COUPLING FINITE ELEMENT AND SPECTRAL METHODS: FIRST RESULTS

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ABSTRACT. A Poisson equation on a rectangular domain is solved by coupling two methods: the domain is divided in two squares; a finite element approximation is used on the first square and a spectral discretization is used on the second. Two kinds of matching conditions on the interface are presented and compared; in both cases, error estimates are proved.

1. INTRODUCTION

To approximate the solutions of partial differential equations, a number of methods can be successfully applied: among them, spectral-type methods, in which the discrete solution is a polynomial of high degree, are known to be very accurate when the solution to be approximated is very smooth (see [16, 8] for a general description of these methods). Their main drawback lies in the difficulty to take into account the singularities of the function to be approximated, and also in the difficulty in handling domains with a complicated boundary. This last problem is usually solved by decomposition into subdomains and/or transformation of coordinates. On the other hand, the finite element method, where the discrete solution, restricted to very small domains called "elements", is a polynomial of low degree, is well suited to problems with complex geometries, but its accuracy is limited by the degree of the polynomials (general properties of finite elements are analyzed in [11]). Several attempts have been made to combine the two methods into a unified framework, and thereby obtaining the advantages of both. The main idea on which these attempts rely consists of a decomposition of the domain into (rather) small subdomains so as to fit the geometric complexity of the boundary, and then use high-degree polynomials on each subdomain to approximate the solution. Two different approaches have been proposed: the spectral element method and the p-version of the finite element method. The spectral element method [25], which consists of using a spectral algorithm on a fixed number of subdomains, is presently developed for a growing number of problems (see, for instance, [15, 19, and 23]); on the opposite side, the so-called p-version of the finite element method, where
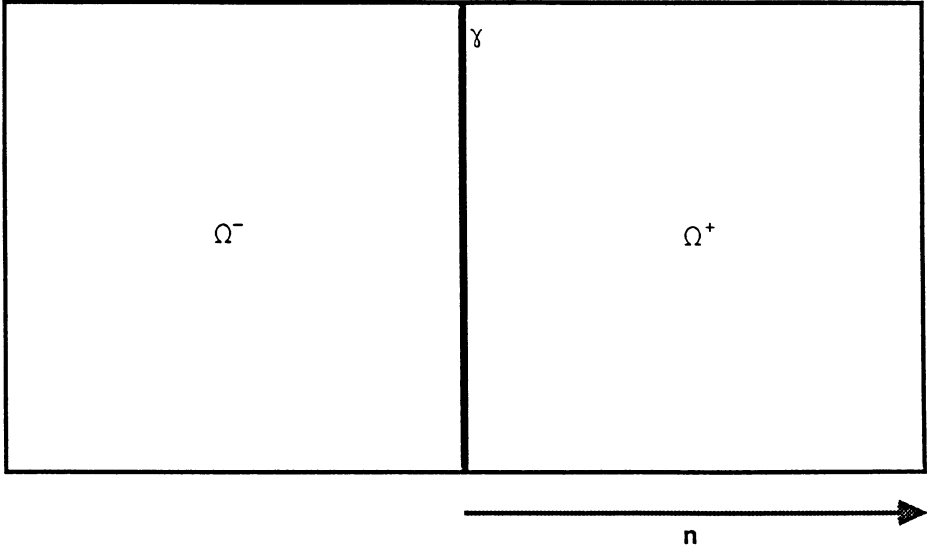
FIGURE 1.1. Decomposition of the domain.

the analysis of the consistency error and of the approximation error. The final error estimates, together with concluding remarks, are given in §5.

The main results of this paper were first presented in [5].

Notation. Let Δ denote any open interval of the real line or any domain in $\mathbb{R}^2$ with a polygonal boundary. For any real number s, we consider the classical Hilbert Sobolev spaces $H^s(\Delta)$, provided with the usual norm $\|\cdot\|_{s,\Delta}$, and also, when s is an integer, with the seminorm $|\cdot|_{s,\Delta}$. For any real number $s \geq 0$ and any p, $1 \leq p \leq +\infty$, $p \neq 2$, we also use the Sobolev spaces $W^{s,p}(\Delta)$, which are no longer Hilbert spaces, provided with the norm $\|\cdot\|_{s,p,\Delta}$. Finally, for any real number $s \geq 0$, $H_0^s(\Delta)$ stands for the closure in $H^s(\Delta)$ of the space of infinitely differentiable functions with compact support in Δ.

Throughout this paper, with any function v defined on Ω, we associate the pair $v^* = (v^-, v^+)$, where v^- (resp. v^+) denotes the restriction of v to Ω^- (resp. Ω^+). The scalar product on $L^2(\Omega^-) \times L^2(\Omega^+)$,

$$(1.2) \qquad (u^*, v^*) = \int_{\Omega^-} u^-(\mathbf{x})v^-(\mathbf{x})\,d\mathbf{x} + \int_{\Omega^+} u^+(\mathbf{x})v^+(\mathbf{x})\,d\mathbf{x},$$

coincides with the usual one on $L^2(\Omega)$. We also provide the product $H^1(\Omega^-) \times H^1(\Omega^+)$ with the norm

$$(1.3) \qquad \|v^*\| = [(v^*, v^*) + (\nabla v^*, \nabla v^*)]^{1/2};$$

the space of pairs v^* in $H^1(\Omega^-) \times H^1(\Omega^+)$ with v continuous through γ is isomorphic to $H^1(\Omega)$. Finally, we define on $H^1(\Omega^-) \times H^1(\Omega^+)$ the bilinear form

$$(1.4) \qquad a(u^*, v^*) = (\nabla u^*, \nabla v^*) \quad \forall (u^*, v^*) \in [H^1(\Omega^-) \times H^1(\Omega^+)]^2.$$

Clearly, for any function f in $L^2(\Omega)$, problem (1.1) is equivalent to the following one: *Find u in $H_0^1(\Omega)$ such that*

$$(1.5) \qquad\qquad a(u^*, v^*) = (f^*, v^*) \quad \forall v \in H_0^1(\Omega).$$

This variational form is precisely the one which will be used in order to define the discrete problems.

In all that follows, c, c', c'', ... are generic positive constants independent of the discretization parameters.

2. THE DISCRETE SPACES AND PROBLEMS

2.1. Definition of the discrete spaces.

We have to define a discrete space on each subdomain Ω^- and Ω^+, and then we must make precise the matching conditions on the interface.

