GENERALIZED NONINTERPOLATORY RULES FOR CAUCHY PRINCIPAL VALUE INTEGRALS

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ABSTRACT. Consider the Cauchy principal value integral

\[ I(kf; \lambda) = \int_{-1}^{1} k(x) \frac{f(x)}{x - \lambda} \, dx, \quad -1 < \lambda < 1. \]

If we approximate \( f(x) \) by \( \sum_{j=0}^{N} a_j p_j(x; w) \) where \( \{p_j\} \) is a sequence of orthonormal polynomials with respect to an admissible weight function \( w \) and \( a_j = (f, p_j) \), then an approximation to \( I(kf; \lambda) \) is given by \( \sum_{j=0}^{N} a_j I(kp_j; \lambda) \).

If, in turn, we approximate \( a_j \) by \( a_{jm} = \sum_{j=1}^{m} w(x_m)f(x_m)p_j(x_m) \), then we get a double sequence of approximations \( \{Q_{mN}(f; \lambda)\} \) to \( I(kf; \lambda) \). We study the convergence of this sequence by relating it to the sequence of approximations associated with \( I(wf; \lambda) \) which has been investigated previously.

1. Introduction

In a recent paper, Rabinowitz and Lubinsky [9] studied the convergence properties of a method proposed by Rabinowitz [7] and Henrici [3] for the numerical evaluation of Cauchy principal value (CPV) integrals of the form

\[ I(wf; \lambda) = \int_{-1}^{1} w(x) \frac{f(x)}{x - \lambda} \, dx, \quad -1 < \lambda < 1, \]

where \( w \in A \), the set of all admissible weight functions, i.e., all functions \( w \) on \( J = [-1, 1] \) such that \( w \geq 0 \) and \( \|w\|_1 > 0 \). This method is based on approximating \( I(wf; \lambda) \) by

\[ \hat{S}_N(f; \lambda) = \sum_{j=0}^{N} a_j q_j(\lambda), \]

where

\[ a_j = (f, p_j), \]

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\( q_j(\lambda) = I(wp_j; \lambda) \) and \( \{p_j(x; w) : j = 0, 1, 2, \ldots \} \) is the family of orthonormal polynomials with respect to \( w \). In turn, \( \hat{S}_N(f; \lambda) \) is approximated by

\[
\hat{S}_N(f; \lambda) = \sum_{j=0}^{N} a_{jm} q_j(\lambda),
\]

where \( a_{jm} = Q_m(fp_j) \) is an approximation to \( a_j \) based on the numerical integration rule

\[
Q_m(g) = \sum_{i=1}^{m} w_{im} g(x_{im}),
\]

and where we assume that

\[
\lim_{m \to \infty} Q_m(g) = \int_{-1}^{1} w(x) g(x) \, dx
\]

for all \( g \in C(J) \) or all \( g \in R(J) \), the set of all Riemann-integrable functions on \( J \).

Now, this method requires knowledge of the three-term recurrence relation for the polynomials \( p_j \) which is not always available. Furthermore, it is not always easy to find sequences of integration rules \( Q_m(g) \) which satisfy (6), especially if \( w \) is a nonstandard weight or if we do not wish to use Gaussian rules but rather rules which concentrate many integration points in subintervals where \( f \) is not well behaved. Finally, the restriction to admissible weight functions does not allow us to deal with CPV integrals of the form

\[
I(kf; \lambda) = \int_{-1}^{1} k(x) \frac{f(x)}{x - \lambda} \, dx, \quad -1 < \lambda < 1,
\]

where \( k \) is such that \( I(kf; \lambda) \) exists but \( k \) need not be nonnegative. Since the main idea in writing the numerator of the integrand in (7) as the product of two functions, \( k \) and \( f \), is to incorporate the singular or difficult part of the numerator into \( k \) and treat it analytically while treating the smooth factor \( f \) numerically, it would make no sense to rewrite (7) as \( I(wF; \lambda) \) with \( F = w^{-1}kf \) unless \( w \) had the same singularity structure as \( k \), and even then we would usually have the problems mentioned above.

