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CONVERGENCE OF THE NONCONFORMING WILSON ELEMENT FOR A CLASS OF NONLINEAR PARABOLIC PROBLEMS

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ABSTRACT. This paper deals with the convergence properties of the nonconforming quadrilateral Wilson element for a class of nonlinear parabolic problems in two space dimensions. Optimal H^1 and L_2 error estimates for the continuous time Galerkin approximations are derived. It is also shown for rectangular meshes that the gradient of the Wilson element solution possesses superconvergence, and that the L_∞ error on the gradient is of order $h\log(1/h)$.

1. INTRODUCTION

The need to reduce the computational work in the use of conforming elements for the solution of higher-order elliptic problems has led to the invention of nonconforming elements. An issue in the study of nonconforming elements is the creation of a proper test that guarantees convergence of the elements. Recently, Shi [3] gave a certain condition on mesh subdivisions which ensures convergence of the quadrilateral Wilson element applied to a class of elliptic problems. The basic error analysis of Galerkin methods applied to parabolic equations is studied in [1, 6], where conforming finite elements are considered. In this paper we extend the techniques and results in [1, 6] to the nonconforming quadrilateral Wilson element under the aforementioned condition in [3]. It is found that the convergence rates of approximate solutions for the problems under consideration have the same order of accuracy as in the comparable elliptic problems. For the nonconforming rectangular Wilson element, the L_∞ error estimate and superconvergence for the gradient are shown as well. The paper is organized as follows. Section 2 deals with a general description of nonconforming methods for solving nonlinear parabolic problems. Section 3 contains some lemmas concerning error estimates for an auxiliary variational problem. The main results of the paper are established in §4. Optimal L_2 and H^1 error estimates, as well as almost optimal maximum-norm error and superconvergence order estimates for the gradient, are obtained.

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2. Nonconforming methods for solving nonlinear parabolic problems

Let Ω be a polygonal domain in R^2 with boundary $\partial\Omega$. Consider the following nonlinear parabolic initial-boundary value problem:

$$
(2.1)\qquad \begin{aligned}
&u_t - \nabla\cdot(a(x,u)\nabla u) = \sum_{i=1}^{2} b_i(x,u)u_{x_i} + f(x,u) && \text{in } \Omega\times I,\\
&u = 0 && \text{on } \partial\Omega\times I,\\
&u(x,0) = u_0(x) && \text{in } \Omega,
\end{aligned}
$$

where I denotes a finite interval $[0, T]$.

We shall assume that a, b_i, and f are sufficiently smooth functions on $\overline{\Omega}\times R$ which, together with an appropriate number of derivatives, are bounded there, and that the function a satisfies

$$
(2.2)\qquad 0 < c_1 \le a(x,u) \le c_2 \quad \text{for } (x,u)\in\overline{\Omega}\times R,
$$

for some constants c_1 and c_2. The global nature of our assumptions with respect to u does not constitute any serious restriction. In fact, the approximate solutions to be considered will be shown to be uniformly close to the exact solution u, and thus depend only on the nature of a, b_i, and f in a neighborhood of the range of u.

Let $W^{s,p}(G)$, $1\le p\le\infty$, $H^s(G) = W^{s,2}(G)$, and $H^s(\partial G)$, $s\in R$, be the usual Sobolev spaces on G and ∂G. The associated norms and seminorms are denoted as follows:

$$
\begin{aligned}
&\|\cdot\|_{s,p,G} = \|\cdot\|_{W^{s,p}(G)}, && \|\cdot\|_{s,G} = \|\cdot\|_{H^s(G)},\\
&\|\cdot\|_{s,\partial G} = \|\cdot\|_{H^s(\partial G)}, && \|\cdot\|_{s,p} = \|\cdot\|_{W^{s,p}(\Omega)},\\
&\|\cdot\|_{s} = \|\cdot\|_{H^s(\Omega)}, && \|\cdot\| = \|\cdot\|_{L_2(\Omega)},
\end{aligned}
$$

$$
|\cdot|_{s,G} = |\cdot|_{H^s(G)},\qquad |\cdot|_s = |\cdot|_{H^s(\Omega)},\qquad \|\cdot\|_{L_\infty} = \|\cdot\|_{L_\infty(\Omega)}.
$$

Let X be a normed vector space with norm $\|\cdot\|_X$. For $\varphi\colon I\to X$, define

$$
\|\varphi\|^2_{L_2(X)} = \int_0^T \|\varphi(t)\|_X^2\,dt,\qquad \|\varphi\|_{L_\infty(X)} = \sup_{t\in I}\|\varphi(t)\|_X.
$$

We now consider the numerical approximation of (2.1) by a nonconforming finite element method. Let $\overline{\Omega} = \bigcup_{K\in K_h} K$ be a decomposition of $\overline{\Omega}$ into elements K with diameters $\le h$. With this subdivision we associate a nonconforming finite element space S_h consisting of certain functions v vanishing at the nodes belonging to $\partial\Omega$. In general, the inclusion $S_h\subset H_0^1(\Omega)$ does not hold. The norm associated with S_h is

$$
\|\cdot\|^2_{S_h} = |\cdot|^2_{1,h} = \sum_K |\cdot|^2_{1,K}.
$$

In this paper we shall confine ourselves to the nonconforming spaces S_h of the Wilson element, i.e., the finite element spaces of all functions whose restrictions to each quadrilateral element $K\in K_h$ are the shape functions defined by equations (3.1)–(3.3) of [3]. We also assume the subdivisions K_h to satisfy the regularity conditions in [3]. Let h_K denote the diameter of the element K and $h = \max_{K \in K_h} h_K$. We introduce a further assumption on mesh subdivisions. The distance d_K between the midpoints of the diagonals of $K \in K_h$ is of the order $O(h_K^2)$, uniformly for all elements K as $h \to 0$.