Let h be a real parameter, $0 < h \leq 1$, which will tend to 0. With each value of h, we associate a triangulation $\mathscr{T}_h$ of the domain Ω^-, i.e., a finite set of triangles such that the intersection of two triangles is either empty, or a vertex, or an edge, and such that (cf. Figure 2.1)

$$(2.1) \qquad\qquad \bar{\Omega}^- = \bigcup_{K \in \mathscr{T}_h} K;$$

h is the upper bound of the diameters of the triangles of $\mathscr{T}_h$. We denote by h_K the diameter of any triangle K in $\mathscr{T}_h$, and by ρ_K the diameter of the inscribed circle in K. Next, we assume that the family $(\mathscr{T}_h)_h$ is regular in the following sense (cf. [11, §3.1] or [3, Definition 3.1]): there exists a constant $\tau > 0$ such that, for any h and for any K in $\mathscr{T}_h$, the following inequality holds:

$$(2.2) \qquad\qquad \rho_K \geq \tau h_K.$$

Let k be a fixed integer ≥ 1. For any closed subset A of $\mathbb{R}$ (resp. $\mathbb{R}^2$), we denote by $P_k(A)$ the set of the restrictions to A of polynomials of one variable (resp. two variables) with total degree $\leq k$. With any triangulation $\mathscr{T}_h$, we have the associated finite-dimensional space X_h defined by

$$(2.3) \quad X_h = \{v_h \in \mathscr{C}^0(\bar{\Omega}^-)\,;\, \forall K \in \mathscr{T}_h,\, v_{h|K} \in P_k(K) \text{ and } v_h = 0 \text{ on } \partial\Omega^- \setminus \gamma\}.$$

We also need the finite-dimensional trace space

$$(2.4) \qquad\qquad x_h = \{v_{h|\gamma}\,,\, v_h \in X_h\}.$$

In order to build an appropriate basis of X_h and x_h, we consider each triangle K as the support of a Lagrange finite element $(K, P_k(K), \Xi_K)$, where Ξ_K is the set of all points in K with barycentric coordinates i/k, j/k, and $(k-i-j)/k$, $0 \leq i, j \leq k$, $i + j \leq k$; it is well known [11, Theorem 2.2.1] that this set of points is $P_k(K)$-unisolvent. Next, we set

$$(2.5) \qquad\qquad \Xi_h = \bigcup_{K \in \mathscr{T}_h} \Xi_K,$$

and also

$$(2.6) \qquad\qquad \xi_h = \{\mathbf{a} \in \Xi_h \cap \gamma\}.$$

With each point $\mathbf{a}$ in $\Xi_h \cap (\Omega^- \cup \gamma)$, we associate the unique function $q_\mathbf{a}$ of X_h which is equal to 1 in $\mathbf{a}$ and vanishes at any other point of Ξ_h. Then the set $\{q_\mathbf{a}, \mathbf{a} \in \Xi_h \cap (\Omega^- \cup \gamma)\}$ is a basis of the space X_h and the set $\{q_{\mathbf{a}|\gamma}, \mathbf{a} \in \xi_h\}$ is a basis of the space x_h.

Next, let N be an integer ≥ 1, which will tend to $+\infty$. For any integer $n \geq 0$, we denote by $Q_n(\bar\Omega^+)$ the set of the restrictions to $\bar\Omega^+$ of polynomials of two variables with degree $\leq n$ with respect to each variable. For each integer N, we consider the finite-dimensional space X_N defined by

$$(2.7) \qquad X_N = \{v_N \in Q_N(\bar\Omega^+) ; v_N = 0 \text{ on } \partial\Omega^+ \setminus \gamma\}.$$

Let $(L_n)_{n\in\mathbb{N}}$ be the family of Legendre polynomials on $[0, 1]$, i.e., of orthogonal polynomials on $[0, 1]$ such that L_n, $n \in \mathbb{N}$, is of degree n and satisfies $L_n(0) = 1$. We recall that the set $\{L_m \otimes L_n, 0 \leq m, n \leq N\}$ is a basis of $Q_N(\bar\Omega^+)$. However, we shall also characterize the polynomials of X_N by pointwise values: let ζ_j, $0 \leq j \leq N$, be the roots of the polynomial $\zeta(1 - \zeta)L'_N(\zeta)$, with $0 = \zeta_0 < \zeta_1 < \cdots < \zeta_N = 1$; we set (cf. Figure 2.1)

$$(2.8) \qquad \Xi_N = \{(\zeta_i, \zeta_j), 0 \leq i, j \leq N\},$$

and

$$(2.9) \qquad \xi_N = \Xi_N \cap \gamma = \{(0, \zeta_j), 1 \leq j \leq N - 1\}.$$
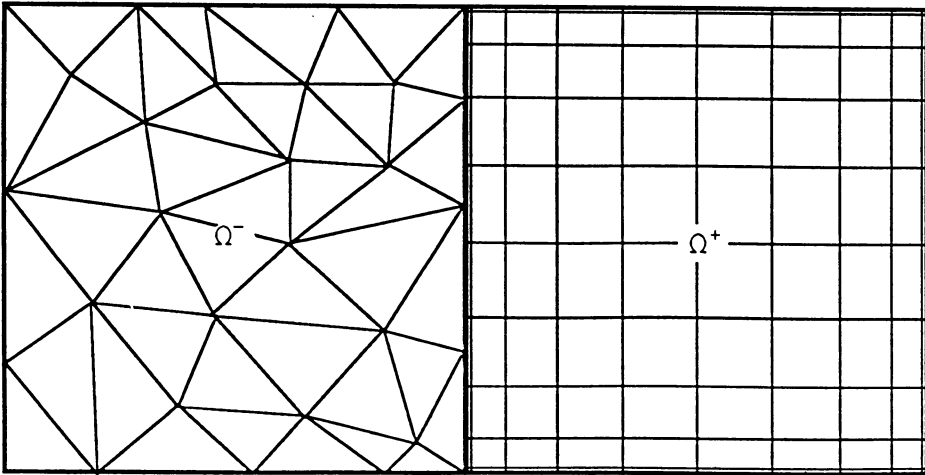

FIGURE 2.1. The triangulation $\mathscr{T}_h$ and the set Ξ_N.

Finally, with each value of h and N, we associate the discretization parameter $\delta = (h, N^{-1})$. The pair u^*, where u is the solution of problem (1.1), will be approximated in a subspace of $X_h \times X_N$ consisting of all pairs which satisfy a matching condition on the interface γ. More precisely, we are going to consider two kinds of matching conditions, with which we associate two kinds

of discrete spaces, both denoted by V_δ :

(1) pointwise matching condition: the space V_δ^P is defined by

$$(2.10) \qquad V_\delta^P = \{v_\delta^* = (v_h, v_N) \in X_h \times X_N ; \forall \mathbf{a} \in \xi_h, v_h(\mathbf{a}) = v_N(\mathbf{a})\};$$

(2) integral matching condition: the space V_δ^I is defined by

$$(2.11) \qquad V_\delta^I = \left\{ v_\delta^* = (v_h, v_N) \in X_h \times X_N ; \forall q_h \in x_h, \right.$$
$$\left. \int_\gamma (v_h - v_N)(0, y) q_h(y)\, dy = 0 \right\}.$$

Here, we compare the two kinds of spaces; the integral matching condition will turn out to be better.

Remark 2.1. We immediately note that both methods are nonconforming since a function v_δ, associated with a pair v_δ^* of V_δ, is generally discontinuous through γ and consequently does not belong to $H_0^1(\Omega)$. Indeed, for $N \geq k$, the function v_δ associated with the pair $v_\delta^* = (v_h, v_N)$ in $X_h \times X_N$ belongs to $H_0^1(\Omega)$ if and only if its restriction to γ is a polynomial of $P_k(\overline{\gamma}) \cap H_0^1(\gamma)$; in the particular case $k = 1$, this implies that both v_h and v_N vanish on γ.

Remark 2.2. From a numerical point of view, to enforce the pointwise matching condition, one has to interpolate polynomials of X_N at every point of ξ_h and hence must store the values $L_n(\mathbf{a})$, $0 \leq n \leq N$, $\mathbf{a} \in \xi_h$. On the other hand, to enforce the integral matching condition, one needs to store the integrals $\int_\gamma L_n(y) q_\mathbf{a}(0, y)\, dy$, $0 \leq n \leq N$, $\mathbf{a} \in \xi_h$. Consequently, the costs of the two methods are of the same order.

