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# GENERALIZED NONINTERPOLATORY RULES FOR CAUCHY PRINCIPAL VALUE INTEGRALS

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ABSTRACT. Consider the Cauchy principal value integral

$$I(kf;\lambda) = \fint_{-1}^{1} k(x)\frac{f(x)}{x-\lambda}\,dx, \quad -1<\lambda<1.$$

If we approximate $f(x)$ by $\sum_{j=0}^{N} a_j p_j(x;w)$ where $\{p_j\}$ is a sequence of orthonormal polynomials with respect to an admissible weight function $w$ and $a_j = (f, p_j)$, then an approximation to $I(kf;\lambda)$ is given by $\sum_{j=0}^{N} a_j I(kp_j;\lambda)$. If, in turn, we approximate $a_j$ by $a_{jm} = \sum_{i=1}^{m} w_{im} f(x_{im}) p_j(x_{im})$, then we get a double sequence of approximations $\{Q_m^N(f;\lambda)\}$ to $I(kf;\lambda)$. We study the convergence of this sequence by relating it to the sequence of approximations associated with $I(wf;\lambda)$ which has been investigated previously.

## 1. INTRODUCTION

In a recent paper, Rabinowitz and Lubinsky [9] studied the convergence properties of a method proposed by Rabinowitz [7] and Henrici [3] for the numerical evaluation of Cauchy principal value (CPV) integrals of the form

$$(1) \qquad I(wf;\lambda) = \fint_{-1}^{1} w(x)\frac{f(x)}{x-\lambda}\,dx, \qquad -1<\lambda<1,$$

where $w \in A$, the set of all admissible weight functions, i.e., all functions $w$ on $J=[-1,1]$ such that $w \geq 0$ and $\|w\|_1 > 0$. This method is based on approximating $I(wf;\lambda)$ by

$$(2) \qquad \hat{S}_N(f;\lambda) = \sum_{j=0}^{N} a_j q_j(\lambda),$$

where

$$(3) \qquad a_j = (f, p_j),$$

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$q_j(\lambda) = I(wp_j; \lambda)$ and $\{p_j(x; w) : j = 0, 1, 2, \ldots\}$ is the family of orthonormal polynomials with respect to $w$. In turn, $\hat{S}_N(f; \lambda)$ is approximated by

$$\hat{Q}_m^N(f; \lambda) = \sum_{j=0}^{N} a_{jm} q_j(\lambda), \tag{4}$$

where $a_{jm} = Q_m(fp_j)$ is an approximation to $a_j$ based on the numerical integration rule

$$Q_m(g) = \sum_{i=1}^{m} w_{im} g(x_{im}), \tag{5}$$

and where we assume that

$$\lim_{m \to \infty} Q_m(g) = \int_{-1}^{1} w(x) g(x)\, dx \tag{6}$$

for all $g \in C(J)$ or all $g \in R(J)$, the set of all Riemann-integrable functions on $J$.

Now, this method requires knowledge of the three-term recurrence relation for the polynomials $p_j$ which is not always available. Furthermore, it is not always easy to find squences of integration rules $Q_m(g)$ which satisfy (6), especially if $w$ is a nonstandard weight or if we do not wish to use Gaussian rules but rather rules which concentrate many integration points in subintervals where $f$ is not well behaved. Finally, the restriction to admissible weight functions does not allow us to deal with CPV integrals of the form

$$I(kf; \lambda) = ⨍_{-1}^{1} k(x) \frac{f(x)}{x - \lambda}\, dx, \qquad -1 < \lambda < 1, \tag{7}$$

where $k$ is such that $I(kf; \lambda)$ exists but $k$ need not be nonnegative. Since the main idea in writing the numerator of the integrand in (7) as the product of two functions, $k$ and $f$, is to incorporate the singular or difficult part of the numerator into $k$ and treat it analytically while treating the smooth factor $f$ numerically, it would make no sense to rewrite (7) as $I(wF; \lambda)$ with $F = w^{-1}kf$ unless $w$ had the same singularity structure as $k$, and even then we would usually have the problems mentioned above.