The nonconforming approximation of the solution of (2.1) is the following.

Find a differentiable map $U: I \to S_h$ such that

$$(U_t, V) + A_h(U; U, V) = \sum_{i=1}^{2} (b_i(U)U_{x_i}, V)_h + (f(U), V),$$
$$(U(0), V) = (u_0, V) \quad \forall V \in S_h, \tag{2.3}$$

where

$$(\varphi, \psi) = \int_\Omega \varphi\psi\, dx, \qquad (\varphi, \psi)_h = \sum_K \int_K \varphi\psi\, dx,$$
$$A_h(p; \varphi, \psi) = (a(p)\nabla\varphi, \nabla\psi)_h = \sum_K (a(p)\nabla\varphi, \nabla\psi)_K,$$
$$a(p) = a(x, p), \quad f(U) = f(x, U), \quad U(0) = U(x, 0).$$

Here, $(\cdot, \cdot)_K$ denotes the inner product in $L_2(K)^2$. Without loss of generality we shall assume the initial condition in (2.1) to be $u_0(x) = 0$ throughout this paper.

3. Lemmas

Let u and U be the solutions of problems (2.1) and (2.3), respectively. In order to estimate the error $U - u$, we introduce an auxiliary variational problem:

Find a map $\tilde{u}: I \to S_h$ such that

$$A_h(u; \tilde{u}, V) = A(u; u, V) \quad \forall V \in S_h, \tag{3.1}$$

where

$$A(w, u, V) = -\int_\Omega \nabla \cdot (a(w)\nabla u) V\, dx.$$

Note that the initial condition $\tilde{u}(0) = 0$ holds, since $u_0(x) = 0$. In the sequel, c will be used as a generic constant. It may have different values at different places.

Lemma 3.1. *Assume $\psi \in H^1(\Omega)$ and $v_h \in S_h$; then*

$$|T_h^{(r)}(\psi, v_h)| := \left| \sum_K \int_{\partial K} \psi v_h N_r\, ds \right| \le ch^2 \|\psi\|_1 |v_h|_{2,h}, \tag{3.2}$$

where $n = (N_1, N_2)$ is the unit exterior normal vector along the boundary ∂K of K, and the seminorm $|\cdot|_{2,h}$ is defined by $|\cdot|_{2,h}^2 = \sum_K |\cdot|_{2,K}^2$.

Proof. The proof uses ideas contained in [3, Theorem 1]. We decompose the trial function v_h into the conforming and nonconforming parts, $v_h = y_h + z_h$. Since $y_h \in H_0^1(\Omega)$,

$$T_h^{(r)}(\psi, y_h) = 0, \qquad T_h^{(r)}(\psi, v_h) = T_h^{(r)}(\psi, z_h). \tag{3.3}$$

Let us decompose $T_h^{(r)}(\psi, z_h)$ in the form

$$\begin{aligned} T_h^{(r)}(\psi, z_h) &= \sum_K \int_{\partial K} \psi z_h N_r \, ds \\ &= \sum_K \int_{\partial K} R_0\psi R_0 z_h N_r \, ds + \sum_K \int_{\partial K} R_0 \psi P_0 z_h N_r \, ds \\ &\quad + \sum_K \int_{\partial K} P_0 \psi z_h N_r \, ds = \sum_{j=1}^{3} T_j , \end{aligned} \tag{3.4}$$

where $P_0 v$ is the piecewise constant approximation of v defined in [3, equation (2.6)] and $R_0 v = v - P_0 v$ is the associated remainder term. We estimate each of the terms T_i, $i = 1, 2, 3$, on the right-hand side of (3.4).

(i) By virtue of interpolation theory we have

$$\begin{aligned} |T_1| &\le \sum_K \left| \int_{\partial K} R_0 \psi R_0 z_h N_r \, ds \right| \\ &\le \sum_K \left(\int_{\partial K} (R_0 \psi)^2 \, ds \right)^{1/2} \left(\int_{\partial K} (R_0 z_h)^2 \, ds \right)^{1/2} \\ &\le ch |\psi|_1 |z_h|_{1,h} . \end{aligned} \tag{3.5}$$

Combining [3, (3.11)–(3.12); 2, (2.11); and 3, (2.3)] yields

$$|z_h|_{1,K} \le ch_K |v_h|_{2,K} .$$

Consequently,

$$|T_1| \le ch^2 |\psi|_1 |v_h|_{2,h} . \tag{3.6}$$

(ii) Similarly, we have

$$|P_0 z_h| \le ch_K |v_h|_{2,K} . \tag{3.7}$$

Thus,

$$\begin{aligned} |T_2| &\le \sum_K \left| \int_{\partial K} R_0 \psi P_0 z_h N_r \, ds \right| \\ &\le \sum_K \left(\int_{\partial K} (R_0 \psi)^2 \, ds \right)^{1/2} \left(\int_{\partial K} (P_0 z_h)^2 \, ds \right)^{1/2} \\ &\le ch^2 |\psi|_1 |v_h|_{2,h} . \end{aligned} \tag{3.8}$$

(iii) For the term T_3, we use the inequalities [3, (3.17)–(3.20) and (3.5)] and note that $d_K = O(h_K^2)$ to get

$$\left| \int_{\partial K} z_h N_r \, ds \right| \le c d_K h_K |v_h|_{2,K} \le ch^3 |v_h|_{2,K} . \tag{3.9}$$

Thus,

$$|T_3| \le \sum_K \left| \int_{\partial K} P_0 \psi z_h N_r \, ds \right| \le ch^2 \|\psi\|_1 |v_h|_{2,h} . \tag{3.10}$$

Combining relations (3.3), (3.4), (3.6), (3.8), and (3.10), we see that (3.2) follows. □

Corollary 3.1. *For* $\psi \in H^1(\Omega)$ *and* $v_h \in S_h$,

$$|T_h^{(r)}(\psi, v_h)| \le ch\|\psi\|_1 |v_h|_{1,h}, \qquad r = 1, 2. \tag{3.11}$$

The following three lemmas are contained in [3, 4, and 5], respectively.