However, when k is equal to 1, for a given value of N, it is possible to choose the triangulation $\mathcal{T}_h$ such that the sets ξ_h and ξ_N coincide. Then, since the polynomials of X_N are characterized by their values at the points of Ξ_N, enforcing the pointwise matching condition would be less expensive. But this would require very strong restrictions on the triangulation $\mathcal{T}_h$; in particular, the parameters h and N would be linked by a relation of the type $h \geq cN^{-1}$. Moreover, the triangulation could not be uniformly regular since it is well known (see [30, Theorem 6.21.3]) that the ζ_j, $1 \leq j \leq N-1$, satisfy $\zeta_j = \sin^2 \theta_j$ with $(2j-1)\pi/4N < \theta_j < (j+1)\pi/2(N+1)$; hence, the points of ξ_N are not at all equally distributed (they cluster near $+1$). That is why we do not recommend such a choice.

2.2. The discrete problems.   We are now in a position to define the discrete problems. We recall (see [13, §2.7 or 18, Chapter 25]) that there exist positive weights ρ_j, $0 \leq j \leq N$, such that the Gauss-Lobatto quadrature formula

$$(2.12) \qquad \int_0^1 \Phi(\zeta)\, d\zeta \simeq \sum_{j=0}^N \Phi(\zeta_j) \rho_j$$

is exact for all polynomials of degree $\leq 2N - 1$.

With each point $\mathbf{a} = (\zeta_i, \zeta_j)$ in Ξ_N, we associate the weight $\rho_{\mathbf{a}} = \rho_i \rho_j$. We now introduce the following discrete bilinear form on $L^2(\Omega^-) \times \mathscr{C}^0(\bar{\Omega}^+)$:

$$(2.13) \qquad (u^*, v^*)_\delta = \sum_{K \in \mathscr{T}_h} \int_K u^-(\mathbf{x}) v^-(\mathbf{x}) \, d\mathbf{x} + \sum_{\mathbf{a} \in \Xi_N} u^+(\mathbf{a}) v^+(\mathbf{a}) \rho_{\mathbf{a}},$$

which coincides with the usual scalar product on $L^2(\Omega^-) \times Q_{N-1}(\bar{\Omega}^+)$. Finally, we define on $H^1(\Omega^-) \times \mathscr{C}^1(\bar{\Omega}^+)$ the bilinear form

$$(2.14) \qquad a_\delta(u^*, v^*) = (\nabla u^*, \nabla v^*)_\delta \quad \forall (u^*, v^*) \in [H^1(\Omega^-) \times \mathscr{C}^1(\bar{\Omega}^+)]^2.$$

Then, for any pair f^* given in $L^2(\Omega^-) \times \mathscr{C}^0(\bar{\Omega}^+)$, for each kind of matching condition, the discrete problem is the following: *Find u_δ, with u_δ^* in V_δ, such that*

$$(2.15) \qquad a_\delta(u_\delta^*, v_\delta^*) = (f^*, v_\delta^*)_\delta \quad \forall v_\delta^* \in V_\delta.$$

Remark 2.3. Of course, in definition (2.13), one could, by using a quadrature formula, replace each integral $\int_K u^-(\mathbf{x}) v^-(\mathbf{x}) \, d\mathbf{x}$ by its approximation. The resulting algorithm will be thoroughly analyzed in [14].

We recall [9, Lemma 3.2] the property

$$(2.16) \qquad \sum_{j=0}^N L_N(\zeta_j)^2 \rho_j = (2 + N^{-1}) \int_0^1 L_N(\zeta)^2 \, d\zeta.$$

Since the quadrature formula (2.12) is exact for all polynomials of degree $\leq 2N - 1$, the discrete scalar product $(\cdot, \cdot)_\delta$ is uniformly equivalent to $(\cdot, \cdot)$ on $L^2(\Omega^-) \times Q_N(\bar{\Omega}^+)$. Consequently, the form a_δ satisfies the following properties of continuity:

$$(2.17) \qquad |a_\delta(u^*, v^*)| \leq c \|u^*\| \|v^*\| \quad \forall (u^*, v^*) \in [H^1(\Omega^-) \times Q_N(\bar{\Omega}^+)]^2,$$

and of ellipticity:

$$(2.18) \qquad a_\delta(u^*, u^*) \geq (\nabla u^*, \nabla u^*) \quad \forall u^* \in H^1(\Omega^-) \times Q_N(\bar{\Omega}^+).$$

Now, since both $\partial \Omega^- \cap \partial \Omega$ and $\partial \Omega^+ \cap \partial \Omega$ have a positive measure, it follows from the Poincaré-Friedrichs inequality that the seminorm $v^* \to (\nabla v^*, \nabla v^*)^{1/2}$ is a norm equivalent to $\|\cdot\|$ on the space

$$\{v^* \in H^1(\Omega^-) \times H^1(\Omega^+) ; v = 0 \text{ on } \partial \Omega\},$$

which yields in our particular case

$$(2.19) \qquad a_\delta(u^*, u^*) \geq c \|u^*\|^2 \quad \forall u^* \in H^1(\Omega^-) \times Q_N(\bar{\Omega}^+).$$

Thus we have proved the following result.

Proposition 2.4. *For pointwise as well as integral matching conditions, problem (2.15) has a unique solution u_δ with u_δ^* in V_δ.*

The purpose of what follows is to give an estimate for the error between the solutions u and u_δ of problems (1.1) and (2.15), respectively. We begin with a classical bound (see [11, Theorem 4.2.2]).

Proposition 2.5. *For pointwise as well as integral matching conditions, the solutions* u *and* u_δ *of problems* (1.1) *and* (2.15) *satisfy*

$$
\|u^* - u_\delta^*\| \leq c \left\{ \inf_{v_\delta^* \in V_\delta} \{\|u^* - v_\delta^*\| + \sup_{w_\delta^* \in V_\delta} [a(v_\delta^*, w_\delta^*) - a_\delta(v_\delta^*, w_\delta^*)]/\|w_\delta^*\|\} \right.
$$

(2.20)
$$
+ \sup_{w_\delta^* \in V_\delta} [(f^*, w_\delta^*) - (f^*, w_\delta^*)_\delta]/\|w_\delta^*\|
$$

$$
\left. + \sup_{w_\delta^* = (w_h, w_N) \in V_\delta} \int_\gamma (\partial u/\partial n)(0, y)(w_N - w_h)(0, y)\, dy/\|w_\delta^*\| \right\}.
$$

Proof. Let v_δ^* be any element in V_δ. Using (2.19), we have

$$
c\|u_\delta^* - v_\delta^*\|^2 \leq a_\delta(u_\delta^* - v_\delta^*, u_\delta^* - v_\delta^*)
$$
$$
= -a(v_\delta^*, u_\delta^* - v_\delta^*) + a(v_\delta^*, u_\delta^* - v_\delta^*)
$$
$$
- a_\delta(v_\delta^*, u_\delta^* - v_\delta^*) + (f^*, u_\delta^* - v_\delta^*)_\delta.
$$

