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Uniform Enclosure of High Order for Boundary Value Problems by Monotone Discretization*

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Abstract. In the investigation of boundary value problems the construction of a two-sided inclusion of the solution can be as important as a numerical approximation of the solution itself. In the present paper we analyze a monotone discretization technique of higher order based upon piecewise interpolation and shifting such that bounding upper and lower solutions are obtained. The monotone discretization under consideration takes advantage of the property of the operator to be of monotone kind.

1. Introduction. In the investigation of boundary value problems the construction of a two-sided inclusion of the solution can be as important as a numerical approximation of the solution itself. In the present paper we analyze a monotone discretization technique of higher order based upon piecewise interpolation and shifting such that bounding upper and lower solutions are obtained. Therefore no a priori information is needed. The principle of monotone discretization and a detailed investigation of the first-order technique can be found in [6].

The monotone discretization under consideration takes advantage of the property of the operator to be of monotone kind. This property is guaranteed by weak maximum principles. Using these principles, discretization techniques which produce enclosing lower and upper solutions are discussed in [1], [2], [4], [9], [10]. However, the approaches used in these papers are different from that adopted here, in that the enclosure is constructed by means of correction terms, or using semi-infinite programming techniques, or adapting free parameters in specific representations of the discrete solution.

For monotone iteration schemes in discrete systems, see for example [11]. A combination of monotone discretization with monotone iteration techniques is given in [5].

2. The Basic Principle of Monotone Discretization to Generate Uniform Enclosures. Throughout this paper we deal with the following type of weakly nonlinear two-point boundary value problems:

$$\begin{aligned} &-u''(x) + g(x, u(x)) = 0 \quad \text{in } \Omega := (a, b), \\ &u(a) = u(b) = 0. \end{aligned} \tag{2.1}$$

Here, $\Omega \subset R$ denotes a given interval and $g\colon \overline{\Omega} \times R \to R$ denotes some continuous differentiable function satisfying

$$g(x,t) \le g(x,s) \quad \forall x \in \overline{\Omega},\ t \le s. \tag{2.2}$$

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Under these conditions, problem (2.1) possesses a unique solution. Furthermore, property (2.2) guarantees the following monotonicity principle to hold (see, e.g., [9]).

LEMMA 2.1. *Let $u, v \in C^1(\overline{\Omega})$ be piecewise twice continuously differentiable. Then*

$$-u''(x) + g(x, u(x)) \leq -v''(x) + g(x, v(x)) \quad a.e. \ in\ \Omega,$$
$$u(a) \leq v(a), \qquad u(b) \leq v(b)$$

implies $u(x) \leq v(x)$ $\forall x \in \overline{\Omega}$.

The method of monotone discretization (see [5], [6]) rests on two facts:

(i) First, the nonlinear function $g(x, u(x))$ is replaced by a piecewise defined simple function such that the modified problem has a known analytic solution.

(ii) Second, the function replacing the nonlinearity in (2.1) is chosen to underestimate or to overestimate the original one.

We remark that the first principle is used to obtain a finite-dimensional representation, i.e., a discretization. The second principle, in combination with Lemma 2.1, guarantees the enclosing property of the generated discrete solution.

Let some grid $Z(\overline{\Omega}) := \{x_i : i = 0(1)N\}$ be given on the interval $\overline{\Omega}$, i.e.,

$$a =: x_0 < x_1 < \cdots < x_{N-1} < x_N := b.$$

The corresponding step sizes and subintervals we denote by

$$h_i := x_i - x_{i-1} \quad \text{and} \quad \Omega_i := (x_{i-1}, x_i), \qquad i = 1(1)N.$$

We introduce the space $C_h(\overline{\Omega})$ of piecewise continuous functions on Ω with discontinuities only at the grid points $\{x_i\}$, i.e.,

$$C_h(\overline{\Omega}) := \{v \in L_2(\Omega) : v|_{\Omega_i} \in C(\overline{\Omega}_i),\ i = 1(1)N\}$$

equipped with the norm

$$|||v||| := \max_{1 \leq i \leq N} \|v|_{\Omega_i}\|_{C(\overline{\Omega}_i)} = \operatorname*{ess\,sup}_{x \in \Omega} |v(x)|.$$