In this paper, we shall try to overcome these shortcomings in [9] by using ideas of noninterpolatory product integration [8] combined with a device found in [1] for expressing CPV integrals with respect to one function, say \( k \), in terms of CPV integrals with respect to a second function, say \( w \), positive in \((-1, 1) \). The point is that we can then choose a convenient weight function \( w \) for expressing our inner products and for evaluating the approximations to these inner products, for example \( w(x) \equiv 1 \) or \( w(x) = (1 - x^2)^{-1/2} \). In fact, this latter weight function is particularly useful, as we shall see. We shall first describe the method in §2 and then study some convergence questions in §3.

2. A GENERALIZED NONINTERPOLATORY RULE

Consider the CPV integral \( I(kf; \lambda) \) given by (7) where \( k \in DT(N_\delta(\lambda)) \cap L_1(J) \) and \( f \in DT(N_\delta(\lambda)) \cap R(J) \), which ensures that \( I(kf; \lambda) \) exists. Here

\[
N_\delta(\lambda) = [\lambda - \delta, \lambda + \delta] \subset (-1, 1)
\]
and, for any interval \( I \) of length \( l(I) \),

\[
DT(I) = \left\{ g : \int_0^{l(I)} \omega_f(g; t) t^{-1} \, dt < \infty \right\},
\]

where the modulus of continuity of \( g \) on \( I \) is given by

\[
\omega_f(g; t) = \sup_{|x_1 - x_2| \leq t} |g(x_1) - g(x_2)|.
\]

Assume now that we have a convenient weight function \( w \in DT(N_\delta(\lambda)) \cap A \) such that \( w(\lambda) > 0 \). We then have a three-term recurrence relation for the sequence of orthonormal polynomials \( \{p_j(x; w)\} \) of the form

\[
p_{-1} = 0, \quad p_0 = 1, \quad p_{j+1}(x) = (A_j x - \alpha_j) p_j(x) - \beta_j p_{j-1}(x), \quad j \geq 0.
\]

If we expand \( f \) in an orthogonal series in terms of the \( p_j(x; w) \), which for the moment, we assume converges uniformly in \( J \),

\[
f(x) = \sum_{j=0}^{\infty} a_j p_j(x; w),
\]

then we can approximate \( f(x) \) by \( \sum_{j=0}^{N} a_j p_j(x; w) \) and \( I(k f ; \lambda) \) by

\[
S_N(f; \lambda) = \sum_{j=0}^{N} a_j M_j(k; \lambda),
\]

where \( M_j(k; \lambda) = I(k p_j; \lambda) \). In turn, we then approximate \( S_N(f; \lambda) \) by

\[
Q_m^N(f; \lambda) = \sum_{j=0}^{N} a_{jm} M_j(k; \lambda).
\]

The \( M_j(k; \lambda) \) satisfy the following nonhomogeneous recurrence relation

\[
M_{j+1}(k; \lambda) = (A_j \lambda - \alpha_j) M_j(k; \lambda) - \beta_j M_{j-1}(k; \lambda) + A_j N_j(k),
\]

where

\[
N_j(k) = \int_{-1}^{1} k(x) p_j(x; w) \, dx.
\]

Relation (12) follows by replacing \( p_{j+1} \) in \( I(k p_{j+1}; \lambda) \) by the right-hand side of (8) and using the well-known device

\[
\int_{-1}^{1} k(x) \frac{x p_j(x)}{x - \lambda} \, dx = \int_{-1}^{1} k(x) \frac{(x - \lambda) p_j(x)}{x - \lambda} \, dx + \lambda \int_{-1}^{1} k(x) \frac{p_j(x)}{x - \lambda} \, dx.
\]

Hence, if we know the \( N_j(k) \), we can evaluate (11) in a stable manner by backward recurrence.