Lemma 3.2. *Let* $u \in H^2(\Omega)$ *for all* $t \in I$ *and* $\tilde{u}$ *be the solutions of* (2.1) *and* (3.1), *respectively. Then for* $t \in I$

$$\|\tilde{u} - u\| + h|\tilde{u} - u|_{1,h} \le ch^2\|u\|_2. \tag{3.12}$$

Lemma 3.3. *Under the hypotheses of Lemma* 3.2 *and the additional assumption that* $\overline{\Omega}$ *is decomposed into rectangular elements, we have that*

$$u \in W^{2,\infty}(\Omega) \quad \forall t \in I$$

implies

$$\|\tilde{u} - u\|_{L_\infty} \le ch^2 \log\frac{1}{h}\|u\|_{2,\infty}. \tag{3.13}$$

Lemma 3.4. *If, in addition to the hypotheses of Lemma* 3.3, $u \in H^3(\Omega)$ *for* $t \in I$, *then for* $t \in I$

$$\left\{\frac{1}{N}\sum_{x_0 \in G} |\nabla(\tilde{u} - u)(x_0)|^2\right\}^{1/2} \le ch^2\|u\|_3, \tag{3.14}$$

where G *denotes the set of nice stress points and* $N = O(h^{-2})$ *is the cardinal number of* G.

(The nice stress points relative to the given subdivision are the centers of the rectangular elements of that subdivision.)

We now give the following estimates for the error $\tilde{u} - u$ between solutions of (3.1) and (2.1).

Lemma 3.5. *Under the hypotheses of Lemma* 3.2, *we have for* $t \in I$

$$\left(\sum_K \|\tilde{u} - u\|^2_{-1/2,\partial K}\right)^{1/2} \le ch^2\|u\|_2, \tag{3.15}$$

$$\left(\sum_K \|\tilde{u}_t - u_t\|_{-1/2,\partial K}\right)^{1/2} \le ch^2(\|u\|_2 + \|u_t\|_2). \tag{3.16}$$

Proof. (i) Let $\varphi_K \in H^1(K)$ be the solution of the elliptic problem

$$\int_K a(u)\nabla\varphi_K \cdot \nabla v\, dx = \int_{\partial K} \gamma_K v\, ds \quad \forall v \in H^1(K), \; t \in I. \tag{3.17}$$

Then $\gamma_K \in H^{1/2}(\partial K)$ may be chosen so that

$$\int_{\partial K} \gamma_K \eta \, ds = \|\eta\|^2_{-1/2,\partial K}$$

and

$$\|\gamma_K\|_{1/2,\partial K} = \|\eta\|_{-1/2,\partial K},$$

where $\eta = \tilde{u} - u$. The existence of γ_K is guaranteed by the Hahn-Banach theorem, since $H^{1/2}(\partial K)$ is the dual of $H^{-1/2}(\partial K)$.

It is well known that if K is convex, then $\varphi_K \in H^2(K)$ and

$$\|\varphi_K\|_{2,K} \le c\|\gamma_K\|_{1/2,\partial K} \le c\|\eta\|_{-1/2,\partial K}. \tag{3.18}$$

Setting $v = \eta$ in (3.17) and summing over $K \in K_h$, we find

$$\sum_K \|\eta\|^2_{-1/2,\partial K} = A_h(u;\eta,\varphi) = A_h(u;\eta,\varphi - \pi\varphi) + A_h(u;\eta,\pi\varphi), \tag{3.19}$$

where the function φ is defined by $\varphi|_K = \varphi_K$ and $\pi\varphi \in S_h$ denotes the S_h-interpolant of φ. By the continuity of $A_h(u;\cdot,\cdot)$, the interpolation property, Lemma 3.2, and (3.18), we get

$$\begin{aligned} |A_h(u;\eta,\varphi-\pi\varphi)| &\le c|\eta|_{1,h}|\varphi-\pi\varphi|_{1,h} \le ch^2\|u\|_2|\varphi|_{2,h} \\ &\le ch^2\|u\|_2\left(\sum_K \|\eta\|^2_{-1/2,\partial K}\right)^{1/2}. \end{aligned} \tag{3.20}$$

For the second term on the right-hand side of (3.19), we have, by (3.1),

$$\begin{aligned} A_h(u;\eta,\pi\varphi) &= A_h(u;\tilde{u},\pi\varphi) - A_h(u;u,\pi\varphi) \\ &= A(u;u,\pi\varphi) - A_h(u;u,\pi\varphi) = -D_h(u;u,\pi\varphi), \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} D_h(w;u,v_h) &= A_h(w;u,v_h) - A(w;u,v_h) \\ &= \sum_K \int_K [a(w)\nabla u\cdot\nabla v_h + \nabla\cdot(a(w)\nabla u)v_h]\,dx \\ &= \sum_K D_K(w;u,v_h). \end{aligned} \tag{3.22}$$

Using Green's formula, we obtain

$$D_K(w;u,v) = \int_{\partial K} a(w)\frac{\partial u}{\partial n} v_h \, ds. \tag{3.23}$$

It follows from Lemma 3.1 and (3.18) that

$$\begin{aligned} |A_h(u;\eta,\pi\varphi)| &\le ch^2\|u\|_2|\pi\varphi|_{2,h} \le ch^2\|u\|_2|\varphi|_{2,h} \\ &\le ch^2\|u\|_2\left(\sum_K \|\eta\|^2_{-1/2,\partial K}\right)^{1/2}. \end{aligned} \tag{3.24}$$

Inequality (3.15) now follows from (3.19) and inequalities (3.20) and (3.24).

