The Validity of Shapiro’s Cyclic Inequality

By B. A. Troesch

Abstract. A cyclic sum $S_N(x) = \sum x_i/(x_{i+1} + x_{i+2})$ is formed with $N$ components of a vector $x$, where in the sum $x_{N+1} = x_1$, $x_{N+2} = x_2$, and where all denominators are positive and all numerators are nonnegative. It is known that there exist vectors $x$ for which $S_N(x) < N/2$ if $N > 14$ and even, and if $N > 24$. It has been proved that the inequality $S_N(x) > N/2$ holds for $N < 13$. Although it has been conjectured repeatedly that the inequality also holds for odd $N$ between 15 and 23, this has apparently never been proved. Here we will confirm that the inequality indeed holds for all odd $N \leq 23$. This settles the question for all $N$.

1. Introduction. The problem suggested by H. S. Shapiro in 1954 [12] has attracted wide interest; the history of the problem up to 1970 is described vividly by D. S. Mitrinović in his book “Analytic Inequalities” [8, pp. 132ff.]. When the problem was published, it appeared very reasonable to conjecture that $N/2$ is the minimum that the cyclic sum $S_N$ can attain. It came therefore as a surprise that for some $N$ actually $S_N(x) < N/2$ is possible ([5], reporting a result by Lighthill). This led to the considerable interest in the problem.

It has been proved that $S_N(x) \geq N/2$ for all admissible vectors $x$, if $N \leq 13$ [14]. On the other hand, there exist vectors $x$ such that $S_N < N/2$, if $N \geq 14$ and even, and also for all $N \geq 24$ ([7] contains a slight misprint). The difference in behavior for $N$ even against $N$ odd is explained in [11].

In this investigation it will be shown that $S_N \geq N/2$ for the remaining cases, namely $15 \leq N \leq 23$ and odd. This settles the question of Shapiro’s inequality for all $N$. From a result in [1], it follows that only the case $N = 23$ need to be investigated: if the inequality $S_N(x) \geq N/2$ holds for $N = 23$, it automatically holds for all lower odd $N$.

Unfortunately, the only feasible method to show that $S_{23} \geq 23/2$ appears to be based on the discussion and some numerical computation of many different cases. This approach has been used in [9] for $N = 10$, in [6] for $N = 12$, and in [14] for $N = 13$. The largest $N$ where a purely algebraic proof has been successful is $N = 8$ [3].

It is crucial to consider the cases separately depending on which components of $x$ are zero, and which components are different from zero. The reason for this is clear: $S_N$ is a function of the $N$ variables $x_1, x_2, \ldots, x_N$, where $x_k \geq 0$. At the stationary points of $S_N$ we have $\partial S_N/\partial x_k = 0$ when $x_k > 0$, while at the boundary of the admissible domain where $x_k = 0$ the derivative of $S_N$ need not vanish. Although no two consecutive components of $x$ are permitted to vanish, the
number of possibilities nevertheless grows very rapidly with \( N \), and turns out to be over 2500 for \( N = 23 \). It seems very undesirable to let the computer investigate all these cases.

2. General Description of the Method. The approach, the results, and the notation described in \[14\] will be used. The number and the positions of the zero components in the vector \( \mathbf{x} \) is essential; the string of consecutive nonzero components is called a segment. There are three observations that immediately reduce the number of cases to be considered down to 100 cases. First, it is shown in \[14\] that there is no loss of generality if the segments are rearranged, for instance in order of decreasing segment length. Furthermore, a case with \( S_N < N/2 \) must necessarily contain a segment of length 6 at least (\[14, \text{Section 4}\]). And last, segments of length 2 need not be considered, because it can be shown that there is always another case that has a lower sum \( S \).