In this paper, we shall try to overcome these shortcomings in [9] by using ideas of noninterpolatory product integration [8] combined with a device found in [1] for expressing CPV integrals with respect to one function, say $k$, in terms of CPV integrals with respect to a second function, say $w$, positive in $(-1, 1)$. The point is that we can then choose a convenient weight function $w$ for expressing our inner products and for evaluating the approximations to these inner products, for example $w(x) \equiv 1$ or $w(x) = (1 - x^2)^{-1/2}$. In fact, this latter weight function is particularly useful, as we shall see. We shall first describe the method in §2 and then study some convergence questions in §3.

## 2. A generalized noninterpolatory rule

Consider the CPV integral $I(kf; \lambda)$ given by (7) where $k \in DT(N_\delta(\lambda)) \cap L_1(J)$ and $f \in DT(N_\delta(\lambda)) \cap R(J)$, which ensures that $I(kf; \lambda)$ exists. Here

$$N_\delta(\lambda) = [\lambda - \delta, \ \lambda + \delta] \subset (-1, 1)$$

and, for any interval $I$ of length $l(I)$,

$$DT(I) = \left\{ g : \int_0^{l(I)} \omega_I(g; t)t^{-1}\, dt < \infty \right\},$$

where the modulus of continuity of $g$ on $I$ is given by

$$\omega_I(g; t) = \sup_{\substack{|x_1 - x_2| \le t \\ x_1, x_2 \in I}} |g(x_1) - g(x_2)|.$$

Assume now that we have a convenient weight function $w \in DT(N_\delta(\lambda)) \cap A$ such that $w(\lambda) > 0$. We then have a three-term recurrence relation for the sequence of orthonormal polynomials $\{p_j(x; w)\}$ of the form

$$(8)\quad p_{-1} = 0, \quad p_0 = 1, \quad p_{j+1}(x) = (A_j x - \alpha_j)p_j(x) - \beta_j p_{j-1}(x), \qquad j \ge 0.$$

If we expand $f$ in an orthogonal series in terms of the $p_j(x; w)$, which for the moment, we assume converges uniformly in $J$,

$$f(x) = \sum_{j=0}^{\infty} a_j p_j(x; w), \tag{9}$$

then we can approximate $f(x)$ by $\sum_{j=0}^{N} a_j p_j(x; w)$ and $I(kf; \lambda)$ by

$$S_N(f; \lambda) = \sum_{j=0}^{N} a_j M_j(k; \lambda), \tag{10}$$

where $M_j(k; \lambda) = I(kp_j; \lambda)$. In turn, we then approximate $S_N(f; \lambda)$ by

$$Q_m^N(f; \lambda) = \sum_{j=0}^{N} a_{jm} M_j(k; \lambda). \tag{11}$$

The $M_j(k; \lambda)$ satisfy the following nonhomogeneous recurrence relation

$$M_{j+1}(k; \lambda) = (A_j\lambda - \alpha_j)M_j(k; \lambda) - \beta_j M_{j-1}(k; \lambda) + A_j N_j(k), \tag{12}$$

where

$$N_j(k) = \int_{-1}^{1} k(x)p_j(x; w)\, dx.$$

Relation (12) follows by replacing $p_{j+1}$ in $I(kp_{j+1}; \lambda)$ by the right-hand side of (8) and using the well-known device

$$⨍_{-1}^{1} k(x)\frac{xp_j(x)}{x - \lambda}\, dx = \int_{-1}^{1} k(x)\frac{(x - \lambda)p_j(x)}{x - \lambda}\, dx + \lambda ⨍_{-1}^{1} k(x)\frac{p_j(x)}{x - \lambda}\, dx.$$

Hence, if we know the $N_j(k)$, we can evaluate (11) in a stable manner by backward recurrence.

