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APPROXIMATION BY MEDIANTS

WIEB BOSMA

ABSTRACT. The distribution is determined of some sequences that measure how well a number is approximated by its mediants (or intermediate continued fraction convergents). The connection with a theorem of Fatou, as well as a new proof of this, is given.

0. INTRODUCTION

Let x denote an irrational number. From the expansion of x into a regular continued fraction

$$x = B_0 + \cfrac{1}{B_1 + \cfrac{1}{B_2 + \cdots}} = [B_0; B_1, B_2, \ldots] \tag{0.1}$$

one gets the *convergents* P_n/Q_n of x by truncation,

$$\frac{P_n}{Q_n} = [B_0; B_1, B_2, \ldots, B_n], \qquad n \geq 0. \tag{0.2}$$

These convergents satisfy the relation

$$\frac{P_n}{Q_n} = \frac{B_n P_{n-1} + P_{n-2}}{B_n Q_{n-1} + Q_{n-2}}, \qquad n \geq 2, \tag{0.3}$$

and provide very good approximations to x; for instance, defining $\{\Theta_n(x)\}_{n=0}^{\infty}$ by

$$\left| x - \frac{P_n}{Q_n} \right| = \frac{\Theta_n(x)}{Q_n^2}, \tag{0.4}$$

it is a classical result that $\Theta_n(x) \leq 1$ always holds. In [1] it was shown that for almost all x the sequence $\{\Theta_n(x)\}_{n=0}^{\infty}$ has a limiting distribution $\frac{1}{\log 2}F(z)$, where

$$F(z) = \begin{cases} 0, & \text{for } z \leq 0, \\ z, & \text{for } 0 \leq z \leq \frac{1}{2}, \\ 1 - z + \log(2z), & \text{for } \frac{1}{2} \leq z \leq 1, \\ \log 2, & \text{for } 1 \leq z. \end{cases} \tag{0.5}$$

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Here we will consider a similar question for the *mediants* (or secondary convergents, or intermediate convergents) of x; these are defined by

$$\frac{L_n^{(B)}}{M_n^{(B)}} = \frac{BP_{n-1} + P_{n-2}}{BQ_{n-1} + Q_{n-2}} \tag{0.6}$$

for integers B, $0 < B < B_n$ $(n \geq 2)$. In particular, we will derive in §1 for almost all x the limiting distribution of the sequences $\{\Theta_n^{(B)}(x)\}_{n=0}^{\infty}$ for every B, where $\Theta_n^{(B)}(x)$ is given by

$$\left| x - \frac{L_n^{(B)}}{M_n^{(B)}} \right| = \frac{\Theta_n^{(B)}(x)}{(M_n^{(B)})^2}. \tag{0.7}$$

Note that some care is needed because $L_n^{(B)}/M_n^{(B)}$ and hence $\Theta_n^{(B)}$ does not exist for every n and B. The values of $\Theta_n^{(B)}$ are not bounded by 1 but satisfy

$$\frac{B}{B+1} \leq \Theta_n^{(B)} \leq B + 1; \tag{0.8}$$

thus these values are uniformly bounded for fixed x if and only if the partial quotients B_n are bounded. In §1 we study the distribution of $\Theta_n^{(B)}$ for fixed B. In order to be able to study the distribution of the values of $\Theta_n^{(B)}$ for all B simultaneously (in §2), we will consider sets of the form $\{\Theta | \Theta \leq C\}$ (for any positive real constant C), with $\Theta = Q|Qx - P|$, where P/Q ranges over the rationals that are either convergents or mediants of x. Finally, in §3 and §4 we collect some (previously known) results, especially concerning the approximation by *nearest* mediants, that follow from the method employed. In particular, we show how to retrieve Fatou's theorem, stating that every rational number P/Q for which $Q|Qx - P| \leq 1$ is either a convergent or a nearest mediant of x.

In the following we will always assume rationals P/Q (and L/M) to be in lowest terms, i.e., that $\gcd(P, Q) = 1$ and that $Q > 0$. Whenever a result is stated for almost all x, this is meant to be in the Lebesgue sense.

1. Approximation by mediants

The main tool we will use is a variation on a theme that first appeared in [1] and was used in several papers thereafter. The theme consists of considering the sequence $\{(T_n(x), V_n(x))\}_{n=0}^{\infty}$ for an irrational number x, where $T_n(x)$ is given by

$$T_n = T_n(x) = [0; B_{n+1}, B_{n+2}, \ldots] \tag{1.1}$$

and $V_n(x)$ by

$$V_n = V_n(x) = [0; B_n, B_{n-1}, \ldots, B_1], \tag{1.2}$$

with B_i as in (0.1). For every x and every n, the pair $(T_n(x), V_n(x)) \in [0, 1] \times [0, 1]$, and for almost all x the sequence $\{(T_n(x), V_n(x))\}_{n=0}^{\infty}$ is distributed over the unit square with density function

$$\frac{1}{\log 2} \frac{1}{(1 + TV)^2}. \tag{1.3}$$

Basically, this is a consequence of the fact that

(1.4) $(\mathcal{M}, \mathcal{B}, \mu, \mathcal{T})$ forms an ergodic system;

here $\mathcal{M}$ is the unit square and $\mathcal{T}$ acts on $\mathcal{M}$ by

$$\mathcal{T}(x, y) = \left(\frac{1}{x} - \left\lfloor\frac{1}{x}\right\rfloor, \frac{1}{\lfloor\frac{1}{x}\rfloor + y}\right),$$