Let us denote by \((c_1, c_2, \ldots, c_l)\) the case where \( c_1 \) is the length of the longest segment, down to \( c_l \), the length of the shortest segment. The list of possibilities then starts out with \((22), (20, 1), (18, 3), (18, 1, 1), (17, 4), (16, 5), (16, 3, 1)\) and ends with \((6, 3, 3, 4 * 1), (6, 3, 6 * 1), (6, 8 * 1)\), namely a 6-segment followed by eight one-segments. It turns out that many additional cases can be eliminated from consideration, if the inequalities to be described below are taken into account, together with the restriction on the pivotal ratio \( u \) which is easily obtained for segments of odd length up to length 9.

The remaining cases are then investigated by a comprehensive search in a small region of a two-parameter plane (see Figure 1). The implementation of the search requires only a few lines of programming.

3. The Properties of a Segment. From the remark above it follows that each segment can be analyzed separately, and then segments with the same leading ratio \( u \) (see below) are concatenated to find the admissible stationary point. According to \[9\], there is at most one of them for each case.

Let us therefore analyze a segment of length \( m \) in more detail. We take as example \( m \) to be odd to enable us to be specific in the signs, where they alternate. Therefore, we set (the zero components are not included in the numbering)

\[
\mathbf{x} = x_1 \ 0 \ x_2 \ 0 \ x_3 \ \cdots \ x_m \ 0 \ x_{m+1} \ 0 \ x_{m+2} \ \cdots
\]

The sum for the \( m \)-segment is

\[
S_m = \frac{x_2}{x_3 + x_4} + \frac{x_3}{x_4 + x_5} + \cdots + \frac{x_{m-1}}{x_m + x_{m+1}} + \frac{x_m}{x_{m+1}} + \frac{z_{m+1}}{x_{m+2}}
\]

A choice of new independent variables

\[
y_1 = x_2, \ y_2 = x_3 + x_4, \ldots, y_{m-1} = x_m + x_{m+1}, \ y_m = x_{m+1}, \ y_{m+1} = x_{m+2}
\]

is used with success in \[9\], \[6\], and \[14\], and solving for \( \mathbf{x} \),

\[
x_{m+2} = y_{m+1}, \ x_{m+1} = y_m, \ x_m = y_{m-1} - y_m, \ldots, x_3 = y_2 - y_3 + y_4 - y_m, \ x_2 = y_1
\]

leads to

\[
S_m = \frac{y_1}{y_2} + \frac{y_2 - y_3 - \cdots - y_m}{y_3} + \frac{y_3 - y_4 - \cdots - y_m}{y_4} + \frac{y_{m-2} - y_{m-1} - y_m}{y_{m-1}} + \frac{y_{m-1} - y_m}{y_m} + \frac{y_m}{y_{m+1}}
\]
or

\[ S_m = c_2 + c_3 + \cdots + c_m + c_{m+1}, \]

which defines the ratios \( c \).

As in [14, Section 3], we set \( r_k = y_k/y_{k+1} \), so that \( c_2 = r_1, c_{m+1} = r_m, y_3 c_3 - y_2 = -y_4 c_4 \), and quite generally, \( y_k c_k - y_{k-1} = -y_{k+1} c_{k+1} \), \( k = 3, 4, \ldots, m \). In terms of the \( r_k \)'s this can be written as

\[ (3.1) \quad c_{k+1} = r_k (r_{k-1} - c_k), \quad k = 3, 4, \ldots, m. \]

For a stationary \( S_m \), namely \( \partial S_m / \partial y_k = 0 \) for \( k = 2, 3, \ldots, m + 1 \), we obtain

\[
\begin{align*}
-y_1 + y_2 & = 0, \\
-y_2 + 0 - y_4 + y_5 - y_7 + & y_9 - \cdots - y_{m-1} + y_m + y_2 & = 0, \\
+ y_4 - y_3 & = 0 - y_5 & \ldots & - y_{m-1} + y_m + y_4 & = 0, \\
-y_5 + y_4 - y_6 & = 0 & \ldots & - y_{m-1} + y_m + y_5 & = 0, \\
& & \ldots & & \ldots \\
+ y_{m-1} & y_{m-1} & y_{m-1} & y_{m-1} & \ldots & - y_m + y_{m+1} & y_{m+1} & y_{m+1} & y_{m+1} & = 0. \\
& y_3 & y_4 & y_5 & \ldots & y_m & y_{m+1} & y_{m+1} & y_{m+2} & = 0.
\end{align*}
\]