If \( w(x) = (1 - x^2)^{-1} \), so that (except for normalization) \( p_j = T_j \), the Chebyshev polynomial of the first kind, then recurrence relations for \( N_j(k) \) are known for a wide variety of functions [6]. For \( w(x) \equiv 1 \), for which \( p_j = P_j \), the
Legendre polynomial, recurrence relations for $N_j(k)$ for $k(x) = e^{ix}$, $|x - \tau|^\alpha$ and $\log|x - \tau|$ are given by Paget [5], and for a variety of functions by Gatteschi [2]. Since the work of Paget is not readily available, we give his recurrence relations in Appendix 1. In Appendix 2, we give the recurrence relations for evaluating $Q^N_m(f; \lambda)$ when the $N_j(k)$ are known, as well as for evaluating the weights $w^N_{im}(\lambda)$ in the Lagrangian formulation of $Q^N_m(f; \lambda)$, namely

\begin{equation}
Q^N_m(f; \lambda) = \sum_{i=1}^{m} w^N_{im}(\lambda) f(x_{im})
\end{equation}

with

\begin{equation}
w^N_{im}(\lambda) = w_{im} \sum_{j=0}^{N} p_j(x_{im}) M_j(k; \lambda).
\end{equation}

3. Convergence results

We study first the convergence of $S_N(f; \lambda)$ to $I(kf; \lambda)$, for then we can proceed as in [9] to study the convergence of $Q^N_m(f; \lambda)$ to $I(kf; \lambda)$, either as an iterated limit or as a double limit. Since we have results in [9] for the convergence of $\hat{S}_N(f; \lambda)$ to $I(wf; \lambda)$, we shall try to reduce the study of the convergence of $S_N(f; \lambda)$ to that of the convergence of $\hat{S}_N(f; \lambda)$. To this end, we use a device in [1] to relate a CPV integral weighted by $k$ to one weighted by $w$. This is done by writing

\[
I(kf; \lambda) = \int_{-1}^{1} k(x) \frac{f(x)}{x - \lambda} dx = \int_{-1}^{1} w(x) \frac{k(x)}{w(x)} \frac{f(x)}{x - \lambda} dx
\]

\[
= \int_{-1}^{1} w(x) \frac{f(x)}{x - \lambda} \left[ \frac{k(x)}{w(x)} - \frac{k(\lambda)}{w(\lambda)} \right] dx + \frac{k(\lambda)}{w(\lambda)} \int_{-1}^{1} w(x) \frac{f(x)}{x - \lambda} dx
\]

\[
= \int_{-1}^{1} f(x) k(x, \lambda) dx - \frac{k(\lambda)}{w(\lambda)} \int_{-1}^{1} f(x) w(x, \lambda) dx + \frac{k(\lambda)}{w(\lambda)} I(wf; \lambda).
\]

Here, we have used the divided difference notation,

\[
h[x, y] = \frac{h(x) - h(y)}{x - y}.
\]

Consequently, if we have conditions on $f$ and $w$ which ensure convergence of $\hat{S}_N(f; \lambda)$ to $I(wf; \lambda)$, we need only find the additional conditions on $f$, $k$ and $w$ to insure the convergence of

\[
\sum_{j=0}^{N} a_j \int_{-1}^{1} p_j(x) w(x, \lambda) dx \to \int_{-1}^{1} f(x) w(x, \lambda) dx \equiv I_1
\]

and

\[
\sum_{j=0}^{N} a_j \int_{-1}^{1} p_j(x) k(x, \lambda) dx \to \int_{-1}^{1} f(x) k(x, \lambda) dx \equiv I_2,
\]
for then

\[ S_N(f; \lambda) = \sum_{j=0}^{N} a_j M_j(k; \lambda) = \frac{k(\lambda)}{w(\lambda)} \sum_{j=0}^{N} a_j q_j(\lambda) - \frac{k(\lambda)}{w(\lambda)} \sum_{j=1}^{2} \]

\[ = \frac{k(\lambda)}{w(\lambda)} I(wf; \lambda) - \frac{k(\lambda)}{w(\lambda)} I_1 + I_2 = I(kf; \lambda). \]