$\mathcal{B}$ is the collection of Borel subsets of $\mathcal{M}$ and μ is the measure on $\mathcal{M}$ with density function $\frac{1}{\log 2}\frac{1}{(1+xy)^2}$ (see [10]). Using ergodicity and the first of the basic relations

(1.5) $$\Theta_n = \frac{T_n}{1+T_nV_n} \quad\text{and}\quad \Theta_{n-1} = \frac{V_n}{1+T_nV_n},$$

one gets immediately that

$$\lim_{n\to\infty}\frac{1}{n}\#\{j \le n : \Theta_j(x) < z\} = \mu(\mathcal{H}_z),$$

where $\mathcal{H}_z$ is the subspace of $\mathcal{M}$ consisting of points under the hyperbola

$$\frac{T}{1+TV} = z.$$

The variation we need here is, that instead of using the function Θ_n in every point of the unit square, we consider $B_n - 1$ functions, namely $\Theta_n^{(B)}$ with $0 < B < B_n$. More precisely, let $B > 0$; then the function $\Theta_n^{(B)}$ as in (0.7) is defined in $(T_{n-1}, V_{n-1}) \in [0, 1] \times [0, 1]$ precisely when the partial quotient B_n exceeds B, that is, when $T_{n-1} \le \frac{1}{B+1}$. So $\Theta_n^{(B)}$ is defined on the rectangle

(1.6) $$\mathcal{R}^{(B)} = \left\{(T, V) : 0 \le T \le \frac{1}{B+1}, 0 \le V \le 1\right\}.$$

Instead of (0.6) and (0.7) one would like to have formulas expressing $\Theta_n^{(B)}$ in terms of B, T, and V only. This can be done as follows. Combining (0.4), (0.6), and (0.7), one easily gets

(1.7) $$\Theta_n^{(B)} = -B^2\Theta_{n-1} - B\left(V_{n-1}\Theta_{n-1} - \frac{\Theta_{n-2}}{V_{n-1}}\right) + \Theta_{n-2}.$$

Then use (1.5) to express Θ_{n-1} and Θ_{n-2} in terms of T_{n-1} and V_{n-1} and one arrives at

(1.8) $$\Theta_n^{(B)} = \frac{(1 - BT_{n-1})(B + V_{n-1})}{1 + T_{n-1}V_{n-1}}.$$

This provides the preliminaries for the proof of the following theorem.

(1.9) **Theorem.** *Let $B > 0$ be an integer.*

(i) *For every x and for every $n \ge 1$ such that $0 < B < B_n$, there holds*

$$\frac{B}{B+1} \le \Theta_n^{(B)}(x) \le B + 1.$$

(ii) *For almost all* x, *the sequence* $\{\Theta_n^{(B)}(x)\}_{n=1}^{\infty}$ *is distributed according to the distribution function*

$$\frac{1}{\log\frac{B+2}{B+1}} G^{(B)}(z),$$

where

$$G^{(B)}(z) = \begin{cases} G_0^{(B)}(z) = 0, & \text{for } z \le \dfrac{B}{B+1}, \\ G_1^{(B)}(z) = -1 + \dfrac{B+1}{B} z - \log\left(\dfrac{B+1}{B} z\right), & \text{for } \dfrac{B}{B+1} \le z \le \dfrac{B+1}{B+2}, \\ G_2^{(B)}(z) = \dfrac{1}{B(B+1)} z + \log\left(\dfrac{B(B+2)}{(B+1)^2}\right), & \text{for } \dfrac{B+1}{B+2} \le z \le B, \\ G_3^{(B)}(z) = 1 - \dfrac{1}{B+1} z + \log\left(\dfrac{B+2}{(B+1)^2} z\right), & \text{for } B \le z \le B+1, \\ G_4^{(B)}(z) = \log\dfrac{B+2}{B+1}, & \text{for } B+1 \le z. \end{cases}$$