If $w(x) = (1 - x^2)^{-1/2}$, so that (except for normalization) $p_j = T_j$, the Chebyshev polynomial of the first kind, then recurrrence relations for $N_j(k)$ are known for a wide variety of functions [6]. For $w(x) \equiv 1$, for which $p_j = P_j$, the

Legendre polynomial, recurrence relations for $N_j(k)$ for $k(x) = e^{i\tau x}$, $|x-\tau|^\alpha$ and $\log|x-\tau|$ are given by Paget [5], and for a variety of functions by Gatteschi [2]. Since the work of Paget is not readily available, we give his recurrence relations in Appendix 1. In Appendix 2, we give the recurrence relations for evaluating $Q_m^N(f;\lambda)$ when the $N_j(k)$ are known, as well as for evaluating the weights $w_{im}^N(\lambda)$ in the Lagrangian formulation of $Q_m^N(f;\lambda)$, namely

$$Q_m^N(f;\lambda) = \sum_{i=1}^{m} w_{im}^N(\lambda) f(x_{im}) \tag{13}$$

with

$$w_{im}^N(\lambda) = w_{im} \sum_{j=0}^{N} p_j(x_{im}) M_j(k;\lambda). \tag{14}$$

## 3. Convergence results

We study first the convergence of $S_N(f;\lambda)$ to $I(kf;\lambda)$, for then we can proceed as in [9] to study the convergence of $Q_m^N(f;\lambda)$ to $I(kf;\lambda)$, either as an iterated limit or as a double limit. Since we have results in [9] for the convergence of $\hat{S}_N(f;\lambda)$ to $I(wf;\lambda)$, we shall try to reduce the study of the convergence of $S_N(f;\lambda)$ to that of the convergence of $\hat{S}_N(f;\lambda)$. To this end, we use a device in [1] to relate a CPV integral weighted by $k$ to one weighted by $w$. This is done by writing

$$\begin{aligned} I(kf;\lambda) &= \fint_{-1}^{1} k(x)\frac{f(x)}{x-\lambda}\,dx = \fint_{-1}^{1} w(x)\frac{k(x)}{w(x)}\frac{f(x)}{x-\lambda}\,dx \\ &= \int_{-1}^{1} w(x)\frac{f(x)}{x-\lambda}\left[\frac{k(x)}{w(x)} - \frac{k(\lambda)}{w(\lambda)}\right]dx + \frac{k(\lambda)}{w(\lambda)}\fint_{-1}^{1} w(x)\frac{f(x)}{x-\lambda}\,dx \\ &= \int_{-1}^{1} f(x)k[x,\lambda]\,dx - \frac{k(\lambda)}{w(\lambda)}\int_{-1}^{1} f(x)w[x,\lambda]\,dx + \frac{k(\lambda)}{w(\lambda)} I(wf;\lambda). \end{aligned}$$

Here, we have used the divided difference notation,

$$h[x,y] = \frac{h(x)-h(y)}{x-y}.$$

Consequently, if we have conditions on $f$ and $w$ which ensure convergence of $\hat{S}_N(f;\lambda)$ to $I(wf;\lambda)$, we need only find the additional conditions on $f$, $k$ and $w$ to insure the convergence of

$$\sum\nolimits_1 \equiv \sum_{j=0}^{N} a_j \int_{-1}^{1} p_j(x) w[x,\lambda]\,dx \quad \text{to} \quad \int_{-1}^{1} f(x) w[x,\lambda]\,dx \equiv I_1$$

and

$$\sum\nolimits_2 \equiv \sum_{j=0}^{N} a_j \int_{-1}^{1} p_j(x) k[x,\lambda]\,dx \quad \text{to} \quad \int_{-1}^{1} f(x) k[x,\lambda]\,dx \equiv I_2,$$

for then

$$S_N(f;\lambda) = \sum_{j=0}^{N} a_j M_j(k;\lambda) = \frac{k(\lambda)}{w(\lambda)} \sum_{j=0}^{N} a_j q_j(\lambda) - \frac{k(\lambda)}{w(\lambda)} \sum{}_1 + \sum{}_2$$

$$\to \frac{k(\lambda)}{w(\lambda)} I(wf;\lambda) - \frac{k(\lambda)}{w(\lambda)} I_1 + I_2 = I(kf;\lambda).$$