Clearly, sufficient conditions for the convergence of \( \sum_1 \) and \( \sum_2 \) are that (9) holds uniformly in \( J \) and that \( w \) and \( k \in DT(J) \), for then

\[ |I_1 - \sum_1| \leq 2\|r_N\|_{\infty} \int_{0}^{2} \omega_j(w; t)t^{-1} dt, \]

where \( r_N(x) = \sum_{j=N+1}^{\infty} a_j p_j(x; w) \), and similarly for \( |I_2 - \sum_2| \). Hence, provided \( |k(\lambda)| < \infty \) and \( w(\lambda) > 0 \), we have convergence of \( S_N(f; \lambda) \) whenever \( \hat{S}_N(f; \lambda) \) converges. Furthermore, if \( \hat{S}_N(f; \lambda) \) converges uniformly with respect to \( \lambda \) on some closed subset \( \Delta \) of \((-1, 1)\) and \( w(\lambda) > 0 \) and \( |k(\lambda)| < \infty \) on \( \Delta \), then we will have uniform convergence of \( S_N(f; \lambda) \) on \( \Delta \). However, we can weaken these conditions in various directions. Thus, it is not necessary that \( w \) and \( k \in DT(J) \), only that \( w, k \in DT(N_{\delta}(\lambda)) \cap L_1(J) \). For then, we can replace (15) by

\[ |I_1 - \sum_1| = \left| \int_{-1}^{1} r_N(x)w[x, \lambda] dx \right| \]

\[ \leq \|r_N\|_{\infty} \left[ \int_{N_{\delta}(\lambda)} |w[x, \lambda]| dx + \int_{J-N_{\delta}(\lambda)} |w[x, \lambda]| dx \right], \]

where both integrals are finite, and similarly for \( \sum_2 \). The first integral in (16) is finite since

\[ \int_{N_{\delta}(\lambda)} |w[x, \lambda]| dx = \int_{-\delta}^{\delta} \left| \frac{w(x) - w(\lambda)}{x-\lambda} \right| dx \]

\[ = \int_{-\delta}^{\delta} \left| \frac{w(t+\lambda) - w(\lambda)}{t} \right| dt \leq 2 \int_{0}^{\delta} \omega_{N_{\delta}(\lambda)}(w; t)t^{-1} dt \]

while, for the second integral, we have

\[ \int_{J-N_{\delta}(\lambda)} |w[x, \lambda]| dx = \int_{J-N_{\delta}(\lambda)} \left| \frac{w(x) - w(\lambda)}{x-\lambda} \right| dx \]

\[ \leq \delta^{-1} \int_{J-N_{\delta}(\lambda)} |w(x) - w(\lambda)| dx \]

\[ < \delta^{-1} \[\|w\|_1 + 2w(\lambda)\] < \infty. \]

Another possibility is to require only that (9) holds uniformly in \( N_{\delta}(\lambda) \). Then, if both \( w^{-1} \in L_1(J) \) and \( k^2/w \in L_1(J) \), a well-known condition in product integration theory [10], we have convergence of \( S_N(f; \lambda) \). We summarize these remarks in a theorem and several corollaries.
**Theorem 1.** Assume that for some \( \lambda \in (-1, 1) \),

\[
\int_{-1}^{1} r_N(x)w[x, \lambda] \, dx \to 0, \quad \int_{-1}^{1} r_N(x)k[x, \lambda] \, dx \to 0 \quad \text{as} \quad N \to \infty,
\]

that \( w(\lambda) > 0 \) and that \( |k(\lambda)| < \infty \). Then

\[
S_N(f; \lambda) \to I(kf; \lambda)
\]

if and only if

\[
S_N(f; \lambda) \to I(\lambda f; \lambda).
\]

Let \( \Delta \) be a closed subset of \((-1, 1)\) and assume that (17) holds uniformly in \( \Delta \), and that \( w(\lambda) > 0 \) and \( |k(\lambda)| < \infty \) for all \( \lambda \in \Delta \); then (18) holds uniformly in \( \Delta \) if and only if (19) holds uniformly in \( \Delta \).

