c \int_{\partial\hat{\Gamma}_i^f} U^2\,ds$$

holds for any $x \in [0,1]$. By standard finite element techniques, the Poincaré and trace inequalities,

$$\begin{aligned}\int_0^1 (u(x,y,0) - U(x,y,0))^2\,dy &\le c\hat{h}\,|u-U|^2_{1/2,\partial\hat{\Gamma}_i^f} \\ &\le c\hat{h}\,|u-U|^2_{1,\partial\hat{\Omega}} \le C\hat{h}D_{\partial\hat{\Omega}}(u-U,u-U).\end{aligned}$$

Inequality (4.25) follows from the argument used to bound (7.5) in the proof of Lemma 4.1 for Method 2.

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