Next, it follows from (1.1) that, for any w^* in $H^1(\Omega^-) \times H^1(\Omega^+)$ such that w vanishes on $\partial\Omega$,

$$
(f^*, w^*) = \int_\Omega f(\mathbf{x})w(\mathbf{x})\, d\mathbf{x} = -\int_{\Omega^-} (\Delta u)(\mathbf{x})w^-(\mathbf{x})\, d\mathbf{x} - \int_{\Omega^+} (\Delta u)(\mathbf{x})w^+(\mathbf{x})\, d\mathbf{x}
$$
$$
= a(u^*, w^*) + \int_\gamma (\partial u/\partial n)(0, y)(w^+ - w^-)(0, y)\, dy.
$$

Setting $w^* = u_\delta^* - v_\delta^* = (w_h, w_N)$ and combining this result with the previous inequality, we obtain

$$
c\|u_\delta^* - v_\delta^*\|^2 \leq a(u^* - v_\delta^*, u_\delta^* - v_\delta^*) + a_\delta(v_\delta^*, u_\delta^* - v_\delta^*) - a_\delta(v_\delta^*, u_\delta^* - v_\delta^*)
$$
$$
- (f^*, u_\delta^* - v_\delta^*) + (f^*, u_\delta^* - v_\delta^*)_\delta
$$
$$
+ \int_\gamma (\partial u/\partial n)(0, y)(w_N - w_h)(0, y)\, dy,
$$

and (2.20) follows.   □

We are now interested in deriving a bound for
(1) the consistency error term

$$
\sup_{w_\delta^* = (w_h, w_N) \in V_\delta} \int_\gamma (\partial u/\partial n)(0, y)(w_N - w_h)(0, y)\, dy/\|w_\delta^*\|,
$$

(2) the approximation error term

$$
\inf_{v_\delta^* \in V_\delta} \|u^* - v_\delta^*\|,
$$

since estimating the other terms in (2.20) is standard in spectral methods.

3. ANALYSIS OF THE CONSISTENCY ERROR

The aim of this section is to study the term

$$
\int_\gamma (\partial u/\partial n)(0, y)(w_N - w_h)(0, y)\, dy,
$$

for any pair $w_\delta^* = (w_h, w_N)$ in V_δ, where u is a given function on Ω which we shall assume to be sufficiently smooth. The analysis involves only one-dimensional approximation operators.

3.1. Pointwise matching.   We recall that there exists an interpolation operator ι_h from $\{v \in \mathscr{C}^0(\overline{\gamma}) ; \ v(0) = v(1) = 0\}$ into x_h such that, for any function v continuous on $\overline{\gamma}$ and vanishing at 0 and 1, $\iota_h v$ is the only element of x_h which satisfies

$$(3.1) \qquad (\iota_h v)(\mathbf{a}) = v(\mathbf{a}) \quad \forall \mathbf{a} \in \xi_h .$$

Moreover, there exists a constant c such that, for any real number ε, $0 < \varepsilon \le 1/2$, if the function v belongs to $H^{1/2+\varepsilon}(\gamma)$, the following interpolation error estimate holds [4]:

$$(3.2) \qquad \|v - \iota_h v\|_{0,\gamma} \le (c/\sqrt{\varepsilon}) h^{1/2+\varepsilon} \|v\|_{1/2+\varepsilon,\gamma} .$$

We are now in a position to prove the following result.

Proposition 3.1. *For any function u in $H_0^1(\Omega) \cap H^2(\Omega)$, the following estimate holds for any $w_\delta^* = (w_h, w_N)$ in V_δ^P:*

$$(3.3) \quad \int_\gamma (\partial u/\partial n)(0,y)(w_N - w_h)(0,y)\,dy \le c h^{1/2} \sqrt{|\log(hN^2)|}\ \|u^-\|_{2,\Omega^-} \|w_\delta^*\| .$$

Proof. Since u belongs to $H^2(\Omega)$, the trace $\partial u/\partial n$ belongs to $H^{1/2}(\gamma)$, hence we have

$$\|\partial u/\partial n\|_{1/2,\gamma} \le c \|u\|_{2,\Omega^-} .$$

Next, we estimate

$$\int_\gamma (\partial u/\partial n)(0,y)(w_N - w_h)(0,y)\,dy \le \|\partial u/\partial n\|_{0,\gamma} \|w_h - w_N\|_{0,\gamma}$$

$$\le c\|u\|_{2,\Omega^-} \|w_h - w_N\|_{0,\gamma} .$$

But, in view of the definition (2.10) of V_δ^P, w_h is equal to $\iota_h w_N$ on γ, so that by (3.2),

$$\int_\gamma (\partial u/\partial n)(0,y)(w_N - w_h)(0,y)\,dy \le (c'/\sqrt{\varepsilon}) h^{1/2+\varepsilon} \|u\|_{2,\Omega^-} \|w_N\|_{1/2+\varepsilon,\gamma} .$$

Applying the inverse inequality [9] gives

$$\int_\gamma (\partial u/\partial n)(0,y)(w_N - w_h)(0,y)\,dy \le (c'/\sqrt{\varepsilon}) N^{2\varepsilon} h^{1/2+\varepsilon} \|u\|_{2,\Omega^-} \|w_N\|_{1/2,\gamma} .$$

Choosing $\varepsilon = 1/|\log(hN^2)|$, we obtain the desired result. $\square$

Remark 3.2. Of course, the estimate (3.3) is not what we want, since convergence is obtained only if the discretization parameters are linked by the following condition:

$$(3.4) \qquad \lim \left[h^{1/2} \sqrt{|\log(hN^2)|} \right] = 0$$

(in fact, in (3.3) and in this condition, h can be replaced by $\tilde{h}$ which is the greatest of the lengths of the edges of triangles K in $\mathscr{T}_h$ contained in $\bar{\gamma}$).

Remark 3.3. The estimate (3.3) is independent of k; indeed, we do not know how to improve it for large values of k.

3.2. Integral matching. This case turns out to be simpler. We denote by π_h the orthogonal projection operator from $L^2(\gamma)$ onto x_h. We have for any v in $H_0^1(\gamma)$,

$$\|v - \pi_h v\|_{0,\gamma} \leq \|v - \iota_h v\|_{0,\gamma},$$

so that, for any v in $H_0^1(\gamma) \cap H^l(\gamma)$, $1 \leq l \leq k+1$,

$$(3.5) \qquad \|v - \pi_h v\|_{0,\gamma} \leq ch^l \|v\|_{l,\gamma}.$$

By interpolation, this inequality also holds for any v in $H_0^l(\gamma)$, $1/2 < l < 1$. Finally, recalling that the interpolation space with index $1/2$ between $H_0^1(\gamma)$ and $L^2(\gamma)$ is $H_{00}^{1/2}(\gamma)$ (see [20, Chapter 1, Theorem 11.7]) and denoting by $\|\cdot\|_{1/2^*,\gamma}$ the norm of $H_{00}^{1/2}(\gamma)$, we also obtain for any v in $H_{00}^{1/2}(\gamma)$,

$$(3.6) \qquad \|v - \pi_h v\|_{0,\gamma} \leq ch^{1/2} \|v\|_{1/2^*,\gamma}.$$

Now, we prove the following.