Proof. From (1.8) we see that $\Theta_n^{(B)} < z$ if and only if (T_{n-1}, V_{n-1}) is in $\mathscr{R}^{(B)}$ and satisfies

$$\frac{(1 - BT_{n-1})(B + V_{n-1})}{1 + T_{n-1}V_{n-1}} < z$$

or, equivalently,

$$V_{n-1} < \frac{B^2 T_{n-1} + z - B}{1 - (B+z)T_{n-1}}.$$

So, for given x and fixed B we have to find all pairs $(T_{n-1}(x), V_{n-1}(x))$ in $\mathscr{R}^{(B)}$ under the hyperbola

$$V = \frac{B^2 T + z - B}{1 - (B+z)T}.$$

Denote by $\mathscr{H}^{(B)}$ the set of points (T, V) under the hyperbola

$$\mathscr{H}^{(B)}(z):\ V < \frac{B^2 T + z - B}{1 - (B+z)T}. \tag{1.10}$$

Since $\mathscr{R}^{(B)} \cap \mathscr{H}^{(B)}(z)$ is empty for $z < \frac{B}{B+1}$ and $\mathscr{R}^{(B)} \cap \mathscr{H}^{(B)}(z) = \mathscr{R}^{(B)}$ for $z > B+1$, we are done with part (i). For the second part we use the ergodicity given in (1.4), which implies that for almost all x:

$$\lim_{n\to\infty} \frac{1}{n} \#\{j \le n : \Theta_n^{(B)}(x) < z\} = \frac{1}{\mu(\mathscr{R}^{(B)})} \mu(\mathscr{R}^{(B)} \cap \mathscr{H}^{(B)}(z)).$$

Therefore, we are left with the computation of $\mu(\mathscr{R}^{(B)} \cap \mathscr{H}^{(B)}(z))$ as a function of z, which equals, by (1.3),

$$\frac{1}{\log 2} \iint_{\mathscr{R}^{(B)} \cap \mathscr{H}^{(B)}(z)} \frac{1}{(1+TV)^2}\, dV\, dT. \tag{1.11}$$

For $\frac{B}{B+1} \leq z \leq \frac{B+1}{B+2}$ one gets

$$\mathscr{R}^{(B)} \cap \mathscr{H}^{(B)}(z) = \left\{(T, V) : \frac{B-z}{B^2} \leq T \leq \frac{1}{B+1}, 0 \leq V \leq \frac{B^2T+z-B}{1-(B+z)T}\right\},$$

and we find

$$\begin{aligned}
&\frac{1}{\log 2}\iint_{\mathscr{R}^{(B)}\cap\mathscr{H}^{(B)}(z)} \frac{1}{(1+TV)^2}\,dV\,dT \\
&= \frac{1}{\log 2}\int_{\frac{B-z}{B^2}}^{\frac{1}{B+1}} \left[\frac{V}{1+TV}\right]_{V=0}^{V=\frac{B^2T+z-B}{1-(B+z)T}} dT \\
&= \frac{1}{\log 2}\int_{\frac{B-z}{B^2}}^{\frac{1}{B+1}} \left(\frac{z}{(1-BT)^2} - \frac{B}{1-BT}\right) dT \\
&= \frac{1}{\log 2}\left[\frac{z}{B(1-BT)} + \log(1-BT)\right]_{\frac{B-z}{B^2}}^{\frac{1}{B+1}} \\
&= \frac{1}{\log 2}\left(\frac{B+1}{B}z - \log\left(\frac{B+1}{B}z\right) - 1\right).
\end{aligned}$$

For $\frac{B+1}{B+2} \leq z \leq B$,

$$\begin{aligned}
&\mathscr{R}^{(B)} \cap \mathscr{H}^{(B)}(z) \\
&= \left\{(T, V) : \frac{B-z}{B^2} \leq T \leq \frac{B+1-z}{B^2+B+z}, 0 \leq V \leq \frac{B^2T+z-B}{1-(B+z)T}\right\} \\
&\cup \left\{(T, V) : \frac{B+1-z}{B^2+B+z} \leq T \leq \frac{1}{B+1}, 0 \leq V \leq 1\right\},
\end{aligned}$$

and this gives

$$\begin{aligned}
&\frac{1}{\log 2}\iint_{\mathscr{R}^{(B)}\cap\mathscr{H}^{(B)}(z)} \frac{1}{(1+TV)^2}\,dV\,dT \\
&= \frac{1}{\log 2}\left(\frac{z}{B(B+1)} + \log\frac{B(B+1)}{B^2+B+z}\right) \\
&+ \frac{1}{\log 2}\left(\log\frac{B+2}{B+1} - \log\frac{(B+1)^2}{B^2+B+z}\right) \\
&= \frac{1}{\log 2}\left(\frac{z}{B(B+1)} + \log\frac{B(B+2)}{(B+1)^2}\right)
\end{aligned}$$

by a computation similar to the above.

Finally, for $B \leq z \leq B+1$,

$$\begin{aligned}
\mathscr{R}^{(B)} \cap \mathscr{H}^{(B)}(z) &= \left\{(T, V) : 0 \leq T \leq \frac{B+1-z}{B^2+B+z}, 0 \leq V \leq \frac{B^2T+z-B}{1-(B+z)T}\right\} \\
&\cup \left\{(T, V) : \frac{B+1-z}{B^2+B+z} \leq T \leq \frac{1}{B+1}, 0 \leq V \leq 1\right\},
\end{aligned}$$

and the double integral (1.11) equals

$$\frac{1}{\log 2}\left(1-\frac{z}{B+1}+\log\frac{(B+1)z}{B^2+B+z}\right)+\frac{1}{\log 2}\left(\log\frac{B+2}{B+1}-\log\frac{(B+1)^2}{B^2+B+z}\right)$$
$$=\frac{1}{\log 2}\left(1-\frac{z}{B+1}+\log\frac{B+2}{(B+1)^2}z\right).$$

To find the distribution function $G^{(B)}$, we have to normalize, i.e., we have to divide in each of the cases by

$$\mu(\mathscr{R}^{(B)})=\frac{1}{\log 2}\log\frac{B+2}{B+1}.$$

This completes the proof of (1.9). □

Remark. The special case $B=1$ of Theorem (1.9) yields the result that was found as Lemma 2.24 in [7].