Clearly, sufficient conditions for the convergence of $\sum_1$ and $\sum_2$ are that (9) holds uniformly in $J$ and that $w$ and $k \in DT(J)$, for then

$$\left|I_1 - \sum{}_1\right| \le 2\|r_N\|_\infty \int_0^2 \omega_J(w;t)t^{-1}\,dt, \tag{15}$$

where $r_N(x) = \sum_{j=N+1}^{\infty} a_j p_j(x;w)$, and similarly for $|I_2 - \sum_2|$. Hence, provided $|k(\lambda)| < \infty$ and $w(\lambda) > 0$, we have convergence of $S_N(f;\lambda)$ whenever $\hat{S}_N(f;\lambda)$ converges. Furthermore, if $\hat{S}_N(f;\lambda)$ converges uniformly with respect to $\lambda$ on some closed subset $\Delta$ of $(-1, 1)$ and $w(\lambda) > 0$ and $|k(\lambda)| < \infty$ on $\Delta$, then we will have uniform convergence of $S_N(f;\lambda)$ on $\Delta$. However, we can weaken these conditions in various directions. Thus, it is not necessary that $w$ and $k \in DT(J)$, only that $w, k \in DT(N_\delta(\lambda)) \cap L_1(J)$. For then, we can replace (15) by

$$\begin{aligned} \left|I_1 - \sum{}_1\right| &= \left|\int_{-1}^{1} r_N(x) w[x,\lambda]\,dx\right| \\ &\le \|r_N\|_\infty \left[\int_{N_\delta(\lambda)} |w[x,\lambda]|\,dx + \int_{J-N_\delta(\lambda)} |w[x,\lambda]|\,dx\right], \end{aligned} \tag{16}$$

where both integrals are finite, and similarly for $\sum_2$. The first integral in (16) is finite since

$$\begin{aligned} \int_{N_\delta(\lambda)} |w[x,\lambda]|\,dx &= \int_{\lambda-\delta}^{\lambda+\delta} \left|\frac{w(x) - w(\lambda)}{x - \lambda}\right| dx \\ &= \int_{-\delta}^{\delta} \left|\frac{w(t+\lambda) - w(\lambda)}{t}\right| dt \le 2\int_0^\delta \omega_{N_\delta(\lambda)}(w;t)t^{-1}\,dt \end{aligned}$$

while, for the second integral, we have

$$\begin{aligned} \int_{J-N_\delta(\lambda)} |w[x,\lambda]|\,dx &= \int_{J-N_\delta(\lambda)} \left|\frac{w(x) - w(\lambda)}{x - \lambda}\right| dx \\ &\le \delta^{-1} \int_{J-N_\delta(\lambda)} |w(x) - w(\lambda)|\,dx \\ &< \delta^{-1}[\|w\|_1 + 2w(\lambda)] < \infty. \end{aligned}$$

Another possibility is to require only that (9) holds uniformly in $N_\delta(\lambda)$. Then, if both $w^{-1} \in L_1(J)$ and $k^2/w \in L_1(J)$, a well-known condition in product integration theory [10], we have convergence of $S_N(f;\lambda)$. We summarize these remarks in a theorem and several corollaries.

**Theorem 1.** *Assume that for some* $\lambda \in (-1, 1)$,

$$(17) \quad \int_{-1}^{1} r_N(x) w[x, \lambda]\, dx \to 0, \qquad \int_{-1}^{1} r_N(x) k[x, \lambda]\, dx \to 0 \quad \text{as } N \to \infty,$$

*that* $w(\lambda) > 0$ *and that* $|k(\lambda)| < \infty$. *Then*

$$(18) \qquad S_N(f; \lambda) \to I(kf; \lambda)$$

*if and only if*

$$(19) \qquad \hat{S}_N(f; \lambda) \to I(wf; \lambda).$$

*Let* $\Delta$ *be a closed subset of* $(-1, 1)$ *and assume that* (17) *holds uniformly in* $\Delta$, *and that* $w(\lambda) > 0$ *and* $|k(\lambda)| < \infty$ *for all* $\lambda \in \Delta$; *then* (18) *holds uniformly in* $\Delta$ *if and only if* (19) *holds uniformly in* $\Delta$.