Proposition 3.4. *For any function u in $H_0^1(\Omega) \cap H^2(\Omega)$ such that the function u^- belongs to $H^l(\Omega^-)$, where l is a real number, $2 \leq l \leq k+5/2$, the following estimate holds for any $w_\delta^* = (w_h, w_N)$ in V_δ^l:*

$$(3.7) \qquad \int_\gamma (\partial u/\partial n)(0, y)(w_N - w_h)(0, y)\, dy \leq ch^{l-1} \|u^-\|_{l,\Omega^-} \|w_\delta^*\|.$$

Proof. Let $w_\delta^* = (w_h, w_N)$ be any element in V_δ^l; by the definition (2.11) of V_δ^l, w_h coincides with $\pi_h w_N$ on γ. We compute

$$\int_\gamma (\partial u/\partial n)(0, y)(w_N - w_h)(0, y)\, dy$$

$$= \int_\gamma (\partial u/\partial n)(0, y)(w_N - \pi_h w_N)(0, y)\, dy$$

$$= \int_\gamma [(\partial u/\partial n) - \pi_h(\partial u/\partial n)](0, y)(w_N - \pi_h w_N)(0, y)\, dy,$$

so that

$$\int_\gamma (\partial u/\partial n)(0, y)(w_N - w_h)(0, y)\, dy$$

$$\leq \|(\partial u/\partial n) - \pi_h(\partial u/\partial n)\|_{0,\gamma} \|w_N - \pi_h w_N\|_{0,\gamma}.$$

We note that $\partial u/\partial n$ belongs to $H_{00}^{1/2}(\gamma) \cap H^{l-3/2}(\gamma)$ and that, since w_N vanishes on $\partial\Omega^+ \setminus \gamma$, $w_{N|\gamma}$ belongs to $H_{00}^{1/2}(\gamma)$. Applying (3.5) or (3.6) to bound

the first term, and (3.6) to bound the second, we obtain

$$\int_\gamma (\partial u/\partial n)(0,y)(w_N - w_h)(0,y)\,dy \le ch^{l-1}\|\partial u/\partial n\|_{l-3/2,\gamma}\|w_N\|_{1/2^*,\gamma}$$

(with $\|\partial u/\partial n\|_{1/2,\gamma}$ replaced by $\|\partial u/\partial n\|_{1/2^*,\gamma}$ in the case $l = 2$), and the result follows. $\square$

Remark 3.5. Clearly, the estimate (3.7) is much better than (3.3), since it is independent of N. In fact, the term $\int_\gamma (\partial u/\partial n)(0,y)(w_N - w_h)(0,y)\,dy$ goes to 0 whenever the discretization parameter h decreases to 0.

4. ANALYSIS OF THE APPROXIMATION ERROR

We begin by recalling some properties of the approximation by finite element functions and by polynomials in the two-dimensional case.

First, since for each K in $\mathscr{T}_h$ the set Ξ_K is $P_k(K)$-unisolvent, there exists an interpolation operator $\mathscr{I}_h$ from $\{v \in \mathscr{C}^0(\bar{\Omega}^-); v = 0 \text{ on } \partial\Omega^- \setminus \gamma\}$ into X_h such that, for any function v continuous on $\bar{\Omega}^-$ and vanishing on $\partial\Omega^- \setminus \gamma$, $\mathscr{I}_h v$ is the only element of X_h which satisfies

$$(4.1) \qquad (\mathscr{I}_h v)(\mathbf{a}) = v(\mathbf{a}) \quad \forall \mathbf{a} \in \Xi_h \cap (\Omega^- \cup \gamma).$$

Moreover, if the function v belongs to $H^l(\Omega^-)$ for a real number l, $2 \le l \le k + 1$, the following interpolation error estimate holds [11, Theorem 3.1.5]:

$$(4.2) \qquad \|v - \mathscr{I}_h v\|_{m,\Omega^-} \le ch^{l-m}\|v\|_{l,\Omega^-}, \qquad m = 0 \text{ or } 1.$$

Next, we state the following result which can be derived in the same way as in [22, Theorem 3.2].

Lemma 4.1. *Let ρ be a real number ≥ 1 such that $\rho - 1/2$ is not an integer. There exists a projection operator Π_N^ρ from the space $\{v \in H^\rho(\Omega^+); v = 0 \text{ on } \partial\Omega^+ \setminus \gamma\}$ onto $\{v_N \in Q_N(\bar{\Omega}^+); v_N = 0 \text{ on } \partial\Omega^+ \setminus \gamma\}$ such that, if a function v vanishing on $\partial\Omega^+ \setminus \gamma$ belongs to $H^\sigma(\Omega^+)$ for a real number $\sigma \ge \rho$, the following error estimate holds:*

$$(4.3) \qquad \|v - \Pi_N^\rho v\|_{\mu,\Omega^+} \le cN^{\mu-\sigma}\|v\|_{\sigma,\Omega^+}, \qquad 0 \le \mu \le \rho.$$

We are now going to approximate a function u of $H_0^1(\Omega)$, which is sufficiently smooth, by a function v_δ, with v_δ^* in V_δ. We set

$$(4.4) \qquad v_\delta^* = (\mathscr{I}_h u^- + q_h, \Pi_N^\rho u^+),$$

where q_h will be chosen in X_h so that v_δ satisfies the matching condition under consideration.

4.1. Pointwise matching. We immediately prove the following result.

Proposition 4.2. *For any function u in $H_0^1(\Omega)$ such that the pair u^* belongs to $H^l(\Omega^-) \times H^\sigma(\Omega^+)$, where l and σ are real numbers, $2 \le l \le k + 1$ and $\sigma \ge 2$, there exists a pair v_δ^* in V_δ^P such that*

$$(4.5) \qquad \|u^* - v_\delta^*\| \le c\{h^{l-1}\|u^-\|_{l,\Omega^-} + (h^{\sigma-1} + N^{1-\sigma})\|u^+\|_{\sigma,\Omega^+}\}.$$

Proof. For any function z defined on Ω^+, denote by $\overline{z}$ the function defined on Ω^- by

$$\overline{z}(x, y) = z(-x, y) \quad \forall (x, y) \in \Omega^-.$$

Next, take $2 \leq \rho \leq \inf\{k + 1, \sigma\}$, $\rho - 1/2 \notin \mathbb{N}$, and choose v_δ^* as in (4.4) with q_h equal to $-\mathscr{I}_h(\overline{u}^+ - \overline{\Pi_N^\rho u^+})$. Clearly, we have at any point $\mathbf{a}$ of ξ_h,

$$(\mathscr{I}_h u^-)(\mathbf{a}) + q_h(\mathbf{a}) = u^-(\mathbf{a}) - \overline{u}^+(\mathbf{a}) + (\overline{\Pi_N^\rho u^+})(\mathbf{a}) = (\Pi_N^\rho u^+)(\mathbf{a}),$$

so that v_δ^* belongs to V_δ^P. Moreover, we write

$$\|u^* - v_\delta^*\| \leq \|u^- - \mathscr{I}_h u^-\|_{1,\Omega^-} + \|q_h\|_{1,\Omega^-} + \|u^+ - \Pi_N^\rho u^+\|_{1,\Omega^+}$$

$$\leq \|u^- - \mathscr{I}_h u^-\|_{1,\Omega^-} + \|(\mathrm{id} - \mathscr{I}_h)(\overline{u}^+ - \overline{\Pi_N^\rho u^+})\|_{1,\Omega^-}$$

$$+ 2\|u^+ - \Pi_N^\rho u^+\|_{1,\Omega^+}.$$

Finally, applying (4.2) and Lemma 4.1, we deduce

$$\|u^* - v_\delta^*\| \leq c\{h^{l-1}\|u^-\|_{l,\Omega^-} + h^{\rho-1}\|\overline{u}^+ - \overline{\Pi_N^\rho u^+}\|_{\rho,\Omega^-} + N^{1-\sigma}\|u^+\|_{\sigma,\Omega^+}\}$$

$$\leq c\{h^{l-1}\|u^-\|_{l,\Omega^-} + h^{\rho-1}N^{\rho-\sigma}\|u^+\|_{\sigma,\Omega^+} + N^{1-\sigma}\|u^+\|_{\sigma,\Omega^+}\}.$$