(ii) The second assertion of the lemma can be proven using the same technique as the one leading to (3.15).

Similarly as (3.19) was derived, one can easily obtain

$$\sum_K \|\eta_t\|^2_{-1/2,\partial K} = A_h(u;\eta_t,\varphi-\pi\varphi) + A_h(u;\eta_t,\pi\varphi), \tag{3.25}$$

where

$$\|\varphi\|_{2,K} \le c\|\eta_t\|_{-1/2,\partial K}. \tag{3.26}$$

Obviously,

$$|A_h(u;\eta_t,\varphi-\pi\varphi)| \le ch^2\|u\|_2\left(\sum_K \|\eta_t\|^2_{-1/2,\partial K}\right)^{1/2}. \tag{3.27}$$

To estimate the second term on the right-hand side of (3.25), we rewrite the representation (3.1) as

$$A_h(u;\eta,V) + D_h(u;u,V) = 0 \quad \forall V \in S_h. \tag{3.28}$$

Differentiating (3.28) with respect to t, we see that

$$A_h(u;\eta_t,V) + A_h^*(u;\eta,V) + D_h(u;u_t,V) + D_h^*(u;u,V) = 0 \quad \forall V \in S_h, \tag{3.29}$$

where

$$A_h^*(w;v,V) = \sum_K \int_K (a(w))_t \nabla v\cdot\nabla V\,dx, \tag{3.30}$$

$$D_h^*(w;v,V) = \sum_K \int_{\partial K} (a(w))_t \frac{\partial v}{\partial n} V\,ds. \tag{3.31}$$

Hence,

$$A_h(u;\eta_t,\pi\varphi) = -D_h(u;u_t,\pi\varphi) - D_h^*(u;u,\pi\varphi) - A_h^*(u;\eta,\pi\varphi). \tag{3.32}$$

Applying Lemma 3.1 to the first two terms on the right-hand side of (3.32), we conclude that

$$|D_h(u;u_t,\pi\varphi)| \le ch^2\|u_t\|_2\|\varphi\|_{2,h}, \tag{3.33}$$

$$|D_h^*(u;u,\pi\varphi)| \le ch^2\|u\|_2\|\varphi\|_{2,h}. \tag{3.34}$$

For the last term of (3.32), we have by Green's formula,

$$\begin{aligned}-A_h^*(u;\eta,\pi\varphi) &= \sum_K \int_K \nabla\cdot(a(u)_t\nabla\pi\varphi)\eta\,dx - \sum_K\int_{\partial K}(a(u))_t\eta\frac{\partial\pi\varphi}{\partial n}\,dx\\ &= G_1 + G_2,\end{aligned} \tag{3.35}$$

and consequently, writing $\|\cdot\|^2_{2,h} = \|\cdot\|^2 + |\cdot|^2_{1,h} + |\cdot|^2_{2,h}$,

$$|G_1| \le c\|\eta\|\,\|\pi\varphi\|_{2,h} \le ch^2\|u\|_2\|\varphi\|_{2,h}, \tag{3.36}$$

$$(3.37)\quad |G_2| \le c\sum_K \|\eta\|_{-1/2,\partial K}\left\|\frac{\partial \pi\varphi}{\partial n}\right\|_{1/2,\partial K} \le c\left(\sum_K \|\eta\|^2_{-1/2,\partial K}\right)^{1/2}\|\varphi\|_{2,h},$$

where we have used the trace inequality. Combining the inequalities (3.26), (3.33)–(3.37), and (3.15), we see from (3.32) that

$$(3.38)\qquad |A_h(u;\eta_t,\pi\varphi)| \le ch^2\|u\|_2\left(\sum_K \|\eta_t\|^2_{-1/2,\partial K}\right)^{1/2}.$$

Finally, substituting (3.27) and (3.38) into (3.25), we arrive at (3.16). □

Lemma 3.6. *Assume that the hypotheses of Lemma* 3.2 *are satisfied. Moreover, assume that* $u_t, u_{tt} \in H^2(\Omega)$ *for* $t \in I$; *then, for* $t \in I$,

$$(3.39)\qquad \|\tilde u_t - u_t\| + h|\tilde u_t - u_t|_{1,h} \le ch^2[\|u\|_2 + \|u_t\|_2],$$

$$(3.40)\qquad \|\tilde u_{tt} - u_{tt}\| + h|\tilde u_{tt} - u_{tt}|_{1,h} \le ch^2[\|u\|_2 + \|u_t\|_2 + \|u_{tt}\|_2].$$

Proof. (i) Let $u^* \in S_h$ be such that for $t \in I$

$$(3.41)\qquad A_h(u;u^* - u_t, V) + D_h(u;u_t,V) = 0 \quad \forall V \in S_h.$$

Then by Lemma 3.2,

$$(3.42)\qquad |u^* - u_t|_{1,h} \le ch\|u_t\|_2.$$

Using inequality (2.2) and Corollary 3.1, we deduce from (3.29) and (3.41) with $V = \tilde u_t - u^*$ that

$$\begin{aligned} c_1|\tilde u_t - u^*|^2_{1,h} &\le A_h(u;\tilde u_t - u^*, \tilde u_t - u^*) \\ &= -D^*(u;u,\tilde u_t - u^*) - A_h^*(u;\tilde u - u, \tilde u_t - u^*) \\ &\le c[h\|u\|_2 + |\tilde u - u|_{1,h}]|\tilde u_t - u^*|_{1,h} \le ch\|u\|_2|\tilde u_t - u^*|_{1,h}. \end{aligned}$$