**Corollary 1.** If for some \( \lambda \in (-1, 1) \), \( \sup \| q_j(\lambda) \| < \infty \), \( \sup \| p_j(\cdot; w) \|_{\infty} < \infty \), \( w(\lambda) > 0 \), \( w, k \in DT(N_\delta(\lambda)) \cap L_1(J) \), \( f \in L_1, w(J) \) and \( f[x, \lambda] \in L_1, w(J) \), then (18) holds.

**Proof.** By Theorem 2 in [9], the hypotheses of the corollary suffice for (19) to hold. By Theorem 4 in [4, p. 70], \( \| r_N \|_{\infty} \to 0 \). Hence, as in (16),

\[
\left| \int_{-1}^{1} r_N(x)w[x, \lambda] \, dx \right| \leq \| r_N \|_{\infty} \left[ \int_{-1}^{1} |w[x, \lambda]| \, dx + \int_{N_\delta(\lambda)} |w[x, \lambda]| \, dx \right] \to 0,
\]

and similarly for \( \int_{-1}^{1} r_N(x)k[x, \lambda] \, dx \). Furthermore, since \( k \in DT(N_\delta(\lambda)) \), one has \( |k(\lambda)| < \infty \). Hence, by Theorem 1, (18) holds. \( \Box \)

Before stating the next corollary, we recall the definition of a generalized smooth Jacobi (GSJ) weight function [1]. We say that \( w \in GSJ \) if

\[
w(x) = \psi(x) \prod_{j=0}^{p+1} |x - t_j|^\gamma_j, \quad \gamma_j > -1, \quad j = 0, \ldots, p + 1,
\]

where \(-1 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1, \ p \geq 0 \) and \( \psi > 0, \ \psi \in DT(J) \). Corresponding to such a \( w \), we define the set \( D = J - T \), where \( T = \{t_0, t_1, \ldots, t_{p+1}\} \).

**Corollary 2.** Assume that \( f \in DT(J), \ w \in GSJ \) and \( k \in DT(\Delta) \cap L_1(J) \), where \( \Delta \) is any compact subset of \( D \). If (9) holds uniformly in \( J \), then (18) holds uniformly in \( \Delta \).

**Proof.** By Theorem 3 in [9], (19) holds uniformly in \( \Delta \). \( \Box \)

**Corollary 3.** Assume that \( f \in DT(J) \) and \( w(x) = (1 - x^2)^{-1/2} \), or that \( f \in H_{1/2+\epsilon}(J) \) and \( w(x) \equiv 1 \), where \( H_{\mu}(J) = \{g : \omega_j(g; t) < At^\mu, \ 0 < \mu \leq 1, \ A > 0\} \). If \( k \in DT(\Delta) \cap L_1(J) \), where \( \Delta \) is any compact subset of \((-1, 1)\), then (18) holds uniformly in \( \Delta \).

**Proof.** Under the above hypotheses, (9) holds uniformly in \( J \). \( \Box \)
Corollary 4. Assume that \( f \in DT(J) \), \( w \in GSJ \), \( w^{-1} \in L_1(J) \), \( k^2w^{-1} \in L_1(J) \) and \( k \in DT(\Delta) \cap L_1(J) \) for every compact subset \( \Delta \) of \( D \). Then (18) holds uniformly in any compact subset of \( \Delta \) of \( D \).