Applying the convexity inequality $\alpha\beta \leq \alpha^p/p + \beta^q/q$ ($1/p + 1/q = 1$, $\alpha \geq 0$, $\beta \geq 0$) gives the proposition. □

4.2. Integral matching. As far as the approximation error is concerned, this case is less simple. We are led to make an additional assumption on the family of triangulations: more precisely, we suppose that there exists a positive constant σ such that, for any h and for any triangles K and K' in $\mathscr{T}_h$ such that the lengths of $K \cap \overline{\gamma}$ and of $K' \cap \overline{\gamma}$ are positive, the following inequality holds

$$(4.6) \qquad\qquad h_K \leq \sigma h_{K'}.$$

That means that the family of triangulations is uniformly regular over γ. The following result is proven in [7, Lemma 5.1].

Lemma 4.3. *If assumption (4.6) is satisfied, there exists a lifting operator* R_h *from* x_h *into* X_h *such that the following estimate holds for any* v_h *in* x_h:

$$(4.7) \qquad\qquad \|R_h v_h\|_{1,\Omega^-} \leq c\|v_h\|_{1/2^*,\gamma}.$$

We recall that π_h stands for the orthogonal projection operator from $L^2(\gamma)$ onto x_h. Our goal is to prove a stability result for this operator in the $H_{00}^{1/2}$-norm.

Lemma 4.4. *If assumption (4.6) is satisfied, the following stability property holds for any function* v *in* $H_{00}^{1/2}(\gamma)$:

$$(4.8) \qquad\qquad \|\pi_h v\|_{1/2^*,\gamma} \leq c\|v\|_{1/2^*,\gamma}.$$

Proof. First, let v be any function in $H_0^1(\gamma)$. To bound $\|\pi_h v\|_{1,\gamma}$, we write

$$\|\pi_h v\|_{1,\gamma} \leq \|i_h v\|_{1,\gamma} + \|\pi_h v - i_h v\|_{1,\gamma}.$$

Since the family of the $\{K \cap \bar{\gamma}, \ K \in \mathscr{T}_h\}$ is uniformly regular, denoting by $\check{h}$ the maximum of the lengths of $K \cap \bar{\gamma}$, $K \in \mathscr{T}_h$, we can apply the standard inverse inequality [11, Theorem 3.2.6], which gives

$$
\begin{aligned}
\left\| \pi_h v \right\|_{1,\gamma} &\leq \left\| i_h v \right\|_{1,\gamma} + c\check{h}^{-1} \left\| \pi_h v - i_h v \right\|_{0,\gamma} \\
&\leq \left\| v \right\|_{1,\gamma} + \left\| v - i_h v \right\|_{1,\gamma} + c\check{h}^{-1} \left(\left\| v - i_h v \right\|_{0,\gamma} + \left\| v - \pi_h v \right\|_{0,\gamma} \right).
\end{aligned}
$$

Using the estimate for the interpolation error [11, Theorem 3.15] together with (3.5), we obtain

$$
(4.9) \qquad\qquad \left\| \pi_h v \right\|_{1,\gamma} \leq c \left\| v \right\|_{1,\gamma}.
$$

Finally, by the definition of the space $H_{00}^{1/2}(\gamma)$, we interpolate this result with the estimate

$$
\left\| \pi_h v \right\|_{0,\gamma} \leq \left\| v \right\|_{0,\gamma}
$$

with index $1/2$ (cf. [20, Chapter 1, Theorem 5.1]) and we obtain the desired result. $\square$

Remark 4.5. It is known that the stability result (4.8) holds if a less restrictive assumption than (4.6) is satisfied. However, the necessary condition for (4.7) to be true is not clear.

Proposition 4.6. *If assumption* (4.6) *is satisfied, for any function* u *in* $H_0^1(\Omega)$ *such that the pair* u^* *belongs to* $H^l(\Omega^-) \times H^\sigma(\Omega^+)$, *where* l *and* σ *are real numbers,* $2 \leq l \leq k+1$ *and* $\sigma \geq 2$, *there exists a pair* v_δ^* *in* V_δ^l *such that*

$$
(4.10) \qquad \left\| u^* - v_\delta^* \right\| \leq c \left\{ h^{l-1} \left\| u^- \right\|_{l,\Omega^-} + N^{1-\sigma} \left\| u^+ \right\|_{\sigma,\Omega^+} \right\}.
$$

Proof. We choose v_δ as in (4.4) with ρ equal to 1 and q_h equal to

$$
R_h \pi_h (\Pi_N^1 u^+ - \mathscr{S}_h u^-).
$$

Then, we have

$$
\mathscr{S}_h u^- + q_h - \pi_h \Pi_N^1 u^+ = 0,
$$

so that v_δ^* belongs to V_δ^l. Moreover, we estimate

$$
\begin{aligned}
\left\| u^* - v_\delta^* \right\| &\leq \left\| u^- - \mathscr{S}_h u^- \right\|_{1,\Omega^-} + \left\| R_h \pi_h \left(\Pi_N^1 u^+ - \mathscr{S}_h u^- \right) \right\|_{1,\Omega^-} \\
&\quad + \left\| u^+ - \Pi_N^1 u^+ \right\|_{1,\Omega^+} \\
&\leq \left\| u^- - \mathscr{S}_h u^- \right\|_{1,\Omega^-} + \left\| R_h \pi_h \left(u^- - \mathscr{S}_h u^- \right) \right\|_{1,\Omega^-} \\
&\quad + \left\| R_h \pi_h \left(u^+ - \Pi_N^1 u^+ \right) \right\|_{1,\Omega^-} + \left\| u^+ - \Pi_N^1 u^+ \right\|_{1,\Omega^+}.
\end{aligned}
$$

Lemmas 4.3 and 4.4 imply

$$
\begin{aligned}
\left\| u^* - v_\delta^* \right\| &\leq \left\| u^- - \mathscr{S}_h u^- \right\|_{1,\Omega^-} + c \left(\left\| u^- - \mathscr{S}_h u^- \right\|_{1/2^*,\gamma} + \left\| u^+ - \Pi_N^1 u^+ \right\|_{1/2^*,\gamma} \right) \\
&\quad + \left\| u^+ - \Pi_N^1 u^+ \right\|_{1,\Omega^+}.
\end{aligned}
$$

Using the trace theorem, we obtain simply

$$\left\| u^* - v_\delta^* \right\| \leq c \left(\left\| u^- - \mathscr{I}_h u^- \right\|_{1,\Omega^-} + \left\| u^+ - \Pi_N^1 u^+ \right\|_{1,\Omega^+} \right).$$

Finally, using the estimates (4.2) and (4.3) gives the result. □

Remark 4.7. Here also, the error is better for the integral matching condition than for the pointwise one. Indeed, in (4.10), the two discretization parameters enter in a completely independent way.