**Corollary 1.** *If for some* $\lambda \in (-1, 1)$, $\sup_j |q_j(\lambda)| < \infty$, $\sup_j \|p_j(\cdot; w)\|_\infty < \infty$, $w(\lambda) > 0$, $w$, $k \in DT(N_\delta(\lambda)) \cap L_1(J)$, $f \in L_{1,w}(J)$ *and* $f[x, \lambda] \in L_{1,w}(J)$, *then* (18) *holds.*

*Proof.* By Theorem 2 in [9], the hypotheses of the corollary suffice for (19) to hold. By Theorem 4 in [4, p. 70], $\|r_N\|_\infty \to 0$. Hence, as in (16),

$$\left| \int_{-1}^{1} r_N(x) w[x, \lambda]\, dx \right| \le \|r_N\|_\infty \left[ \int_{N_\delta(\lambda)} |w[x, \lambda]|\, dx + \int_{J - N_\delta(\lambda)} |w[x, \lambda]|\, dx \right] \to 0,$$

and similarly for $\int_{-1}^{1} r_N(x) k[x, \lambda]\, dx$. Furthermore, since $k \in DT(N_\delta(\lambda))$, one has $|k(\lambda)| < \infty$. Hence, by Theorem 1, (18) holds. □

Before stating the next corollary, we recall the definition of a generalized smooth Jacobi (GSJ) weight function [1]. We say that $w \in \mathrm{GSJ}$ if

$$(20) \qquad w(x) = \psi(x) \prod_{j=0}^{p+1} |x - t_j|^{\gamma_j}, \qquad \gamma_j > -1,\ j = 0, \ldots, p+1,$$

where $-1 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $p \ge 0$ and $\psi > 0$, $\psi \in DT(J)$. Corresponding to such a $w$, we define the set $D = J - T$, where $T = \{t_0, t_1, \ldots, t_{p+1}\}$.

**Corollary 2.** *Assume that* $f \in DT(J)$, $w \in GSJ$ *and* $k \in DT(\Delta) \cap L_1(J)$, *where* $\Delta$ *is any compact subset of* $D$. *If* (9) *holds uniformly in* $J$, *then* (18) *holds uniformly in* $\Delta$.

*Proof.* By Theorem 3 in [9], (19) holds uniformly in $\Delta$. □

**Corollary 3.** *Assume that* $f \in DT(J)$ *and* $w(x) = (1 - x^2)^{-1/2}$, *or that* $f \in H_{1/2+\varepsilon}(J)$ *and* $w(x) \equiv 1$, *where* $H_\mu(J) = \{g : \omega_J(g; t) < At^\mu,\ 0 < \mu \le 1,\ A > 0\}$. *If* $k \in DT(\Delta) \cap L_1(J)$, *where* $\Delta$ *is any compact subset of* $(-1, 1)$, *then* (18) *holds uniformly in* $\Delta$.

*Proof.* Under the above hypotheses, (9) holds uniformly in $J$. □

**Corollary 4.** *Assume that* $f \in DT(J)$, $w \in GSJ$, $w^{-1} \in L_1(J)$, $k^2 w^{-1} \in L_1(J)$ *and* $k \in DT(\tilde{\Delta}) \cap L_1(J)$ *for every compact subset* $\tilde{\Delta}$ *of* $D$. *Then* (18) *holds uniformly in any compact subset of* $\Delta$ *of* $D$.