The first and last equation give \( r_1 = r_2, r_m = r_{m+1} \), and adding all equations gives \( u = r_2 = r_m = r_{m+1}, \) where \( r_{m+1} \) is the leading element of the next segment. This shows, as mentioned in [14, Section 2], that at the stationary point all segments have the same pivotal element \( u \), a fact which is very helpful in the investigation.

With the notation above, the remaining equations become

\[
\begin{align*}
-c_3 - 1 + r_3 & = 0, \\
1 & - c_4 - 1 + r_4 = 0, \\
-1 & r_3 r_4 = 0, \\
-c_5 - 1 + r_5 & = 0, \\
1 & r_4 r_5 + \cdots + 1 = 0, \\
\ldots \ldots & \ldots \ldots
\end{align*}
\]

\[ (3.2) \]

Next, we leave the first equation as is, add the first equation to the second equation multiplied by \( r_3 \), add the second equation to the third equation multiplied by \( r_4 \),
and so on:

\[ c_3 = r_3 - 1, \]
\[ c_3 = r_3(r_4 - c_4), \]
\[ c_4 = r_4(r_5 - c_5), \]
or in general,

\[ (3.3) \quad c_{k-1} = r_{k-1}(r_k - c_k), \quad k = 4, \ldots, m. \]

By returning to Eq. (3.1) it is easy to show as follows that the \( c_k \)'s are symmetrical within a segment. Since \( r_{m-1} = r_3, \ r_{m-2} = r_4, \ldots \) (see [14, Section 3]) and \( c_{m+1} = u \), the last equation in (3.1), namely \( c_{m+1} = r_m(r_{m-1} - c_m) \), becomes \( 1 = r_3 - c_m \), and hence, \( c_m = c_3 \) from Eqs. (3.3). Next, again from Eq. (3.1), \( c_m = r_{m-1}(r_{m-2} - c_{m-1}) \) or \( c_3 = r_3(r_4 - c_{m-1}) \) shows that \( c_{m-1} = c_4 \), and so on. This reduces the number of independent variables by nearly a factor of two.

The equations can now be solved recursively by assuming values for \( u \) and \( r_3 \), using Eqs. (3.3) and (3.1) in turn:

\[ c_3 = r_3 - 1, \]
\[ c_4 = r_3(u - c_3), \]
\[ r_4 = c_4 + \frac{c_3}{r_3}, \]
\[ c_5 = r_4(r_3 - c_4), \]
\[ r_5 = c_5 + \frac{c_4}{r_4}, \]

and so on. The symmetry in \( c \) requires that for \( m \) odd the condition

\[ (3.4a) \quad c_{(m+1)/2} = c_{(m+5)/2}, \]

and for \( m \) even,

\[ (3.4b) \quad c_{(m+2)/2} = c_{(m+4)/2}, \]

must hold. For a fixed \( r_3 \), the values for \( u \) are changed to find the values for which this last condition is satisfied. Varying \( r_3 \) leads to curves in the \( r_3 - u \) plane that have stationary values for \( S_m \) and are candidates for stationary values for the cyclic sum \( S_N \).

There are two fortunate circumstances: the recursion formulas are identical for segments of any length, and the search can be restricted to a rather small region, as shown in Figure 1. To show this, we establish several bounds.