5. FINAL ESTIMATES AND CONCLUSION

First, we recall an estimate which follows at once from a standard result in spectral methods [10, Lemma 3.2; 24, (3.22)], in a slightly improved form due to [6, Appendix B].

Lemma 5.1. *For any function f in $L^2(\Omega)$ such that the function $(y - y^2)f^+$ belongs to $H^\rho(\Omega^+)$, where ρ is a real number > 1, the following estimate holds for any $w_\delta^* = (w_h, w_N)$ in V_δ:*

$$(5.1) \qquad (f^*, w_\delta^*) - (f^*, w_\delta^*)_\delta \leq c N^{1-\rho} \|(y - y^2)f^+\|_{\rho,\Omega^+} \|w_N\|_{1,\Omega^+}.$$

Our main results are stated in the two following theorems.

Theorem 5.2. *Assume that the solution u of problem (1.1) is such that the pair u^* belongs to $H^2(\Omega^-) \times H^\sigma(\Omega^+)$, where σ is a real number ≥ 2. Assume moreover that the function f of $L^2(\Omega)$ is such that the function $(y - y^2)f^+$ belongs to $H^\rho(\Omega^+)$, where ρ is a real number > 1. Then, in the case of the pointwise matching condition, the solutions u and u_δ of problems (1.1) and (2.15) satisfy*

$$(5.2) \qquad \begin{aligned} \|u^* - u_\delta^*\| \leq c \Big\{ & h^{1/2}\sqrt{|\log(hN^2)|}\|u^-\|_{2,\Omega^-} + (h^{\sigma-1} + N^{1-\sigma})\|u^+\|_{\sigma,\Omega^+} \\ & + N^{1-\rho}\|(y - y^2)f^+\|_{\rho,\Omega^+} \Big\}. \end{aligned}$$

Theorem 5.3. *Assume that the solution u of problem (1.1) is such that the pair u^* belongs to $H^l(\Omega^-) \times H^\sigma(\Omega^+)$, where l and σ are real numbers, $2 \leq l \leq k + 1$ and $\sigma \geq 2$. Assume moreover that assumption (4.6) is satisfied and that the function f of $L^2(\Omega)$ such that the function $(y - y^2)f^+$ belongs to $H^\rho(\Omega^+)$, where ρ is a real number > 1. Then, in the case of the integral matching condition, the solutions u and u_δ of problems (1.1) and (2.15) satisfy*

$$(5.3) \quad \|u^* - u_\delta^*\| \leq c\{h^{l-1}\|u^-\|_{l,\Omega^-} + N^{1-\sigma}\|u^+\|_{\sigma,\Omega^+} + N^{1-\rho}\|(y - y^2)f^+\|_{\rho,\Omega^+}\}.$$

Proof. We set $\tilde{\delta} = (h, (N - 1)^{-1})$. Of course, we apply Proposition 2.5 and, in (2.20), we choose $v_\delta^* = (v_h, v_N)$ equal to the pair defined in Propositions 4.2 and 4.6, respectively, but with δ replaced by $\tilde{\delta}$. Since v_N belongs to $Q_{N-1}(\Omega^+)$ and the quadrature formula (2.12) is exact for all polynomials of degree $\leq 2N - 1$, this implies that, for any w_δ^* in V_δ,

$$a(v_\delta^*, w_\delta^*) = a_\delta(v_\delta^*, w_\delta^*).$$

Then the estimates (5.2) and (5.3) follow from (2.20), Propositions 3.1 and 3.4, respectively, Propositions 4.2 and 4.6, respectively, and Lemma 5.1. $\quad\square$

By a classical duality method, it is possible to derive an improved estimate for $\|u - u_\delta\|_{0,\Omega}$ in the case of the integral matching condition.

Proposition 5.4. *Under the assumptions of Theorem 5.3, in the case of the integral matching condition, the solutions u and u_δ of problems (1.1) and (2.15) satisfy*

$$(5.4) \quad \|u - u_\delta\|_{0,\Omega} \le c\{h^{l-1}(h + N^{-1})\|u^-\|_{l,\Omega^-} + N^{1-\sigma}(h + N^{-1})\|u^+\|_{\sigma,\Omega^+}$$
$$+ N^{1-\rho}\|(y - y^2)f^+\|_{\rho,\Omega^+}\}.$$

Proof. We have

$$\|u - u_\delta\|_{0,\Omega} = \sup_{g \in L^2(\Omega)} \int_\Omega (u - u_\delta)(\mathbf{x})g(\mathbf{x})\,d\mathbf{x} / \|g\|_{0,\Omega}.$$

Let g be any function in $L^2(\Omega)$. The unique solution w in $H_0^1(\Omega)$ of the problem

$$(5.5) \qquad\qquad -\Delta w = g \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega,$$

satisfies

$$(5.6) \qquad\qquad\qquad \|w\|_{2,\Omega} \le c\|g\|_{0,\Omega}.$$

Setting $u_\delta^* = (u_h, u_N)$, we compute

$$\int_\Omega (u - u_\delta)(\mathbf{x})g(\mathbf{x})\,d\mathbf{x} = a(u^* - u_\delta^*, w^*) + \int_\gamma (\partial w/\partial n)(0, y)(u_N - u_h)(0, y)\,dy.$$

Hence, for any w_δ^* in $V_{\tilde\delta}$, with $\tilde\delta = (h, (N-1)^{-1})$, we have

$$\int_\Omega (u - u_\delta)(\mathbf{x})g(\mathbf{x})\,d\mathbf{x} = a(u^* - u_\delta^*, w^* - w_\delta^*) + (f^*, w_\delta^*) - (f^*, w_\delta^*)_\delta$$
$$+ \int_\gamma (\partial w/\partial n)(0, y)(u_N - u_h)(0, y)\,dy.$$

Choosing w_δ^* as defined in Proposition 4.6, and using Lemma 5.1, we obtain

$$\int_\Omega (u - u_\delta)(\mathbf{x})g(\mathbf{x})\,dx$$
$$(5.7) \qquad \le c\{(h + N^{-1})\|u^* - u_\delta^*\| + N^{1-\rho}\|(y - y^2)f^*\|_{\rho,\Omega^+}\}\|w\|_{2,\Omega}$$
$$+ \int_\gamma (\partial w/\partial n)(0, y)(u_N - u_h)(0, y)\,dy.$$

It remains to estimate the last term in (5.7). We note that $u_{h|\gamma}$ is equal to $\pi_h u_N$, so that

$$\int_\gamma (\partial w/\partial n)(0, y)(u_N - u_h)(0, y)\, dy$$

$$= \int_\gamma [(\partial w/\partial n) - \pi_h(\partial w/\partial n)](0, y)(u_N - \pi_h u_N)(0, y)\, dy$$

$$= \int_\gamma [(\partial w/\partial n) - \pi_h(\partial w/\partial n)](0, y)[(u - \pi_h u) - (\mathrm{id} - \pi_h)(u - u_N)](0, y)\, dy$$

$$\leq \|(\partial w/\partial n) - \pi_h(\partial w/\partial n)\|_{0,\gamma}(\|u - \pi_h u\|_{0,\gamma} + \|(\mathrm{id} - \pi_h)(u - u_N)\|_{0,\gamma}).$$

Using (3.5) and (3.6) yields

$$\int_\gamma (\partial w/\partial n)(0, y)(u_N - u_h)(0, y)\, dy$$

$$\leq ch^{1/2}\|\partial w/\partial n\|_{1/2^*,\gamma}(h^{l-1/2}\|u^-\|_{l,\Omega^-} + h^{1/2}\|u^+ - u_N\|_{1,\Omega^+}),$$

which, together with (5.6) and (5.7), gives (5.4). □

The detailed analysis we have performed allows us to compare the two algorithms, corresponding to different matching conditions. Indeed, whatever the regularity of the exact solution is, we obtain better convergence results in the case of the integral matching condition. Since we have already noted that the computational cost of the two methods is of the same order, we believe that this last algorithm has to be preferred. Numerical tests [14, 19] which are currently being implemented are expected to confirm the theoretical results.