*Proof*. Let $h$ be the distance of $\Delta$ from $T$. Then we can find a compact set $\tilde{\Delta}$ such that $\Delta \subset \tilde{\Delta} \subset D$ and the distance of $\Delta$ from $J - \tilde{\Delta}$ is $h/2$. Since by Theorem 3 in [9], (19) holds uniformly in $\Delta$, we must show (16). Now, by Theorem 2 in [4, p. 95] and by the properties of $p_n(x; w)$, we have $r_N(x) \to 0$ uniformly in $\tilde{\Delta}$. Since $w \in DT(\tilde{\Delta})$,

$$\left|\int_{\tilde{\Delta}} r_N(x) w[x, \lambda]\, dx\right| \le \|r_N\|_{\tilde{\Delta}} \int_{\tilde{\Delta}} |w[x, \lambda]|\, dx \to 0.$$

Furthermore,

$$\begin{aligned} &\left|\int_{J-\tilde{\Delta}} r_N(x) w[x, \lambda]\, dx\right| \\ &\le \left(\int_{J-\tilde{\Delta}} w(x) r_N^2(x)\, dx\right)^{1/2} \left(\int_{J-\tilde{\Delta}} \frac{w^2[x, \lambda]}{w(x)}\, dx\right)^{1/2}. \end{aligned} \tag{21}$$

Since $f \in L_{2,w}$, the first integral in the right-hand side tends to $0$. As for the second integral, we have that

$$\int_{J-\tilde{\Delta}} \frac{(w(x) - w(\lambda))^2}{w(x)(x-\lambda)^2}\, dx \le \frac{4}{h^2} \int_{-1}^{1} (w(x) - 2w(\lambda) + w(\lambda) w(x)^{-1})\, dx < \infty.$$

Similarly, since $k \in DT(\tilde{\Delta})$, one has $\int_{\tilde{\Delta}} r_N(x) k[x, \lambda]\, dx \to 0$.

As for $\int_{J-\tilde{\Delta}} r_N(x) k[x, \lambda]\, dx$, we use an inequality analogous to (21) and the fact that

$$\int_{J-\tilde{\Delta}} \frac{k^2[x, \lambda]}{w(x)}\, dx \le \frac{4}{h^2} \int_{-1}^{1} \frac{k^2(x) - 2k(x)k(\lambda) + k(\lambda)\, dx}{w(x)} < \infty,$$

since $kw^{-1} = (kw^{-1/2}) w^{-1/2} \in L_1(J)$ by the Cauchy-Schwarz inequality. □

As particular cases of Corollary 4, we note that if $w(x) = (1 - x^2)^{-1/2}$, we only require of $k$ that $|k(x)| \le C(1 - x^2)^{-3/4+\varepsilon}$, while if $w(x) \equiv 1$, we require that $|k(x)| \le C(1 - x^2)^{-1/2+\varepsilon}$. As in Corollary 3, this again shows the superiority of the Chebyshev weight.

Once we have shown that (18) holds, we can proceed to the study of the convergence of $Q_m^N(f; \lambda)$. We shall state here three theorems corresponding to Theorems 6–8 in [9]. We do not give any proofs, since they are almost identical to the proofs in [9].

**Theorem 2.** *Assume that* $f \in R(J)$, *that* $I(kf; \lambda)$ *exists and that* $w \in A$, $k \in L_1(J)$ *and* $\lambda \in (-1, 1)$ *are such that* (18) *holds. Let* $\{Q_m(g)\}$ *be a sequence of integration rules such that* (6) *holds for all* $g \in R(J)$. *Then*

$$\lim_{N\to\infty} \lim_{m\to\infty} Q_m^N(f; \lambda) = I(kf; \lambda). \tag{22}$$

**Theorem 3.** *Suppose that for* $m = 1, 2, \dots$, *the rule* $Q_m(g)$ *has precision* $\pi_m > N_m$, *that* $\mu_m \equiv \min(N_m, \pi_m - N_m) \to \infty$ *as* $m \to \infty$ *and that*

$$\sum_{i=1}^{m} |w_{im}^{N_m}(\lambda)| \le C \log \mu_m, \qquad m = 1, 2, \dots .$$