Combining this inequality with the triangular inequality

$$|\tilde u_t - u_t|_{1,h} \le |\tilde u_t - u^*|_{1,h} + |u^* - u_t|_{1,h}$$

and using (3.42) yield

$$(3.43)\qquad |\tilde u_t - u_t|_{1,h} \le ch[\|u\|_2 + \|u_t\|_2].$$

For the L_2-norm we proceed by duality. Note that

$$(3.44)\qquad \|\eta_t\| = \sup_{g\in L_2}\frac{|(g,\eta_t)|}{\|g\|}.$$

Let g be an arbitrary function in $L_2(\Omega)$. Let $\varphi \in H^2(\Omega)\cap H^1_0(\Omega)$ be the solution of

$$(3.45)\qquad \begin{aligned} -\nabla\cdot(a(u)\nabla\varphi) &= g \quad \text{in } \Omega, \\ \varphi &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Recall that

$$(3.46)\qquad \|\varphi\|_2 \le c\|g\|.$$

From (3.29) we have

$$(3.47)\quad \begin{aligned}(g, \eta_t) &= A_h(u; \eta_t, \varphi - \pi\varphi) - A_h^*(u; \eta, \pi\varphi) - D_h(u; u_t, \pi\varphi)\\ &\quad - D_h^*(u; u, \pi\varphi) - [A_h(u; \eta_t, \varphi) - (g, \eta_t)]\\ &= \sum_{j=1}^{5} I_j.\end{aligned}$$

Using arguments similar to those used to derive inequalities (3.27) and (3.38), one has

$$(3.48)\qquad I_1 + I_2 + I_3 + I_4 \le ch^2\|u\|_2\|\varphi\|_2.$$

Noting that $u_t \in H_0^1(\Omega)$, we have from (3.45) and (3.22) that

$$I_5 = -D_h(u; \varphi, \eta) = -D_h(u; \varphi, \tilde{u}_t),$$

and

$$(3.49)\qquad |I_5| \le ch^2\|\varphi\|_2|\tilde{u}_t|_{2,h}.$$

The inverse inequality implies

$$\begin{aligned}|\tilde{u}_t|_{2,h} &\le |\tilde{u}_t - \Pi u_t|_{2,h} + |\Pi u_t|_{2,h}\\ &\le ch^{-1}[|\tilde{u}_t - u_t|_{1,h} + |u_t - \Pi u_t|_{1,h}] + |\Pi u_t - u_t|_{2,h} + |u_t|_2\\ &\le c\|u_t\|_2.\end{aligned}$$

Together with (3.44) and (3.46)–(3.49), this proves the first estimate of the lemma.

(ii) We now prove (3.40). Differentiating (3.29) with respect to t, we get

$$(3.50)\quad \begin{aligned}&A_h(u; \eta_{tt}, V) + 2A_h^*(u; \eta_t, V) + A_h^{**}(u; \eta, V) + D_h(u; u_{tt}, V)\\ &\quad + 2D_h^*(u; u_t, V) + D_h^{**}(u; u, V) = 0 \quad \forall V \in S_h,\end{aligned}$$

where

$$A_h^{**}(u; \eta, V) = \sum_K \int_K (a(u))_{tt} \nabla\eta \cdot \nabla V\, dx,$$

$$D_h^{**}(u; u, V) = \sum_K \int_{\partial K} (a(u))_{tt} \frac{\partial u}{\partial n} V\, ds.$$

Define $u^{**} \in S_h$ by

$$A_h(u; u^{**} - u_{tt}, V) + D_h(u; u_{tt}, V) = 0 \quad \forall V \in S_h,\ t \in I.$$

Observing

$$(3.51)\qquad |u^{**} - u_{tt}|_{1,h} \le ch\|u_{tt}\|_2,$$

and setting $V = \tilde{u}_{tt} - u^{**}$ in (3.50), we find

$$(3.52)\quad \begin{aligned}c_1|\tilde{u}_{tt} - u^{**}|_{1,h}^2 &\le A_h(u; \tilde{u}_{tt} - u^{**}, \tilde{u}_{tt} - u^{**})\\ &\le -2D_h^*(u; u_t, \tilde{u}_{tt} - u^{**}) - D_h^{**}(u; u, \tilde{u}_{tt} - u^{**})\\ &\quad - 2A_h^*(u; \eta_t, \tilde{u}_{tt} - u^{**}) - A_h^{**}(u; \eta, \tilde{u}_{tt} - u^{**}).\end{aligned}$$

On the other hand, again by (3.45), it follows from (3.50) that

$$\begin{aligned}(g, \eta_{tt}) = {} & A_h(u; \eta_{tt}, \varphi - \pi\varphi) - 2A_h^*(u; \eta_t, \pi\varphi) - A_h^{**}(u; \eta, \pi\varphi) \\ & - D_h(u; u_{tt}, \pi\varphi) - 2D_h^*(u; u_t, \pi\varphi) \\ & - D_h^{**}(u; u, \pi\varphi) - D_h(u; \varphi, \eta_{tt}).\end{aligned} \tag{3.53}$$

Applying the technique used to estimate (3.43), (3.47)–(3.52), and (3.53), inequality (3.40) follows. □

Corollary 3.2. *Assume that* $u, u_t \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ *for* $t \in I$. *Then* $\nabla \tilde{u}$, $\tilde{u}_t$, *and* $\nabla \tilde{u}_t$ *are uniformly bounded, i.e., there exists a constant* $\tilde{c}$ *independent of* h *and* t *such that*

$$\|\nabla \tilde{u}\|_{L_\infty(L_\infty)} + \|\tilde{u}_t\|_{L_\infty(L_\infty)} + \|\nabla \tilde{u}_t\|_{L_\infty(L_\infty)} \le \tilde{c}, \tag{3.54}$$

where $\|\nabla\varphi\|_{L_\infty(L_\infty)} = \sup_{t\in I} \max_K \|\nabla\varphi\|_{L_\infty(K)}$.