Proof. Let \( h \) be the distance of \( \Delta \) from \( T \). Then we can find a compact set \( \Delta \) such that \( \Delta \subset \Delta \subset D \) and the distance of \( \Delta \) from \( J - \Delta \) is \( h/2 \). Since by Theorem 3 in [9], (19) holds uniformly in \( \Delta \), we must show (16). Now, by Theorem 2 in [4, p. 95] and by the properties of \( p_n(x ; w) \), we have \( r_N(x) \to 0 \) uniformly in \( \Delta \). Since \( w \in DT(\Delta) \),

\[
\left| \int_{\Delta} r_N(x)w[x, \lambda] \, dx \right| \leq \|r_N\|_\Delta \left| \int_{\Delta} |w[x, \lambda]| \, dx \right| \to 0.
\]

Furthermore,

\[
\left| \int_{-\Delta} r_N(x)w[x, \lambda] \, dx \right|
\leq \left( \int_{-\Delta} w(x)r_N^2(x) \, dx \right)^{1/2} \left( \int_{-\Delta} \frac{w^2[x, \lambda]}{w(x)} \, dx \right)^{1/2}.
\]

Since \( f \in L_2(w) \), the first integral in the right-hand side tends to 0. As for the second integral, we have that

\[
\int_{-\Delta} \frac{(w(x) - w(\lambda))^2}{w(x)(x - \lambda)^2} \, dx \leq \frac{4}{h^2} \int_{-1}^1 (w(x) - 2w(\lambda) + w(\lambda)w(x)^{-1}) \, dx < \infty.
\]

Similarly, since \( k \in DT(\Delta) \), one has \( \int_{\Delta} r_N(x)k[x, \lambda] \, dx \to 0 \).

As for \( \int_{-\Delta} r_N(x)k[x, \lambda] \, dx \), we use an inequality analogous to (21) and the fact that

\[
\int_{-\Delta} \frac{k^2[x, \lambda]}{w(x)} \, dx \leq \frac{4}{h^2} \int_{-1}^1 \frac{k^2(x) - 2k(x)k(\lambda) + k(\lambda) \, dx}{w(x)} < \infty,
\]

since \( kw^{-1} = (kw^{-1/2})w^{-1/2} \in L_1(J) \) by the Cauchy-Schwarz inequality. \( \Box \)

As particular cases of Corollary 4, we note that if \( w(x) = (1 - x^2)^{-1/2} \), we only require of \( k \) that \( |k(x)| \leq C(1 - x^2)^{-3/4+\epsilon} \), while if \( w(x) \equiv 1 \), we require that \( |k(x)| \leq C(1 - x^2)^{-1/2+\epsilon} \). As in Corollary 3, this again shows the superiority of the Chebyshev weight.

Once we have shown that (18) holds, we can proceed to the study of the convergence of \( Q_N(f ; \lambda) \). We shall state here three theorems corresponding to Theorems 6–8 in [9]. We do not give any proofs, since they are almost identical to the proofs in [9].

Theorem 2. Assume that \( f \in R(J) \), that \( I(kf ; \lambda) \) exists and that \( w \in A \), \( k \in L_1(J) \) and \( \lambda \in (-1, 1) \) are such that (18) holds. Let \( \{Q_m(g)\} \) be a sequence of integration rules such that (6) holds for all \( g \in R(J) \). Then

\[
\lim_{N \to \infty} \lim_{m \to \infty} Q_N(f ; \lambda) = I(kf ; \lambda).
\]
Theorem 3. Suppose that for \( m = 1, 2, \ldots \), the rule \( Q_m(g) \) has precision \( \pi_m > N_m \), that \( \mu_m \equiv \min(N_m, \pi_m - N_m) \to \infty \) as \( m \to \infty \) and that
\[
\sum_{i=1}^{m} |w_{i,m}^{N_m}(\lambda)| \leq C \log \mu_m, \quad m = 1, 2, \ldots.
\]
Assume that \( f \in C(J) \) satisfies the Dini-Lipschitz condition
\[
\lim_{t \to 0} \omega(f; t) \log t = 0,
\]
that \( I(kf; \lambda) \) exists, that \( M_0(k; \lambda) \) is finite and that \( |k| \) is bounded in \( N_\delta(\lambda) \) for some \( \delta > 0 \). Then
\[
\lim_{m \to \infty} Q_m^m(f; \lambda) = I(kf; \lambda).
\]