2. Approximation by convergents and mediants

In this section we look at the approximation of an irrational number x by all of its mediants and convergents simultaneously.

(2.1) **Lemma.** *Let $G^{(B)}(z)$ be as in (1.9). Then for the function $H(z)$ defined by*

$$H(z)=\sum_{B=1}^{\infty}G^{(B)}(z)$$

we have

$$H(z)=\begin{cases}0, & \text{for } z\le\frac{1}{2},\\ -1+2z-\log(2z), & \text{for } \frac{1}{2}\le z\le 1,\\ 1+\log\frac{z}{2}, & \text{for } 1\le z.\end{cases}$$

Proof. Let $G_i^{(B)}(z)$ be as in (1.9) for $i=0,\dots,4$. Suppose first that $\frac{1}{2}\le z\le 1$; let the positive integer k be determined by $\frac{k}{k+1}\le z<\frac{k+1}{k+2}$. Then

$$\begin{aligned}\sum_{B=1}^{\infty}G^{(B)}(z)&=\sum_{B=1}^{k-1}G_2^{(B)}(z)+G_1^{(k)}(z)+\sum_{B=k+1}^{\infty}G_0^{(B)}(z)\\&=\sum_{B=1}^{k-1}\left(\frac{1}{B(B+1)}z+\log\frac{B(B+2)}{(B+1)^2}\right)\\&\quad+\left(-1+\frac{k+1}{k}z-\log\frac{k+1}{k}z\right)+0\\&=\left(1-\frac{1}{k}\right)z+\log\frac{k+1}{2k}-1+\frac{k+1}{k}z-\log\frac{k+1}{k}z\\&=-1+2z-\log 2z.\end{aligned}$$

For $1 \le z$ we let the integer k be such that $k \le z < k+1$. Then

$$\begin{aligned}\sum_{B=1}^{\infty} G^{(B)}(z) &= \sum_{B=1}^{k-1} G_4^{(B)}(z) + G_3^{(k)}(z) + \sum_{B=k+1}^{\infty} G_2^{(B)}(z) \\ &= \sum_{B=1}^{k-1} \log \frac{B+2}{B+1} + \left(1 - \frac{1}{k+1} z + \log \frac{k+2}{(k+1)^2} z\right) \\ &\quad + \sum_{B=k+1}^{\infty} \left(\frac{z}{B(B+1)} + \log \frac{B(B+2)}{(B+1)^2}\right) \\ &= \log \frac{k+1}{2} + 1 - \frac{1}{k+1} z + \log \frac{k+2}{(k+1)^2} z \\ &\quad + \frac{1}{k+1} z + \log \frac{k+1}{k+2} \\ &= 1 + \log \frac{z}{2}.\end{aligned}$$

This completes the proof of (2.1). □

For any irrational x we introduce the following notation for the collection of all convergents and mediants of x:

$$\mathscr{A}(x) = \left\{ \frac{L}{M} : \frac{L}{M} = \frac{P_n}{Q_n} \text{ or } \frac{L}{M} = \frac{L_n^{(B)}}{M_n^{(B)}} \text{ for some } n, B \right\}.$$

For any $C > 0$ we will denote by $\mathscr{A}^C(x)$ the subset

$$\mathscr{A}^C(x) = \left\{ \frac{L}{M} \in \mathscr{A}(x) : M|Mx - L| \le C \right\}$$

of $\mathscr{A}(x)$. We enumerate the elements of $\mathscr{A}^C(x)$ after ordering them by increasing denominators; thus every fraction L_n/M_n in $\mathscr{A}^C(x)$ is either a convergent or a mediant of x, and $M_i < M_j$ if $i < j$.

(2.2) **Theorem.** *Let $C > 0$; for almost all x*

$$\lim_{n\to\infty} \frac{1}{n} \# \left\{ j \le n : \frac{L_j}{M_j} \in \mathscr{A}^C(x),\ M_j|M_j x - L_j| \le z \right\}$$

exists and $\{M_j|M_j x - L_j| : \frac{L_j}{M_j} \in \mathscr{A}^C(x)\}$ has limiting distribution $H^{(C)}(z)$ given by

$$H^{(C)}(z) = \begin{cases} \frac{1}{C} z, & \text{for } 0 \le z \le C, \quad \text{if } 0 < C \le 1, \\ \frac{1}{1+\log C} z, & \text{for } 0 \le z \le 1, \\ \frac{1}{1+\log C}(1 + \log z), & \text{for } 1 \le z \le C, \end{cases} \quad \text{if } C \ge 1.$$