Proof. Using the triangle inequality and the inverse inequality, we have

$$\begin{aligned}\|\nabla \tilde{u}\|_{L_\infty} & \le \|\nabla(\tilde{u} - \Pi u)\|_{L_\infty} + \|\nabla(\Pi u - u)\|_{L_\infty} + \|\nabla u\|_{L_\infty} \\ & \le ch^{-1}[|\tilde{u} - u|_{1,h} + |u - \Pi u|_{1,h}] + \|\nabla(\pi u - u)\|_{L_\infty} + \|\nabla u\|_{L_\infty} \\ & \le c[\|u\|_2 + \|\nabla u\|_{L_\infty}] + \|\nabla(u - \pi u)\|_{L_\infty}.\end{aligned} \tag{3.55}$$

We now estimate the last term of (3.55). Since $\pi u \in S_h$, we can write πu as the sum of the conforming part y_h and the nonconforming part z_h,

$$\pi u = y_h + z_h. \tag{3.56}$$

Hence,

$$\begin{aligned}\|\nabla(u - \Pi u)\|_{L_\infty} & \le \|\nabla u - y_h\|_{L_\infty} + \|\nabla z_h\|_{L_\infty} \\ & \le \|\nabla(u - y_h)\|_{L_\infty} + ch^{-1}|z_h|_{1,h} \\ & \le \|\nabla(u - y_h)\|_{L_\infty} + ch^{-1}[|\pi u - u|_{1,h} + |u - y_h|_1] \\ & \le c[\|\nabla u\|_{L_\infty} + \|u\|_2].\end{aligned} \tag{3.57}$$

Together with (3.55), these bounds show our assertions on $\nabla\tilde{u}$. In the same way, the uniform boundedness of $\tilde{u}_t$ and $\nabla\tilde{u}_t$ can be proved. We omit the details. □

4. Error estimates

The main objective of this section is to derive error estimates for $U - u$. With $\tilde{u}$ defined by (3.1), we write the error

$$U - u = (U - \tilde{u}) + (\tilde{u} - u) = \xi + \eta.$$

The lemmas in §3 give estimates on η. It remains to estimate ξ. To do this, we need an additional assumption on U. Assume that there exists a positive constant c^* such that

$$\|U_t\|_{L_\infty(L_\infty)} \le c^* . \tag{4.1}$$

Without loss of generality, assume $c^* \ge 2\tilde{c}$ (see (3.54)).

The following theorem gives the fundamental result on ξ.

Theorem 4.1. *Let u, U, and $\tilde{u}$ be the solutions of problems* (2.1), (2.3), *and* (3.1), *respectively. Assume that $u, u_t \in L_2(H^2(\Omega)) \cap L_\infty(W^{1,\infty}(\Omega))$ and that $u_{tt} \in L_2(H^2(\Omega))$. Then*

$$\|\xi_t\|_{L_\infty(L_2)} + \|\xi\|_{L_\infty(S_h)} + \|\xi_t\|_{L_2(S_h)} \le ch^2 . \tag{4.2}$$

Proof. Using (2.1), (2.3), and (3.1), we observe that

$$\begin{aligned}(\xi_t, V) + A_h(U; \xi, V) = &-(\eta_t, V) + ((a(u) - a(U))\nabla\tilde{u}, \nabla V)_h \\ &+ \sum_{i=1}^{2}(b_i(U)U_{x_i} - b_i(u)u_{x_i}, V)_h \\ &+ (f(U) - f(u), V) \quad \forall V \in S_h .\end{aligned} \tag{4.3}$$

Differentiating (4.3) with respect to t yields

$$\begin{aligned}&(\xi_{tt}, V) + A_h(U; \xi_t, V) \\ &\quad = -(\eta_{tt}, V) - ((a(U))_t\nabla\xi, \nabla V)_h + ((a(u) - a(U))\nabla\tilde{u}_t, \nabla V)_h \\ &\qquad + ((a(u) - a(U))_t\nabla\tilde{u}, \nabla V)_h \\ &\qquad + \sum_{i=1}^{2}((b_i(U)U_{x_i} - b_i(u)u_{x_i})_t, V)_h \\ &\qquad + ((f(U) - f(u))_t, V) \quad \forall V \in S_h .\end{aligned} \tag{4.4}$$

We first discuss estimates of (4.3). Setting $V = \xi_t$ and using (4.1), we see that the left-hand side of (4.3) is

$$\begin{aligned}&(\xi_t, \xi_t) + \frac{1}{2}\frac{d}{dt}A_h(U; \xi, \xi) - (a_p(U)U_t\nabla\xi, \nabla\xi)_h \\ &\qquad \ge \|\xi_t\|^2 + \frac{1}{2}\frac{d}{dt}A_h(U; \xi, \xi) - cc^*|\xi|_{1,h}^2 .\end{aligned} \tag{4.5}$$

Also,

$$-(\eta_t, \xi_t) \le \|\eta_t\|^2 + \|\xi_t\|^2 \tag{4.6}$$

and

$$(f(U) - f(u), \xi_t) \le c[\|\xi\|^2 + \|\eta\|^2 + \|\xi_t\|^2] . \tag{4.7}$$

Applying Corollary 3.2 and the inequality $ab \le a^2/4\varepsilon + \varepsilon b^2$, we get

$$((a(u) - a(U))\nabla \tilde{u}, \nabla \xi_t)_h \le c(\|\xi\|^2 + \|\eta\|^2) + \frac{c_1}{12}|\xi_t|^2_{1,h}. \tag{4.8}$$