Let $G : C(\overline{\Omega}) \to C(\overline{\Omega})$ denote the Nemyckij operator related to the function $g(\cdot, \cdot)$, i.e.,

$$[Gv](x) := g(x, v(x)) \quad \forall x \in \overline{\Omega}.$$

Definition 2.1. Operators $\underline{G}_h, \overline{G}_h : C(\overline{\Omega}) \to C_h(\overline{\Omega})$ are called bounding operators for G, if the inequalities

$$\underline{G}_h v \geq Gv \geq \overline{G}_h v \quad \text{for all } v \in C(\overline{\Omega}) \tag{2.3}$$

hold.

Here in (2.3), as well as later on, the semiordering is defined naturally, i.e., the inequality holds for almost all arguments $x \in \Omega$. The index h used for the bounding operators characterizes the mesh size of the discretization grid, $h = \max_{1 \leq i \leq N} h_i$.

In order to separate the investigations of the bounding property and the properties influenced by the nonlinearity g, we suppose the bounding operators to be constructed via continuous mappings

$$\begin{aligned} &\underline{P}_h, \overline{P}_h : C(\overline{\Omega}) \to C_h(\overline{\Omega}), \\ &\underline{P}_h w \geq w \geq \overline{P}_h w \quad \text{for all } w \in C(\overline{\Omega}) \end{aligned} \tag{2.4}$$

by

$$\underline{G}_h := \underline{P}_h G \quad \text{and} \quad \overline{G}_h := \overline{P}_h G. \tag{2.5}$$

Obviously, operators defined by (2.4) and (2.5) are bounding operators, i.e., they satisfy (2.3) because of property (2.2).

In [6] we studied a first-order method choosing

$$[\underline{P}_h w](x) := \max_{\xi \in \overline{\Omega}_i} w(\xi), \qquad [\overline{P}_h w](x) := \min_{\xi \in \overline{\Omega}_i} w(\xi) \quad \forall x \in \Omega_i.$$

Now, we replace the nonlinear function $g(x, u(x))$ by a piecewise polynomial of degree k to generate a kth order method ($k \geq 1$). Let us define

$$P_k := \{v \in L_2(\Omega) \colon v|_{\Omega_i} \text{ a polynomial of degree } \leq k\}.$$

We use an equidistant auxiliary grid on every subinterval $\overline{\Omega}_i$ given by

$$\sigma_j^i := x_{i-1} + \tfrac{j}{k} h_i, \qquad j = 0(1)k,$$

and define a piecewise interpolation operator $S_k \colon C(\overline{\Omega}) \to P_k$ by $S_k u|_{\Omega_i} :=$ interpolation polynomial of u with knots σ_j^i, $j = 0(1)k$. Further, we introduce the operators $\underline{p}, \overline{p} \colon C(\overline{\Omega}) \to P_0$ by

$$[\underline{p} w](x) := \max_{\xi \in \overline{\Omega}_i} w(\xi), \qquad [\overline{p} w](x) := \min_{\xi \in \overline{\Omega}_i} w(\xi) \quad \forall x \in \Omega_i \tag{2.6}$$

and define

$$\overline{P}_h := S_k + \overline{p}(I - S_k), \tag{2.7}$$

$$\underline{P}_h := S_k + \underline{p}(I - S_k) \tag{2.8}$$

(I is the identity).

In the following we analyze the case $P_h := \underline{P}_h$ and set $p := \underline{p}$ too. The operator P_h is defined according to (2.6)–(2.8) by piecewise polynomial interpolation and shifting of the interpolation polynomial such that it forms an upper (in the case $\overline{P}_h$ a lower) bound to the original function; see Figure 1 for $k = 2$.