Proof. Let $C > 0$ be arbitrary. For $0 \le z \le C$ we have to find all n, B (with $0 < B < B_n$) such that $\Theta_n^{(B)}(x) \le z$ as well as all n for which $\Theta_n(x) \le z$. Let

$\Lambda^{(B)}(z) \subset \mathscr{R}^{(B)}$ denote the subset for which $\Theta_n^{(B)}(x) \le z$ and let $\Lambda^{(0)}(z)$ be the subset of $[0, 1] \times [0, 1]$ for which $\Theta_n(x) \le z$. By the ergodicity of (1.4) and the individual ergodic theorem it follows that for almost all x

$$\lim_{n\to\infty} \frac{1}{n} \#\{j \le n : \Theta_j^{(B)}(x) \le z\} = \frac{1}{\mu(\mathscr{R}^{(B)})} \mu(\Lambda^{(B)}(z))$$

and

$$\lim_{n\to\infty} \frac{1}{n} \#\{j \le n : \Theta_j(x) \le z\} = \mu(\Lambda^{(0)}(z)).$$

In (1.9) we saw that

$$\frac{1}{\mu(\mathscr{R}^{(B)})} \mu(\Lambda^{(B)}(z)) = \frac{1}{\log 2} G^{(B)}(z),$$

and by (0.5),

$$\mu(\Lambda^{(0)}(z)) = \frac{1}{\log 2} F(z).$$

Denoting the whole space by Λ_C, these combine to

$$\begin{aligned} \mu(\Lambda_C) \lim_{n\to\infty} \frac{1}{n} \# &\left\{ j \le n : \frac{L_j}{M_j} \in \mathscr{A}^C(x),\, M_j |M_j x - L_j| \le z \right\} \\ &= \frac{1}{\mu(\mathscr{R}^{(B)})} \mu(\Lambda^{(0)}(z)) + \sum_{B=1}^{\infty} \frac{1}{\mu(\mathscr{R}^{(B)})} \mu(\Lambda^{(B)}(z)) \\ &= \frac{1}{\log 2} F(z) + \frac{1}{\log 2} \sum_{B=1}^{\infty} G^{(B)}(z) \\ &= \frac{1}{\log 2} F(z) + \frac{1}{\log 2} H(z) \end{aligned}$$

as in (2.1). The distribution function $H^{(C)}(z)$ is now found from the definitions of $F(z)$ and $H(z)$ and by scaling:

$$H^{(C)}(z) = \frac{F(z) + H(z)}{F(C) + H(C)}.$$

This proves (2.2). □

3. Approximation by nearest mediants

In this section we look at the approximation of an irrational number x by its nearest mediants, that is, by the mediants with $B = 1$ or with $B = B_n - 1$. Since the case $B = 1$ is contained in Theorem (1.9), we look here at $B = B_n - 1$. Notice that the 'first' mediant ($B = 1$) and the 'final' mediant ($B = B_n - 1$) coincide in case $B_n = 2$; if $B_n = 1$, there are no mediants. The first theorem tells us how the final mediants are distributed for a given partial quotient. By

$$\{\Theta_n^{(B_n - 1)}|_{B_n = D}\}$$

we will denote the sequence consisting of the Θ's belonging to the final mediants for which the partial quotient equals D.

(3.1) **Theorem.** (i) *For every x and for every $n \geq 1$ such that $B_n \geq 2$, there holds*

$$\frac{B_n - 1}{B_n} \leq \Theta_n^{(B_n-1)} \leq \frac{2B_n}{B_n + 2}.$$

(ii) *For almost all x, the sequence $\{\Theta_n^{(B_n-1)}|_{B_n=D}\}$ for $D \geq 2$ is distributed according to the distribution function*

$$\frac{1}{\log \frac{(D+1)^2}{D(D+2)}} J^{(D)}(z),$$

where

$$J^{(D)}(z) = \begin{cases} J_0^{(D)}(z) = 0, \quad \text{for } z \leq \dfrac{D-1}{D}, \\ J_1^{(D)}(z) = -1 + \dfrac{D}{D-1} z - \log\left(\dfrac{D}{D-1} z\right), \\ \qquad\qquad \text{for } \dfrac{D-1}{D} \leq z \leq \dfrac{D}{D+1}, \\ J_2^{(D)}(z) = \dfrac{1}{D(D-1)} z + \log\left(\dfrac{(D-1)(D+1)}{D^2}\right), \\ \qquad\qquad \text{for } \dfrac{D}{D+1} \leq z \leq \dfrac{2(D-1)}{D+1}, \\ J_3^{(D)}(z) = 1 - \dfrac{D+2}{2D} z + \log\left(\dfrac{(D+1)^2}{2D^2} z\right), \\ \qquad\qquad \text{for } \dfrac{2(D-1)}{D+1} \leq z \leq \dfrac{2D}{D+2}, \\ J_4^{(D)}(z) = \log \dfrac{(D+1)^2}{D(D+2)}, \quad \text{for } \dfrac{2D}{D+2} \leq z. \end{cases}$$

Proof. The proof is an imitation of the proof of Theorem (1.9), the difference being that we have to consider pairs (T, V) here in $\mathscr{R}^{(D-1)} \setminus \mathscr{R}^{(D)}$. We leave the details to the reader. □

Let $\mathscr{F}(x)$ denote the collection of final mediants:

$$\mathscr{F}(x) = \left\{ \frac{L}{M} : \frac{L}{M} = \frac{L_n^{(B_n-1)}}{M_n^{B_n-1}} \text{ for some } n \text{ for which } B_n \geq 2 \right\}.$$

We enumerate the elements of $\mathscr{F}(x)$ again after ordering them by increasing denominators; thus every fraction L_n/M_n in $\mathscr{F}$ is a final mediant of x, and $M_i < M_j$ if $i < j$.