Rewrite the third term on the right-hand side of (4.3) as

$$\sum_{i=1}^{2}(b_i(U)\xi_{x_i}, \xi_t)_h + \sum_{i=1}^{2}((b_i(U) - b_i(u))\tilde{u}_{x_i}, \xi_t)_h + \sum_{i=1}^{2}(b_i(u)\eta_{x_i}, \xi_t)_h. \tag{4.9}$$

The first two terms of (4.9) are bounded above by

$$c[\|\eta\|^2 + \|\xi\|^2 + \|\xi_t\|^2 + |\xi|^2_{1,h}]. \tag{4.10}$$

For the last term of (4.9), we get by Green's formula,

$$\begin{aligned} \sum_{i=1}^{2}(b_i(u)\eta_{x_i}, \xi_t)_h &= -\sum_{i=1}^{2}((b_i(u)\xi_t)_{x_i}, \eta)_h \\ &\quad + \sum_{i=1}^{2}\sum_{K}\int_{\partial K} b_i(u)\eta\xi_t N_i\, ds \\ &= Q_1 + Q_2. \end{aligned} \tag{4.11}$$

Obviously,

$$Q_1 \le c\|\eta\|^2 + \frac{c_1}{12}|\xi_t|^2_{1,h}. \tag{4.12}$$

Applying the duality of $H^{1/2}(\partial K)$ and $H^{-1/2}(\partial K)$ and the trace inequality, we conclude that

$$\begin{aligned} Q_2 &\le c\sum_{K}\|\eta\|_{-1/2,\partial K}\|\xi_t\|_{1/2,\partial K} \\ &\le c\sum_{K}\|\eta\|^2_{-1/2,\partial K} + \frac{c_1}{12}(\|\xi_t\|^2 + |\xi_t|^2_{1,h}). \end{aligned} \tag{4.13}$$

Combining (4.5)–(4.13) with (4.3), we obtain

$$\begin{aligned} &\|\xi_t\|^2 + \frac{1}{2}\frac{d}{dt}A_h(U; \xi, \xi) \\ &\le c(c^* + 1)\left[\|\xi\|^2 + |\xi|^2_{1,h} + \|\xi_t\|^2 + \|\eta\|^2 + \|\eta_t\|^2 \right. \\ &\qquad \left. + \sum_{K}\|\eta\|^2_{-1/2,\partial K}\right] + \frac{c_1}{4}|\xi_t|^2_{1,h}. \end{aligned} \tag{4.14}$$

Now we estimate (4.4). Replacing V by ξ_t in (4.4), the coerciveness of the bilinear form $A_h(U; \cdot, \cdot)$ implies that the left-hand side of (4.4) is

$$\geq \frac{1}{2}\frac{d}{dt}\|\xi_t\|^2 + c_1|\xi_t|_{1,h}. \tag{4.15}$$

As for the fifth term on the right-hand side of (4.4), we write it in the form

$$\begin{aligned}
&\sum_{i=1}^{2}((b_i(U)U_{x_i} - b_i(u)u_{x_i})_t, \xi_t)_h \\
&\quad = \sum_{i=1}^{2}(b_i(U)\xi_{x_it}, \xi_t)_h + \sum_{i=1}^{2}((b_i(U) - b_i(u))\tilde{u}_{x_it}, \xi_t)_h \\
&\qquad + \sum_{i=1}^{2}(b_i(u)\eta_{x_it}, \xi_t)_h + \sum_{i=1}^{2}((b_i(U))_t\xi_{x_i}, \xi_t)_h \\
&\qquad + \sum_{i=1}^{2}((b_i(U) - b_i(u))_t\tilde{u}_{x_i}, \xi_t)_h + \sum_{i=1}^{2}((b_i(u))_t\eta_{x_i}, \xi_t)_h.
\end{aligned} \tag{4.16}$$

Using an argument similar to one given in (4.3), we can estimate each term of (4.16). The remaining terms in (4.4) are bounded as before. Following the same analysis that led to (4.14), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\xi_t\|^2 + c_1|\xi_t|_{1,h}^2 \\
&\quad \leq c(c^* + 1)\Big[\|\xi\|^2 + |\xi|_{1,h}^2 + \|\xi_t\|^2 + \|\eta\|^2 \\
&\qquad\qquad + \|\eta_t\|^2 + \|\eta_{tt}\|^2 + \sum_K \|\eta\|_{-1/2,\partial K}^2\Big] \\
&\qquad + \frac{c_1}{4}|\xi_t|_{1,h}^2.
\end{aligned} \tag{4.17}$$

Adding this inequality to (4.14), we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|\xi_t\|^2 + A_h(U; \xi, \xi)] + \frac{c_1}{2}|\xi_t|_{1,h}^2 \\
&\quad \leq c(c^* + 1)\Big[\|\xi\|^2 + |\xi|_{1,h}^2 + \|\xi_t\|^2 + \|\eta\|^2 + \|\eta_t\|^2 \\
&\qquad\qquad + \|\eta_{tt}\|^2 + \sum_K \|\eta\|_{-1/2,\partial K}^2\Big].
\end{aligned} \tag{4.18}$$

Integrating (4.18) with respect to t and noticing that

$$\|\xi\|^2 = \int_\Omega \left(\int_0^t \xi_t\, ds\right)^2 dx \leq T\int_0^t \|\xi_t\|^2\, ds,$$

we have by the initial conditions,

$$
\begin{aligned}
&\|\xi_t\|^2 + |\xi|_{1,h}^2 + \int_0^t |\xi_t|_{1,h}^2\, ds \\
&\quad \le c(1+c^*)\Bigg[\|\xi_t(0)\|^2 + \int_0^t \Bigg(\|\eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2 \\
&\qquad\qquad + \sum_K \|\eta\|_{-1/2,\partial K}^2\Bigg)\, ds \\
&\qquad\qquad + \int_0^t (\|\xi_t\|^2 + |\xi|_{1,h}^2)\, ds\Bigg],
\end{aligned}
\tag{4.19}
$$

which, upon using Gronwall's lemma, gives

$$
\begin{aligned}
&\|\xi_t\| + |\xi|_{1,h} + \|\xi_t\|_{L_2(0,t;S_h)} \\
&\quad \le c(1+c^*)\exp(c(1+c^*)T)\Bigg[\|\xi_t(0)\| + \|\eta\|_{L_2(L_2)} \\
&\qquad\qquad + \|\eta_t\|_{L_2(L_2)} + \|\eta_{tt}\|_{L_2(L_2)} \\
&\qquad\qquad + \int_0^T \Bigg(\sum_K \|\eta\|_{-1/2,K}^2\Bigg)^{1/2} dt\Bigg].
\end{aligned}
\tag{4.20}
$$