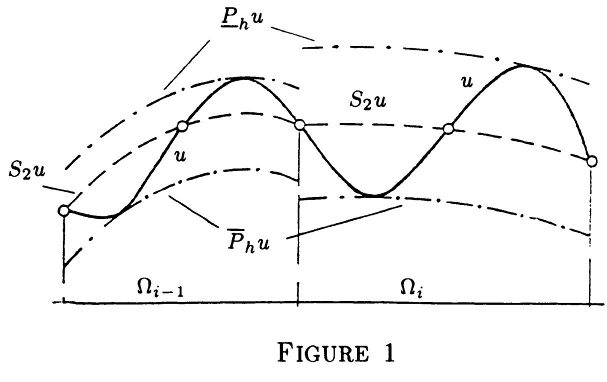

FIGURE 1

Replacing the continuous function $g(\cdot, u(\cdot))$ by a piecewise continuous one, an adequate tool for handling the modified problem is a corresponding weak formulation. We introduce the notation $U := H_0^1(\Omega)$ and let U^* be the related dual space. The bounding operators $\underline{G}_h$, $\overline{G}_h$ can be considered as mappings from U into U^*. Thus our approximate problems corresponding to (2.1) are: Find some $u_h \in U$ with

$$\int_\Omega u_h' v' \, dx + \int_\Omega (G_h u_h) v \, dx = 0 \quad \text{for all } v \in U. \tag{2.9}$$

LEMMA 2.2. *Let there exist solutions $\underline{u}_h, \overline{u}_h$ of* (2.9) *with $G_h := \underline{G}_h$, $G_h := \overline{G}_h$, respectively. Then the solution u of the original problem* (2.1) *is enclosed by $\underline{u}_h \le u \le \overline{u}_h$.*

Proof. By a known standard argument, the solution of our approximate problem (2.9) belongs to the space H^2, thus $u_h \in C^1(\overline{\Omega})$. Applying Lemma 2.1, the desired enclosing property follows from (2.3). □

In the next section we will prove our main result concerning the convergence rate of our enclosing discretization technique, described by (2.9) and the bounding operator defined by (2.5)–(2.8).

THEOREM 2.1. *Assume that all partial derivatives of g of order less than or equal to $k+1$ are continuous on $\Omega_i \times R$, $i = 1(1)N$, and have a continuous extension on $\overline{\Omega}_i \times R$. Then there exists a constant C independent of h such that the following estimation holds:*

$$\max_{x \in \overline{\Omega}} |u(x) - u_h(x)| \le C h^{k+1}.$$

The discrete problem (2.9) is equivalent to a nonlinear system of a finite number of equations.

In Section 4 we discuss the finite-dimensional implementation of our method for arbitrary k and propose a Newton-like iteration technique for solving the generated set of nonlinear equations. These questions are discussed in detail also in [5].

3. Error Estimation. In this section we prove our main result concerning the rate of convergence of our enclosing discretization technique, Theorem 2.1.

Let the operators $L, G \colon U \to U^*$ be defined by

$$\langle Lu, v \rangle := \int_\Omega u'(x) v'(x) \, dx \quad \text{for any } u, v \in U \tag{3.1}$$

and

$$\langle Gu, v \rangle := \int_\Omega g(x, u(x)) v(x) \, dx \quad \text{for any } u, v \in U, \tag{3.2}$$

respectively. Similarly, the bounding operator G_h can be interpreted. In (3.1), (3.2) and in the sequel, $\langle \cdot, \cdot \rangle$ denotes the dual pairing. Using the mappings L, G, the given boundary value problem (2.1) is equivalent to the operator equation

$$Lu + Gu = 0, \tag{3.3}$$

and the discrete problem (2.9) can be written as

$$Lu_h + G_h u_h = 0. \tag{3.4}$$

LEMMA 3.1. *Let there exist a solution u_h of* (3.4). *Then the estimation*

$$\|u - u_h\| \le \frac{1}{\gamma}\|G_h u_h - G u_h\|_* \tag{3.5}$$

holds.

Proof. Let us denote by $\|\cdot\|$ the norm in U and by $\|\cdot\|_*$ the norm in U^*. Equation (3.4) is equivalent to

$$\langle L(u-u_h), v\rangle + \langle Gu - Gu_h, v\rangle + \langle Gu_h - G_h u_h, v\rangle = 0.$$

Taking into account the monotonicity of g and the coerciveness of L, with $v := u - u_h$ we obtain

$$\gamma\|u - u_h\|^2 \le \|Gu_h - G_h u_h\|_* \|u - u_h\|. \quad \square$$

It is rather technical to prove the solvability of the discrete problem. We showed this fact for sufficiently small $h > 0$ using the theory of pseudomonotone operators in [7]; another proof is based on an auxiliary variational inequality [5].