(3.2) **Theorem.** *For almost all x*

$$\lim_{n\to\infty} \frac{1}{n} \# \left\{ j \leq n : \frac{L_j}{M_j} \in \mathscr{F}(x), \ M_j|M_j x - L_j| \leq z \right\}$$

exists and $\{M_j|M_jx - L_j| : \frac{L_j}{M_j} \in \mathscr{F}(x)\}$ *has limiting distribution* $\frac{1}{\log\frac{3}{2}}J(z)$, *where*

$$J(z) = \begin{cases} 0, & \textit{for } z \le \frac{1}{2}, \\ -1 + 2z - \log(2z), & \textit{for } \frac{1}{2} \le z \le \frac{2}{3}, \\ \frac{z}{2} + \log\frac{3}{4}, & \textit{for } \frac{2}{3} \le z \le 1, \\ 1 - \frac{z}{2} + \log\left(\frac{3}{4}z\right), & \textit{for } 1 \le z \le 2, \\ \log\frac{3}{2}, & \textit{for } 2 \le z. \end{cases}$$

Proof. We have to find all n with $\Theta_n^{(B_n-1)}(x) \le z$. Let $\Lambda^{(B_n-1)} \subset \mathscr{R}^{(B_n-1)}$ denote the subset for which $\Theta_n^{(B_n-1)}(x) \le z$. By the ergodicity of (1.4) and the individual ergodic theorem it follows that for almost all x

$$\lim_{n\to\infty}\frac{1}{n}\#\left\{j \le n : \Theta_j^{(B_n-1)}(x) \le z\right\} = \frac{1}{\mu(\mathscr{R}^{(B_n-1)}\setminus\mathscr{R}^{(B_n)})}\mu(\Lambda^{(B_n-1)}(z)).$$

From (3.1) we can see that

$$\mu(\Lambda^{(B_n-1)}(z)) = \frac{1}{\log 2}J^{(B_n-1)}(z).$$

This gives

$$\lim_{n\to\infty}\frac{1}{n}\#\left\{j \le n : \frac{L_j}{M_j} \in \mathscr{F}(x),\ M_j|M_jx - L_j| \le z\right\}$$

$$= \frac{\sum_{B_n-1=1}^{\infty}\mu(\Lambda^{(B_n-1)}(z))}{\sum_{B_n-1=1}^{\infty}\log((B_n+1)^2/B_n(B_n+2))} = \frac{1}{\log\frac{3}{2}}\sum_{D=2}^{\infty}J^{(D)}(z).$$

Suppose first that $\frac{1}{2} \le z \le \frac{2}{3}$; then

$$\sum_{D=2}^{\infty}J^{(D)}(z) = J_1^{(2)}(z) + \sum_{D=3}^{\infty}J_0^{(D)}(z)$$
$$= -1 + 2z - \log(2z) + 0.$$

Next, let $\frac{2}{3} \le z \le 1$; let the positive integer k be determined by $\frac{k-1}{k} \le z < \frac{k}{k+1}$. Then (just as in the proof of (2.1))

$$\sum_{D=2}^{\infty}J^{(D)}(z) = J_3^{(2)}(z) + \sum_{D=3}^{k-1}J_2^{(D)}(z) + J_1^{(k)}(z) + \sum_{D=k+1}^{\infty}J_0^{(D)}(z)$$

$$= 1 - z + \log\left(\frac{9}{8}z\right) + \sum_{D=3}^{k-1}\left(\frac{1}{D(D-1)}z + \log\frac{(D-1)(D+1)}{(D)^2}\right)$$

$$+ \left(-1 + \frac{k}{k-1}z - \log\frac{k}{k-1}z\right) + 0$$

$$= \frac{z}{2} + \log\frac{3}{4}.$$

For $1 \le z \le 2$ we let the integer k be such that $\frac{2(k-2)}{k} \le z < \frac{2(k-1)}{k+1}$. Then

$$\begin{aligned}\sum_{D=2}^{\infty} J^{(D)}(z) &= \sum_{D=2}^{k-2} J_4^{(D)}(z) + J_3^{(k-1)}(z) + \sum_{D=k}^{\infty} J_2^{(D)}(z) \\ &= \sum_{D=2}^{k-2} \log \frac{(D+1)^2}{D(D+2)} + \left(1 - \frac{k+1}{2(k-1)} z + \log\left(\frac{k^2}{2(k-1)^2} z\right)\right) \\ &\quad + \sum_{D=k}^{\infty} \left(\frac{z}{D(D-1)} + \log \frac{(D-1)(D+1)}{D^2}\right) \\ &= 1 + \log \frac{3(k-1)}{2k} + 1 - \frac{k+1}{2(k-1)} z + \log\left(\frac{k^2}{2(k-1)^2} z\right) \\ &\quad + \frac{1}{k-1} z + \log \frac{k-1}{k} \\ &= 1 - \frac{z}{2} + \log \frac{3z}{4}.\end{aligned}$$