4. Some Inequalities. The following inequalities are all based on the fact that the \( c \)'s and the \( r \)'s must be positive.

a. Since \( c_3 = r_3 - 1 \), it follows that

\[ (4.1) \quad r_3 > 1. \]

b. Next,

\[ c_3 = \frac{y_2 - y_3 + \cdots - y_m}{y_3} \]
can also be written as
\[ c_3 = u - 1 + \frac{1}{r_3} - \frac{y_6c_6}{y_3}, \]
and hence
\[ u + \frac{1}{r_3} - r_3 - \frac{y_6c_6}{y_3} = 0, \]
or
\[ u > r_3 - \frac{1}{r_3}. \]

c. Similarly, \( c_4 \) can be written from its definition in two ways:
\[ c_4 = r_3 - 1 + \frac{y_6c_6}{y_4} = r_3 - 1 + \frac{1}{r_4} - \frac{y_7c_7}{y_4}. \]
On the other hand, it follows from the second Eq. (3.2) that
\[ c_4 = r_4 + \frac{1}{r_3} - 1, \]
and therefore
\[ r_3 + \frac{1}{r_4} - r_4 = \frac{y_7c_7}{y_4} > 0, \]
\[ (r_3 - r_4) \left(1 + \frac{1}{r_3r_4}\right) > 0, \]
and finally
\[ r_4 < r_3. \]
d. A useful inequality is obtained by the other representation for \( c_4 \) above:
\[ r_3 - r_4 - \frac{1}{r_3} + \frac{y_6c_6}{y_4} = 0 \]
dividing it by \( r_3 \),
\[ 1 - \frac{r_4}{r_3} - \frac{1}{r_3^2} + \frac{y_6c_6}{y_3} = 0, \]
and adding it to Eq. (4.2) gives
\[ u = r_3 - \frac{1}{r_3} + \frac{1}{r_3^2} - \left(1 - \frac{r_4}{r_3}\right). \]
The desired inequality is
\[ u < r_3 - \frac{1}{r_3} + \frac{1}{r_3^2}. \]
e. Equation (4.6) gives also the result
\[ u - r_4 = r_3 - r_4 - \frac{1}{r_3} + \frac{1}{r_3^2} - 1 + \frac{r_4}{r_3} = \left(1 - \frac{1}{r_3}\right) \left(r_3 - \frac{1}{r_3} - r_4\right). \]
If, as we will show next, the second factor is negative, then
\[ r_4 > u. \]
To this end, we write Eq. (3.1) for \( k = 4 \) and \( k = 5 \):
\[ \frac{c_5}{r_4} = (r_3 - c_4), \quad c_6 = r_5(r_4 - c_5); \]
therefore $r_4 > c_5$, or $c_5/r_4 < 1$, so that $r_3 < c_4 + 1$, and then from Eq. (4.4), $r_3 < r_4 + 1/r_3$, as claimed above.

f. Furthermore, by similar considerations, one can show that, after some algebra,

\[ r_5 > 1 \text{ if } r_4 > 1, \]

since $r_5 - 1 = (r_4 - 1/r_4)(1 - 1/r_3) + r_4(r_3 - r_4)$.

Therefore, if $u > 1$, then $r_3 > 1$, $r_4 > 1$, and $r_5 > 1$ follows from Eqs. (4.1) and (4.8). This result then eliminates analytically many cases if the longest segment is a 9-segment.

5. The Curves in the $r_3 - u$ Plane. The recursion formulas and Eq. (3.4) show that an admissible segment of length $m$ is completely determined by $r_3$ and $u$. Of particular interest are the values of

\[ p_m = \prod_{j=1}^{m} r_j, \]

since from the definition of the $r_j$'s in any particular case the product of all $p_k$'s must equal 1 ([14, Section 3]), and the values of

\[ S_m = \sum_{j=2}^{m+1} c_j. \]

The final goal is to show that

\[ S_{23} = \sum S_m \geq 23/2 \]

in all cases.