Lemma 3.1 shows that the error can be estimated using an estimation of some kind of the approximation error. We proceed by replacing the $\|\cdot\|_*$-norm on the right-hand side of (3.5) by the L^∞-norm:

$$\|u - u_h\| \le \frac{\sqrt{b-a}}{\gamma}\|(I - P_h)Gu_h\|_\infty.$$

Next, we investigate $\|(I - P_h)Gu_h\|_{\infty,\Omega_i}$ on every subinterval Ω_i. Using

$$P_h = S_k + p(I - S_k)$$

and the definition of p, we obtain

$$\|(I - P_h)Gu_h\|_{\infty,\Omega_i} \le 2\|(I - S_k)Gu_h\|_{\infty,\Omega_i}.$$

From approximation by polynomials the following estimation is known:

$$\|(I - S_k)w\|_{C^l(\overline{\Omega}_i)} \le C\|w\|_{C^{k+1}(\overline{\Omega}_i)} h^{k+1-l}, \qquad 0 \le l \le k.$$

Using the piecewise constancy of p, we obtain

$$\|(I - P_h)w\|_{C^l(\overline{\Omega}_i)} \le C\|w\|_{C^{k+1}(\overline{\Omega}_i)} h^{k+1-l}, \qquad 0 \le l \le k. \tag{3.6}$$

It remains to show that

$$\|Gu_h\|_{C^{k+1}(\overline{\Omega}_i)} \le C, \qquad i = 1(1)N, \tag{3.7}$$

for $h \to 0$.

First, we remark that u_h is bounded by some constant independent of the step size h. This fact has been shown in [7, Theorem 1.1]. The continuous embedding $U \to C(\overline{\Omega})$ and the continuity of g imply $\|G_h u_h\|_\infty \le C$. With the smoothing property of L^{-1} we obtain from (3.4) the boundedness of $\|u_h\|_{H^2(\Omega)}$; the continuous embedding $H^2(\Omega) \to C^1(\Omega)$ results in

$$\|u_h\|_{C^1(\overline{\Omega})} \le C. \tag{3.8}$$

On the other hand, u_h satisfies the differential equation

$$-u_h'' = (I - P_h)Gu_h - Gu_h$$

in every subinterval Ω_i. With the smoothness of g and (3.8) this leads recursively to

$$\|u_h\|_{C^{k+1}(\overline{\Omega}_i)} \le C, \qquad i = 1(1)N,$$

and finally this proves (3.7). There follows

$$\|u - u_h\| \le Ch^{k+1}.$$

With the continuous embedding $U \to C(\overline{\Omega})$ we obtain

$$\max_{x\in\overline{\Omega}} |u(x) - u_h(x)| \le Ch^{k+1}. \quad \square$$

4. Finite-Dimensional Implementation of the Method. In this section we first describe the nonlinear system of equations generated by the monotone discretization technique and then propose an iteration technique to solve this set of equations.

The solution of our discrete problem

$$Lu_h + G_h u_h = 0$$

is piecewise polynomial of degree $k+2$ and belongs to $C^1(\overline{\Omega})$. Thus, u_h can be represented in the following way:

$$u_h(x) = \sum_{i=1}^{N-1} u_i \varphi_i(x) + \sum_{i=1}^{N} \sum_{j=0}^{k} w_{ij} \psi_{ij}(x). \tag{4.1}$$

In this representation the φ_i denote the piecewise affine functions

$$\varphi_i(x) = \begin{cases} (x - x_{i-1})/h_i & \text{if } x \in \overline{\Omega}_i, \\ (x_{i+1} - x)/h_{i+1} & \text{if } x \in \Omega_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and the functions ψ_{ij} are related to a basis $\{\xi_{ij}\}_{i=1(1)N, j=0(1)k}$ in P_k according to

$$-\psi_{ij}''(x) = \xi_{ij}(x), \qquad \psi_{ij}(x_l) = 0, \quad l = 1(1)N. \tag{4.2}$$

The basis $\{\xi_{ij}\}$ can be defined by

$$\xi_{ij}(x) = \begin{cases} (x - x_{i-1})^j & \text{if } x \in \Omega_i, i = 1(1)N, \\ 0 & \text{otherwise, } j = 0(1)k. \end{cases} \tag{4.3}$$