This completes the proof of (3.2). □

Next, we look at the sequence of Θ's coming from convergents and nearest mediants of a given x. Let $\mathcal{N}(x)$ denote the collection of convergents and nearest mediants:

$$\mathcal{N}(x) = \left\{ \frac{L}{M} : \frac{L}{M} = \frac{P_n}{Q_n} \text{ or } \frac{L}{M} = \frac{L_n^{(1)}}{M_n^{(1)}} \text{ or } \frac{L}{M} = \frac{L_n^{(B_n-1)}}{M_n^{(B_n-1)}} \text{ for some } n \right\},$$

enumerated in order of increasing denominators M.

(3.3) **Theorem.** *For almost all* x

$$\lim_{n\to\infty} \frac{1}{n} \# \left\{ j \le n : \frac{L_j}{M_j} \in \mathcal{N}(x),\ M_j|M_j x - L_j| \le z \right\}$$

exists and $\{M_j|M_j x - L_j| : \frac{L_j}{M_j} \in \mathcal{N}(x)\}$ *has limiting distribution* $\frac{1}{2\log 2} K(z)$, *where*

$$K(z) = \begin{cases} 0, & \text{for } z \le 0, \\ z, & \text{for } 0 \le z \le 1, \\ 2 - z + 2\log z, & \text{for } 1 \le z \le 2, \\ 2\log 2, & \text{for } 2 \le z. \end{cases}$$

Proof. We consider convergents and nearest mediants now, so it is clear from their definitions that

$$K(z) = F(z) + G^{(1)}(z) + J(z) - C(z) \tag{3.4}$$

if we denote by $C(z)$ the function that gives the distribution of Θ's in case that the first and the final mediants coincide, that is if $B_n = 2$ (see the remark before Theorem (3.1)). To find $C(z)$, we have to evaluate

$$\mu(\{\mathscr{R}^{(1)} \setminus \mathscr{R}^{(2)}\} \cap \mathscr{H}^{(1)}(z))$$

(cf. (1.6) and (1.10)). For $z \leq \frac{2}{3}$ this equals

$$\mu(\mathscr{R}^{(1)} \cap \mathscr{H}^{(1)}(z)) = G^{(1)}(z) = J(z).$$

For $\frac{2}{3} \leq z \leq 1$ we find that

$$\{\mathscr{R}^{(1)} \setminus \mathscr{R}^{(2)}\} \cap \mathscr{H}^{(1)}(z) = \left\{(T, V) : \frac{1}{3} \leq T \leq \frac{2-z}{2+z}, 0 \leq V \leq \frac{T+z-1}{1-(1+z)T}\right\}$$
$$\cup \left\{(T, V) : \frac{2-z}{2+z} \leq T \leq \frac{1}{2}, 0 \leq V \leq 1\right\},$$

and a straightforward calculation of

$$\frac{1}{\log 2} \iint_{\mathscr{R}^{(B)} \cap \mathscr{H}^{(B)}(z)} \frac{1}{(1+TV)^2}\, dV\, dT$$

in this case, as in the proof of (1.9), leads to

$$C(z) = \begin{cases} 0, & \text{for } z \leq \frac{1}{2}, \\ -1 + 2z - \log(2z), & \text{for } \frac{1}{2} \leq z \leq \frac{2}{3}, \\ 1 - z + \log\left(\frac{9}{8}z\right), & \text{for } \frac{2}{3} \leq z \leq 1, \\ \log\frac{9}{8}, & \text{for } 1 \leq z. \end{cases}$$

If we use this with (0.5), Theorems (1.9) and (3.2) in (3.4) we immediately get the function $K(z)$ as in the statement of the theorem. □

(3.5) *Remarks.* In [4], Ito proved the part of (3.3) with $z \leq 1$. Using this, he was able to prove that for $0 \leq \lambda \leq 1$:

$$\lim_{n\to\infty} \frac{1}{\log n} \# \left\{(p, q) \middle| \left|x - \frac{p}{q}\right| < \frac{\lambda}{q^2} \text{ with } \gcd(p, q) = 1 \text{ and } q \leq n\right\} = \frac{12}{\pi^2}\lambda$$

(for almost all x). In fact, this holds for arbitrary $\lambda \geq 0$ and is known as Erdős' theorem (see [2]). Jager proved all of Theorem (3.3) in [7]; there, he also gives an alternative proof for the part of Erdős' theorem with $0 \leq \lambda \leq 1$, using Fatou's theorem (see §4 below). Notice that $K(z) = 2F(\frac{z}{2})$.