Theorem 4. Assume that (6) holds for all \( g \in R(J) \), that \( I(kf; \lambda) \) exists and that (18) holds. Then, given a sequence \( \{(m, N_m)\} \) of pairs of positive integers with \( N_m \to \infty \) as \( m \to \infty \), we have that (23) holds if and only if for every \( \epsilon > 0 \), we can find a positive integer \( l \) such that for all \( m \) sufficiently large,
\[
\left| \sum_{j=1}^{N_m} Q_m(f; \lambda) M_j(k; \lambda) \right| < \epsilon.
\]

Appendix 1

In this appendix we give the backward recurrence formulae of Paget [5] for the evaluation of \( S = \sum_{j=0}^{N} c_j N_j(k) \) for the case \( w(x) \equiv 1 \), i.e.,
\[
N_j(k) = \int_{-1}^{1} k(x) P_n(x) dx,
\]
and for three classes of functions \( k \). In each case we construct the sequence \( \{b_j\} \) defined by
\[
b_{N+2} = b_{N+1} = 0, \quad b_j = c_j + u_j b_{j+1} + v_j b_{j+2}, \quad j = N, N-1, \ldots, 0.
\]

1. For \( k(x) = \exp(\tau x) \),
   \[
   u_j = i(2j+1)/\tau, \quad v_j = 1 \quad \text{and} \quad S = 2(b_0 \sin \tau - ib_1 \cos \tau)/\tau.
   \]

2. For \( k(x) = \log|x-\tau|, \quad -1 < \tau < 1 \),
   \[
   u_j = (2j+1)\tau/(j+2), \quad v_j = -(j-1)/(j+2) \quad \text{and}
   \]
   \[
   S = (b_0 - b_1/2)(1 + \tau) \log(1 + \tau) + (b_0 + b_1/2)(1 - \tau) \log(1 - \tau)
   \]
   \[+ 2b_2/3 - 2b_0. \]

3. For \( k(x) = |x-\tau|^\alpha, \quad \alpha > -1, \quad -1 < \tau < 1 \),
   \[
   u_j = (2j+1)\tau/(j + \alpha + 2), \quad v_j = -(j - \alpha - 1)/(j + \alpha + 2) \quad \text{and}
   \]
   \[
   S = \left(\frac{b_0}{\alpha + 1} + \frac{b_1}{\alpha + 2}\right)(1 - \tau)^{\alpha+1} + \left(\frac{b_0}{\alpha + 1} - \frac{b_1}{\alpha + 2}\right)(1 + \tau)^{\alpha+1}.
   \]
Appendix 2

We give here the backward recurrence relations for evaluating

\[ S = \sum_{j=0}^{N} d_j M_j(k; \lambda) \]

where \( M_j(k; \lambda) = I(kp_j; \lambda) \), the \( p_j \) satisfy (8) and the \( M_j(k; \lambda) \) satisfy (12) with initial conditions

\[ M_{-1}(k; \lambda) \equiv 0, \quad M_0(k; \lambda) = I(k; \lambda). \]

If we choose \( d_j = a_{jm} \), then \( Q_m^N(f; \lambda) = S \) and if we choose \( d_j = p_j(x_{im}) \), then \( w_{im}(\lambda) = w_{im}S \).

We construct the sequence \( \{b_j\} \) defined by

\[ b_{N+2} = b_{N+1} = 0, \quad b_j = (\alpha_j - \beta_j) b_{j+1} - \beta_j b_{j+2} + d_j, \quad j = N, N-1, \ldots, 0. \]

Then

\[ S = b_0 I(k; \lambda) + \sum_{j=0}^{N-1} A_j N_j(k). \]

The latter sum can, in turn, be evaluated by backward recurrence as in Appendix 1, or by any other convenient algorithm. As for the evaluation of \( I(k; \lambda) \), see [7].

Bibliography


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