From the differentiability of u_h at the inner grid points x_i we obtain

$$\begin{aligned} &\frac{u_i - u_{i-1}}{h_i} - \frac{u_{i+1} - u_i}{h_{i+1}} \\ &= \sum_{j=0}^{k} (w_{i+1,j}\psi_{i+1,j}'(x_i + 0) - w_{ij}\psi_{ij}'(x_i - 0)), \qquad i = 1(1)N-1. \end{aligned} \tag{4.4}$$

The validity of the differential equation

$$-u_h'' + G_h u_h = 0$$

on every subinterval Ω_i results in

$$\sum_{i=1}^{N} \sum_{j=0}^{k} w_{ij} \xi_{ij} + G_h u_h = 0. \tag{4.5}$$

For arbitrary fixed k, the conditions (4.4), (4.5) form a system of nonlinear equations that are equivalent to (2.9) (or (3.4)).

To develop an iteration method for solving (4.4), (4.5), we linearize the equations (4.5) near the approximate solution u_h^l on the subinterval Ω_i by

$$\begin{aligned} &\sum_{j=0}^{k} w_{ij}\xi_{ij} + G_h u_h^l + g_u(x_{i-1}, u_{i-1}^l)(u_{i-1} - u_{i-1}^l) = 0, \\ &\sum_{j=0}^{k} w_{ij}\xi_{ij} + G_h u_h^l + g_u(x_i, u_i^l)(u_i - u_i^l) = 0, \end{aligned} \tag{4.6}$$

respectively. Using the basis $\{\xi_{ij}\}$ according to (4.3), we obtain the functions

$$\psi_{i0}(x) = \begin{cases} \frac{1}{2}(x - x_{i-1})(x_i - x) & \text{if } x \in \Omega_i, \\ 0 & \text{otherwise} \end{cases}$$

from the definition (4.2). With (4.4), (4.6) and the boundary conditions this results in the iteration scheme

$$\begin{aligned} &\frac{u_i^{l+1} - u_{i-1}^{l+1}}{h_i} - \frac{u_{i+1}^{l+1} - u_i^{l+1}}{h_{i+1}} + \frac{h_i + h_{i+1}}{2} g_u(x_i, u_i^l) u_i^{l+1} \\ &\quad = \sum_{j=0}^{k} (w_{i+1j}^{l+1} \psi'_{i+1j}(x_i + 0) - w_{ij}^{l+1} \psi'_{ij}(x_i - 0)) \\ &\quad + \frac{h_i + h_{i+1}}{2} g_u(x_i, u_i^l) u_i^l, \qquad i = 1(1)N - 1, \end{aligned} \tag{4.7}$$

with

$$u_0^{l+1} = u_N^{l+1} = 0$$

and

$$\sum_{i=1}^{N} \sum_{j=0}^{k} w_{ij}^{l+1} \xi_{ij} + G_h u_h^l = 0. \tag{4.8}$$

By a straightforward modification of the well-known convergence proofs for Newton's method (see [5]) we obtain

THEOREM 4.1. *Let the function $g(\cdot, \cdot)$ be Lipschitz continuously differentiable. There exist some $\varepsilon > 0$ and some $\overline{h} > 0$ such that for any $h \in (0, \overline{h}]$ and arbitrary $u_h^0 \in U_\varepsilon(u_h)$ the iterative method (4.7), (4.8) generates a unique sequence $\{u_h^l\}$ which converges to u_h. More precisely, there exist some constants C_1, $C_2 > 0$ such that*

$$\|u_h^{l+1} - u_h\| \le C_1 h^{1/2} \|u_h^l - u_h\| + C_2 \|u_h^l - u_h\|^2, \qquad l = 0(1)\cdots. \tag{4.9}$$

Remarks. (i) The estimation (4.9) shows that the method (4.7), (4.8) generates a sequence $\{u_h^l\}$ converging for fixed stepsize $h > 0$ linearly to u_h provided u_h^l was close enough to u_h. However, the convergence is asymptotic superlinear in the following sense:

$$\lim_{h \to +0} \overline{\lim_{l \to \infty}} \frac{\|u_h^{l+1} - u_h\|}{\|u_h^l - u_h\|} = 0.$$

(ii) The value $\varepsilon > 0$ in Theorem 4.1 can be selected independently of the stepsize h. In particular, ε does not have to tend to zero for $h \to +0$.