4. Theorems of Legendre and Fatou

The linear part in the distribution function F of (0.5) for $0 \leq z \leq \frac{1}{2}$ reflects the fact that the convergents to any x include all rationals P/Q for which $Q|Qx - P| < \frac{1}{2}$; this is known as Legendre's theorem, and it is part (i) of Theorem (4.1) below, cf. [5, 2, 4]. Since the distribution function in (3.3) is linear up to $z = 1$, one wonders whether this indicates that for every x all rationals satisfying $Q|Qx-P| < 1$ are among the set of convergents and nearest mediants to x. This is indeed the case, and it seems that this was first observed in [3], where it is stated without proof. The first proof, apparently, appeared in a paper by Koksma (see [8 and 9]). Fatou's theorem is part (ii) of Theorem (4.1) below.

(4.1) **Theorem.** *Let* x *be an irrational number and* P/Q *a rational number* ($Q > 0$ *and* $\gcd(P, Q) = 1$).

(i) *If* $Q|Qx - P| < \frac{1}{2}$, *then*

$$\frac{P}{Q} = \frac{P_n(x)}{Q_n(x)} \quad \text{for some } n \geq 0.$$

(ii) *If* $Q|Qx - P| < 1$, *then*

$$\frac{P}{Q} = \frac{BP_{n-1}(x) + P_{n-2}(x)}{BQ_{n-1}(x) + P_{n-2}(x)} \quad \text{for some } n \geq 2 \text{ and } B \in \{0, 1, B_n - 1\}.$$

Proof. The proof consists of two parts; first we show (using Koksma's argument) that if $\frac{P}{Q}$ is not a convergent or mediant, then necessarily $Q|Qx - P| > 1$. For, in this case we can find integers $n > 0$ and B $(0 \leq B < B_n)$ such that $\frac{P}{Q}$ lies between

$$\frac{P'}{Q'} = \frac{BP_{n-1} + P_{n-2}}{BQ_{n-1} + Q_{n-2}} \quad \text{and} \quad \frac{P''}{Q''} = \frac{(B+1)P_{n-1} + P_{n-2}}{(B+1)Q_{n-1} + Q_{n-2}}.$$

If we assume (the other case being similar) that $\frac{P}{Q} < x$, then

$$\frac{P'}{Q'} < \frac{P}{Q} < \frac{P''}{Q''} < x.$$

This implies

$$\frac{1}{QQ'} \leq \frac{P}{Q} - \frac{P'}{Q'} < \frac{P''}{Q''} - \frac{P'}{Q'} = \frac{P_{n-1}Q_{n-2} - P_{n-2}Q_{n-1}}{Q'Q''} = \frac{1}{Q'Q''}$$

since $P_{n-1}Q_{n-2} - P_{n-2}Q_{n-1} = 1$. So we see that $Q > Q''$.

But on the other hand,

$$\frac{1}{QQ''} \leq \frac{P''}{Q''} - \frac{P}{Q} < x - \frac{P}{Q},$$

so if

$$x - \frac{P}{Q} < \frac{1}{Q^2},$$

we would get

$$\frac{1}{QQ''} < \frac{1}{Q^2}$$

and thus $Q'' > Q$, a contradiction.

In the second part of the proof we therefore consider only convergents and mediants of x. By (1.9)(i) we have $\Theta_n^{(B)} > \frac{1}{2}$ for any n if $B > 0$; this finishes the proof of (4.1)(i).

It remains to prove that $Q|Qx - P| < 1$ can only hold for convergents and nearest mediants; thus suppose that

$$Q|Qx - P| < 1;$$

and suppose, moreover, that $B \geq 2$ in

$$\frac{P}{Q} = \frac{BP_{n-1} + P_{n-2}}{BQ_{n-1} + Q_{n-2}}.$$

We will show that in that case, $B = B_n - 1$.

By (1.8), the inequality $Q|Qx - P| = \Theta_n^{(B)} < 1$ is equivalent to

$$(1 - BT_{n-1})(B + V_{n-1}) < 1 + T_{n-1}V_{n-1}.$$

Then

$$T_{n-1} > \frac{B + V_{n-1} - 1}{B^2 + BV_{n-1} + V_{n-1}} > \frac{B-1}{B^2}$$

since $\frac{B+V-1}{B^2+BV+V}$ increases monotonically with V $(V > 0)$. This implies

$$\frac{1}{B_n + T_n} = T_{n-1} > \frac{B-1}{B^2} = \frac{1}{B + 1 + \frac{1}{B-1}},$$

so

$$B_n < B_n + T_n < B + 1 + \frac{1}{B-1} \leq B + 2,$$

in which the last inequality follows from our assumption that $B \geq 2$. Thus we see that $B > B_n - 2$, and since by definition $B < B_n$, we find that $B = B_n - 1$. This completes the proof of (4.1). □

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Department of Pure Mathematics, University of Sydney, Sydney NSW 2006, Australia. *E-mail*: wieb@galois.math.su.oz.au