As mentioned above, the search in the $r_3 - u$ plane can be restricted to a small region because of the inequalities (4.1), (4.3), and (4.7). Furthermore, segments need only be considered if

\[ u < 2.2. \]

Otherwise, it follows from Eqs. (4.7), (4.8), and (4.9) that $r_3 > 2.4$, $r_4 > 2.2$, and $r_5 > 1$. A simple computation then shows that $S_7 > 11.8$, exceeding the allowed limit already. All longer segments have an even larger sum. Similar considerations show that $u < 1.4$ must hold, except in four cases.

The search for cases with possibly $S_{23} < 23/2$ can therefore be restricted to the small region shown in Figure 1. The admissible values for an individual segment lie on smooth curves; in Figure 1, the curves for the 8-segment and for the 11-segment are drawn as examples. Segments up to length 9 have just one curve, as can be proved by Descartes's rule of signs, whereas longer segments have one or two curves, with the exception of the 19-segment, which has three curves.

The computation starts with the longest segment in the case being considered. To find a point $P$ on the $r_3 - u$ curve, the $r_3$ is kept constant and $u$ is changed until Eq. (3.4) is satisfied. The search in $r_3$ with fixed $u$ is less desirable because of the shape of some curves, like the 11-segment curve. The point $P$ can be ignored, if any of the $r$'s or $c$'s turn out to be negative, or if the sum $S_m \geq 23/2$. For segments which are no longer than length 4, the explicit formulas for $p_k$ and $S_k$, given in [14],
Region of admissible solutions, bounded by Eqs. (4.1), (4.3), (4.7), (5.1).

can be computed simultaneously and added to $S_m$. Only the points where this sum is smaller than $23/2$ need to be analyzed further.

Advantage can also be taken of the fact that for 7- and 9-segments, $u > 0.922$, and that $S_5 \geq 3.0$, $S_7 \geq 4.0$, and $S_9 \geq 5.0$.

Among the about twenty cases left with the possibility that $S_{23} < 23/2$, most are resolved by casual inspection of the numerical results. The cases with the smallest sum $S_{23}$ are listed in Table 1, and all other cases have a larger sum, except for the trivial case with all $x_k = 1$.

In order to check the results and the numerical approach, several cases between $N = 14$ and $N = 22$ were computed by the method described above and the same programming implementation, and indeed the values for $S_N < N/2$ were found, for instance, the case $(11, 1, 1)$ led to $S_{16} < 7.989$.

Table 1

<table>
<thead>
<tr>
<th>Case</th>
<th>(20, 1)</th>
<th>(18, 1, 1)</th>
<th>(16, 1, 1, 1)</th>
<th>(14, 4 * 1)</th>
<th>(12, 5 * 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\min S_{23}$</td>
<td>11.513</td>
<td>11.512</td>
<td>11.513</td>
<td>11.520</td>
<td>11.533</td>
</tr>
</tbody>
</table>

6. A Remark. Since $\inf S_N < N/2$ occurs already for $N = 14$, it might be reasonable to expect that for very large $N$ the ratio $S_N/N$ could fall well below the value 1/2. The result in [10] that $S_N/N \geq 0.3307\ldots$ and in [2] that $S_N/N \geq 0.461238\ldots$ for any $N$ were therefore significant. However, in a remarkable paper, Drinfeld [4] proved that $\inf_N(S_N/N) = 0.4945668$. Without the knowledge of Drinfeld's proof, the same result was obtained in [13], including the
formulas identical to those in [4]. But this did not constitute a proof, but rather an example of [4], because a definite distribution of the zero-components of \( x \) was assumed. The assumption appeared reasonable, based on previous experience. It would be desirable to prove that for any \( N \) this particular distribution of nonzero components always gives the lowest sum \( S_N \), except of course for the case with all components equal to 1. A result of this kind would make the investigation reported here essentially trivial.

It seems astounding that \( S_N/N \), which can be made easily as low as 1/2 for any \( N \geq 3 \) by choosing all \( x_k = 1 \), can never fall below that value by more than about 1%.

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