Setting $t=0$ in (4.3), we obtain

$$(U_t(0) - u_t(0), V) = 0 \quad \forall V \in S_h.$$

Hence,

$$\|\xi_t(0)\|^2 = (\xi_t(0), \xi_t(0)) = -(\eta_t(0), \xi_t(0)) \le \|\eta_t(0)\|\,\|\xi_t(0)\|.$$

An application of Lemma 3.2 then yields

$$\|\xi_t(0)\| \le \|\eta_t(0)\| \le ch^2.$$

Now Lemmas 3.2–3.6 imply

$$\|\xi_t\|_{L_\infty(L_2)} + \|\xi\|_{L_\infty(S_h)} + \|\xi_t\|_{L_2(S_h)} \le c(1+c^*)\exp(c(1+c^*)T)h^2. \tag{4.21}$$

To complete our argument, we must show that for h sufficiently small, $\|U_t\|_{L_\infty(L_\infty)} \le 2\tilde{c} \le c^*$. We use the inverse inequality, (3.54), and (4.21) to see that

$$
\begin{aligned}
\|U_t\|_{L_\infty(L_\infty)} &\le \|\xi_t\|_{L_\infty(L_\infty)} + \|\tilde{u}_t\|_{L_\infty(L_\infty)} \le \tilde{c} + ch^{-1}\|\xi_t\|_{L_\infty(L_2)} \\
&\le \tilde{c} + c(1+c^*)\exp(c(1+c^*)T)h.
\end{aligned}
$$

Then clearly, if h is taken sufficiently small,

$$\|U_t\|_{L_\infty(L_\infty)} \le 2\tilde{c} \le c^*.$$

Hence the constant in (4.21) can be chosen independent of c^*. □

We are now in a position to prove the main results of this paper. First, by using Lemmas 3.2 and 3.6 and Theorem 4.1, we obtain immediately the following

Theorem 4.2. *Let* u *and* U *be the solutions of* (2.1) *and* (2.3), *respectively. Assume that* $u, u_{tt} \in L_\infty(H^2(\Omega)) \cap L_\infty(W^{1,\infty}(\Omega))$ *and that* $u_{tt} \in L_2(H^2(\Omega))$. *Then*

$$\|U-u\|_{L_\infty(L_2)}+\|U_t-u_t\|_{L_\infty(L_2)}+h[\|U-u\|_{L_\infty(S_h)}+\|U_t-u_t\|_{L_\infty(S_h)}] \le ch^2. \tag{4.22}$$

We now turn to the maximum-norm estimate for the error in the gradient.

Theorem 4.3. *In addition to the hypotheses of Theorem* 4.1, *assume that the domain* Ω *is decomposed into rectangular elements. If* $u \in L_\infty(W^{2,\infty}(\Omega))$, *then*

$$\|\nabla(U-u)\|_{L_\infty(L_\infty)} \le ch\log(1/h). \tag{4.23}$$

Proof. Applying Theorem 4.1 and the inverse inequality, we deduce that

$$\|\nabla\xi\|_{L_\infty(L_\infty)} \le ch^{-1}\|\xi\|_{L_\infty(S_h)} \le ch. \tag{4.24}$$

On the other hand, using Lemma 3.3 and the inverse inequality, we have

$$\begin{aligned}\|\nabla\eta\|_{L_\infty(L_\infty)} &\le \|\nabla(\tilde{u}-\Pi u)\|_{L_\infty(L_\infty)} + \|\nabla(\Pi u - u)\|_{L_\infty(L_\infty)} \\ &\le ch^{-1}[\|\eta\|_{L_\infty(L_\infty)} + \|u-\Pi u\|_{L_\infty(L_\infty)}] \\ &\quad + \|\nabla(\Pi u - u)\|_{L_\infty(L_\infty)} \\ &\le ch\log(1/h).\end{aligned} \tag{4.25}$$

Inequality (4.23) is now a direct consequence of (4.24)–(4.25). □

Finally, we derive a superconvergence order estimate for the gradient.

Theorem 4.4. *Under the hypotheses of Theorem* 4.1 *and Lemma* 3.4,

$$\left\{\frac{1}{N}\sum_{x_0\in G}|\nabla(U-u)(x_0)|^2\right\}^{1/2} \le ch^2 \quad \forall t \in I. \tag{4.26}$$

Proof. In view of Lemma 3.4, we only need to estimate $\xi = U - \tilde{u}$. By Theorem 4.1 and the inverse inequality,

$$\begin{aligned}\left\{\frac{1}{N}\sum_{x_0\in G}|\nabla\xi(x_0)|^2\right\}^{1/2} &\le c\left\{\frac{1}{N}\sum_K\|\nabla\xi\|^2_{L_\infty(K)}\right\}^{1/2} \\ &\le c\left\{\frac{1}{N}\sum_K h^{-2}\|\nabla\xi\|^2_{L_2(K)}\right\}^{1/2} \le c|\xi|_{1,h} \le ch^2. \quad \square\end{aligned}$$

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