(iii) The essential part of the method (4.7), (4.8) consists in the determination of $G_h u_h^l$. Here, the shift $p(I - S_k)Gu_h^l$ has been estimated via the remainder in Taylor's formula or via an approximate optimization, respectively, in the practical realization of the method.

5. Numerical Examples. We now illustrate the efficiency of the proposed method by means of results obtained for some test problems.

Example 1.

$$\begin{aligned} &-u'' + u^3 - (9\pi^2 + \sin^2 3\pi x)\sin 3\pi x = 0 \quad \text{in } \Omega = (0,1), \\ &u(0) = u(1) = 0. \end{aligned}$$

Exact solution: $u(x) = \sin 3\pi x$. Table 1 shows the maximum width at the grid points, $\delta_{N,k}^l := \max_{x_i \in Z_N}\{\overline{u}_{N,k}^l(x_i) - \underline{u}_{N,k}^l(x_i)\}$, after $l := l(N,k)$ iterations, using an equidistant grid $Z_N := \{x_i := i/N,\ i = 0(1)N\}$ with $h := 1/N$ and a method of order $k+1$ $(k = 0,1,2)$. The values in parentheses indicate the number of iterations.

TABLE 1

	$k=0$	$k=1$	$k=2$
$N=5$	5.967 685(13)	2.367 227(6)	0.599 570(6)
$N=10$	4.186 350(9)	0.701 473(6)	0.084 508(6)
$N=20$	2.493 946(7)	0.175 436(5)	0.009 893(4)
$N=40$	1.346 794(6)	0.043 601(4)	0.001 214(3)

TABLE 2

	$k=0$	$k=1$	$k=2$
$N=5$	1.181 196(50)	0.779 038(45)	no convergence
$N=10$	0.710 983(15)	0.149 880(8)	0.016 216(11)
$N=20$	0.323 309(8)	0.035 133(5)	0.002 203(5)
$N=40$	0.158 828(6)	0.008 845(4)	0.000 304(4)

Example 2. As a second test problem we take

$$\begin{aligned} &-u'' + \alpha(\sinh u - \operatorname{sgn}(x-0.6)(x-0.8)) = 0 \quad \text{in } \Omega = (0,1), \\ &u(0) = u(1) = 0. \end{aligned}$$

For large values of the parameter α the solution possesses interior layers at $x = 0.6$ and at $x = 0.8$. We choose $\alpha = 100$. Table 2 again shows the maximum distance between the upper and lower solution at the grid points.

In Example 2 the function $\operatorname{sgn}(x-0.6)(x-0.8)$ exhibits a discontinuity at the points $x = 0.6$ and $x = 0.8$. Choosing the grid in such a way that these points are grid points, the proposed enclosing discretization technique is applicable in this case too. Indeed, the mappings $\underline{p}, \overline{p}\colon C(\overline{\Omega}) \subset U^* \to P_0$ (compare (2.6)) are to be modified as follows:

$$[\underline{p}u](x) := \sup_{\xi\in\Omega_i} u(\xi) \quad \text{and} \quad [\overline{p}u](x) := \inf_{\xi\in\Omega_i} u(\xi) \quad \text{for } x \in \Omega_i.$$

The iteration has been stopped if the condition

$$|\delta_{N,k}^{l-1} - \delta_{N,k}^{l}| \le 10^{-6}$$

is fulfilled. As initial point for $k = 0$ we selected in each case $u_{N,k}^1 \equiv 0$, and for $k > 0$, $u_{N,k}^1 := u_{N,k-1}^{l(N,k-1)}$. The results in Tables 1 and 2 show that the number of iterations decreases with the refinement of the grid, according to the asymptotically superlinear convergence noted in Remark (i). The results also confirm the convergence results of Section 3 in a convincing way. For $k = 0$, the error reduces to 1/2 of the preceding error when the number of grid intervals is doubled, for $k = 1$ the reduction is by 1/4, for $k = 2$ by 1/8, in complete agreement with theoretical expectations.

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