*Assume that* $f \in C(J)$ *satisfies the Dini-Lipschitz condition*

$$\lim_{t \to 0} \omega_J(f; t) \log t = 0,$$

*that* $I(kf; \lambda)$ *exists, that* $M_0(k; \lambda)$ *is finite and that* $|k|$ *is bounded in* $N_\delta(\lambda)$ *for some* $\delta > 0$. *Then*

$$\lim_{m \to \infty} Q_m^{N_m}(f; \lambda) = I(kf; \lambda). \tag{23}$$

**Theorem 4.** *Assume that* (6) *holds for all* $g \in R(J)$, *that* $I(kf; \lambda)$ *exists and that* (18) *holds. Then, given a sequence* $\{(m, N_m)\}$ *of pairs of positive integers with* $N_m \to \infty$ *as* $m \to \infty$, *we have that* (23) *holds if and only if for every* $\varepsilon > 0$, *we can find a positive integer* $l$ *such that for all* $m$ *sufficiently large,*

$$\left| \sum_{j=l}^{N_m} Q_m(fp_j) M_j(k; \lambda) \right| < \varepsilon .$$

## Appendix 1

In this appendix we give the backward recurrence formulae of Paget [5] for the evaluation of $S = \sum_{j=0}^{N} c_j N_j(k)$ for the case $w(x) \equiv 1$, i.e.,

$$N_j(k) = \int_{-1}^{1} k(x) P_n(x)\, dx,$$

and for three classes of functions $k$. In each case we construct the sequence $\{b_j\}$ defined by

$$b_{N+2} = b_{N+1} = 0, \quad b_j = c_j + u_j b_{j+1} + v_{j+1} b_{j+2}, \qquad j = N, N-1, \dots, 0.$$

1. For $k(x) = \exp(i\tau x)$,

$$u_j = i(2j+1)/\tau, \quad v_j = 1 \quad \text{and} \quad S = 2(b_0 \sin\tau - ib_1 \cos\tau)/\tau .$$

2. For $k(x) = \log|x - \tau|$, $-1 < \tau < 1$,

$$u_j = (2j+1)\tau/(j+2), \quad v_j = -(j-1)/(j+2) \quad \text{and}$$
$$S = (b_0 - b_1/2)(1+\tau)\log(1+\tau) + (b_0 + b_1/2)(1-\tau)\log(1-\tau) + 2b_2/3 - 2b_0 .$$

3. For $k(x) = |x - \tau|^{\alpha}$, $\alpha > -1$, $-1 < \tau < 1$,

$$u_j = (2j+1)\tau/(j+\alpha+2), \quad v_j = -(j-\alpha-1)/(j+\alpha+2) \quad \text{and}$$
$$S = \left(\frac{b_0}{\alpha+1} + \frac{b_1}{\alpha+2}\right)(1-\tau)^{\alpha+1} + \left(\frac{b_0}{\alpha+1} - \frac{b_1}{\alpha+2}\right)(1+\tau)^{\alpha+1} .$$

## APPENDIX 2

We give here the backward recurrence relations for evaluating

$$S = \sum_{j=0}^{N} d_j M_j(k\,;\lambda)$$

where $M_j(k\,;\lambda) = I(kp_j\,;\lambda)$, the $p_j$ satisfy (8) and the $M_j(k\,;\lambda)$ satisfy (12) with initial conditions

$$M_{-1}(k\,;\lambda) \equiv 0\,, \qquad M_0(k\,;\lambda) = I(k\,;\lambda)\,.$$

If we choose $d_j = a_{jm}$, then $Q_m^N(f\,;\lambda) = S$ and if we choose $d_j = p_j(x_{im})$, then $w_{im}^N(\lambda) = w_{im}S$.

We construct the sequence $\{b_j\}$ defined by $b_{N+2} = b_{N+1} = 0$,

$$b_j = (A_j\lambda - \alpha_j)b_{j+1} - \beta_j b_{j+2} + d_j\,, \qquad j = N\,, N-1\,, \dots\,, 0\,.$$

Then

$$S = b_0 I(k\,;\lambda) + \sum_{j=0}^{N-1} A_j N_j(k)\,.$$

The latter sum can, in turn, be evaluated by backward recurrence as in Appendix 1, or by any other convenient algorithm. As for the evaluation of $I(k\,;\lambda)$, see [7].

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