As already stated, in this paper we are only concerned with a model problem on a model domain. However, in this very simple example, it turns out that the order of accuracy in the finite element domain is simultaneously restricted by the degree of polynomials and by the regularity of the solution, while in the spectral domain it is only limited by the regularity of the solution. That is why we believe that, in more general elliptic problems (for instance the Stokes or Navier-Stokes equations), the finite element domain must be chosen in such a way that it contains a neighborhood of both the singularities of the solution and the singularities of the boundary of the domain (for instance, corners of polygons, which induce singularities of the solution even if the right-hand member is very smooth). Then, local refinements of the mesh can be applied to improve the convergence, in a much simpler way than for the p-version of finite elements. These techniques are presently being developed in [14]. It is important to note here that the balance between the finite element subdomain and the spectral element subdomain is much more easy to handle than a local refinement in the so-called $h - p$ version of finite elements. The matching between the two domains is indeed less stringent in the present method than in the $h - p$ version, where the intersection of two different domains is either a whole edge, or a vertex, or empty.

We conclude this paper by giving an example of a domain with a less trivial boundary, and by explaining how to choose the parameters in this case.

Example. Let Ω be a polygonal domain with vertices $\mathbf{a}_i$, $1 \le i \le K$. For $1 \le i \le K$, we denote by ω_i the measure of the interior angle at $\mathbf{a}_i$ and by $d_i(\Delta)$ the distance of any closed set Δ in $\bar{\Omega}$ to $\mathbf{a}_i$. The domain Ω^+ is chosen as a large rectangle contained in Ω, such that $\bar{\Omega}$ does not contain any vertex $\mathbf{a}_i$, $1 \le i \le K$; then Ω^- is defined as $\Omega \setminus \bar{\Omega}^+$ (cf. Figure 5.1).
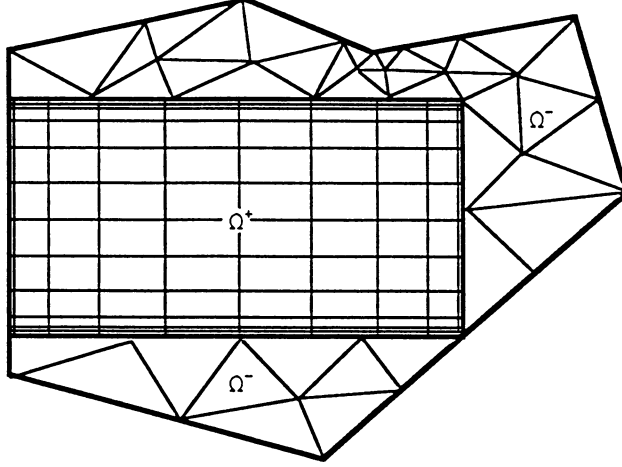

FIGURE 5.1. An example of a domain with a less trivial boundary.

In the domain Ω, we solve the problem (1.1). It is known (see [17, Theorem 5.1.3.5] or [27, Theorem 1]) that, if f belongs to $H^m(\Omega)$ for a real number $m \ge 0$, the solution u satisfies:

(1) The function u^- belongs to the space

$$W_\rho^2(\Omega^-) = \{v : \Omega^- \to \mathbb{R}; \int_{\Omega^-} (\partial^{j+k} v / \partial x^j \partial y^k)^2 \rho(\mathbf{x})\, d\mathbf{x} < +\infty, 0 \le j+k \le 2\},$$

where $\rho(\mathbf{x})$ is a positive bounded weight equivalent to $d_i(\mathbf{x})^{2\alpha_i}$ in a neighborhood of $\mathbf{a}_i$, $1 \le i \le K$, with

$$\begin{cases} \alpha_i = 0 & \text{if } \omega_i < \pi, \\ \alpha_i > 1 - \pi/\omega_i & \text{if } \omega_i > \pi; \end{cases}$$

moreover, one has

$$\left(\sum_{0 \le j+k \le 2} \int_{\Omega^-} (\partial^{j+k} v / \partial x^j \partial y^k)^2 \rho(\mathbf{x})\, d\mathbf{x} \right)^{1/2} \le c \|f\|_{0,\Omega}.$$

(2) The function u^+ belongs to the space $H^{m+2}(\Omega^+)$, and one has

$$\|u\|_{m+2,\Omega^+} \le c \left(\inf_{1 \le i \le K} d_i(\bar{\Omega}^+)^{\pi/\omega_i - m - 1} \right) \|f\|_{m,\Omega}.$$

Next, we consider a regular family of triangulations $(\mathscr{T}_h)_h$ of Ω^- such that, for any $\omega_i > \pi$, $1 \le i \le K$, the diameter of each triangle containing the vertex

$\mathbf{a}_i$ is bounded by $ch^{1/(1-\alpha_i)}$ and the diameter of each triangle K contained in a fixed neighborhood of $\mathbf{a}_i$ is bounded by $ch\, d_i(K)$; a way of constructing such a triangulation was first described in [27]. The finite element functions are chosen to be piecewise linear continuous functions (i.e., k is equal to 1). Using [27, Theorem 2] together with Theorem 5.3, we see that the discrete solution u_δ of problem (2.15) satisfies the error estimate

$$\|u^* - u_\delta^*\| \le c(f) \left\{ h + N^{-m-1} \left(\inf_{1 \le i \le K} d_i^{\pi/\omega_i - m - 1} \right) + N^{-m} \right\}.$$

Of course, in a practical computation, the parameters will be chosen so as to make the three error terms h, $N^{-m-1}(\inf_{1\le i\le K} d_i^{\pi/\omega_i - m - 1})$, and N^{-m} of the same order. For instance, assume that m is equal to 3, that the d_i are equal to $2\cdot10^{-1}$, and that the constants are bounded by 10; in order to obtain a precision of 10^{-2}, we choose h equal to 10^{-3} and N equal to $23 \simeq (10^{3/4})\cdot(5^{7/8})$, which requires $\mathscr{O}(10^6)$ operations in the finite element domain and $\mathscr{O}(1.2\times10^4)$ in the spectral domain; in order to obtain a precision of 10^{-3}, we choose h equal to 10^{-4} and N equal to $41 \simeq 10\cdot(5^{7/8})$, which requires $\mathscr{O}(10^8)$ operations in the finite element domain and $\mathscr{O}(7\times10^4)$ in the spectral domain.

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