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PLANA'S SUMMATION FORMULA FOR $\sum_{m=1,3,\dots}^{\infty} m^{-2}\sin(m\alpha)$, $m^{-3}\cos(m\alpha)$, $m^{-2}A^m$, $m^{-3}A^m$

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ABSTRACT. The four series

$$\sum_0^\infty \sin(2k+1)\alpha/(2k+1)^2, \quad \sum_0^\infty \cos(2k+1)\alpha/(2k+1)^3,$$

$$\sum_0^\infty A^{2k+1}/(2k+1)^2, \quad \text{and} \quad \sum_0^\infty A^{2k+1}/(2k+1)^3$$

are very slowly convergent for $0 \le \alpha \le \pi$ and as $A \to 1^-$. Direct summation involves thousands of terms to get the accuracy desired. Plana's summation formula along with Romberg's method of integration significantly and consistently improves the convergence and accuracy for the above series.

1. INTRODUCTION

The analysis of unilateral plate contact problems [3–5] encountered several very slowly convergent series which were not easily evaluated other than by direct summation. This is unsatisfactory from the viewpoint of computer time, aesthetics, and accuracy. The particular series encountered in [3–5] were

$$\text{(1a,b)} \qquad \sum_{m=1,3,\dots}^{\infty} \frac{\sin(m\alpha)}{m^2}, \quad \sum_{m=1,3,\dots}^{\infty} \frac{\cos(m\alpha)}{m^3} \qquad (0 \le \alpha \le \pi)$$

and

$$\text{(2a,b)} \qquad \sum_{m=1,3,\dots}^{\infty} \frac{A^m}{m^2}, \quad \sum_{m=1,3,\dots}^{\infty} \frac{A^m}{m^3} \qquad (0 < A \le 1).$$

The mathematical formulation of the unilateral contact problems stated in [3–5] used finite Fourier transforms to reduce four coupled series equations to two coupled singular integral equations. The known functions in these integral equations were expressed in terms of many infinite series. The physical problem involves a transition point past which a flexible plate is no longer in contact with a rigid knife edge support. In the vicinity of this transition point, therefore, as an artifact of a linearized plate theory, certain physical quantities exhibit a singular inverse square root behavior.

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The issue in the solution of the above integral equations was to sum the series, several of which were very slowly convergent (as noted above), very precisely in order to accurately identify and portray the singular behavior in question. If the setup of the known functions in the integral equations had to be done only once, crude summation would still not be too much of a drawback. However, unilateral contact problems are nonlinear and require iterative solutions. The CPU time then quickly becomes excessive if direct summation is used. The above series are summed in this paper using Plana's summation formula, which is stated below. Comparisons with several well-known acceleration schemes are presented at the end of the paper.

2. Plana's summation formula [12, 16]

Let f be a function for which the sum $\sum_m^\infty f(k)$ is of interest. The conditions imposed on f are that

I. $f(z)$ be analytic for $\operatorname{Re} z \geq m$;
II. $\lim_{y\to\pm\infty} e^{-2\pi|y|} f(x+iy) = 0$ holds uniformly for $m \leq x \leq n$, no matter how large n is;
III. $\lim_{x\to\infty} \int_{-\infty}^{\infty} e^{-2\pi|y|} |f(x+iy)|\, dy = 0$.

Theorem. *Let f satisfy the above conditions, and let either the series $\sum_m^\infty f(k)$ or the integral $\int_m^\infty f(x)\,dx$ be convergent. Then*

$$(3) \qquad \sum_m^\infty f(k) = \frac{1}{2} f(m) + \int_m^\infty f(x)\,dx - 2\int_0^\infty \frac{q(m,y)}{e^{2\pi y}-1}\,dy,$$

where

$$(4) \qquad q(m,y) = [f(m+iy) - f(m-iy)]/2i.$$

(If f is real when $y = 0$, then $q(x, y)$ is the imaginary part of $f(z)$.)

The development of this formula dates back to Plana in 1820 with subsequent works featuring such notable mathematicians as Abel, Cauchy, and Kronecker. The theory and history behind the formula are discussed in detail in Lindelöf [16].

The particular form of Plana's summation formula (P.S.F.) used in this paper is given by (3) with m equal to zero. The series in (1a,b) and (2a,b) converge very slowly if straight summation is used, especially as α tends to zero and as A tends to 1^-, respectively. In this paper the series are first converted into integral form using Plana's summation formula and then evaluated using the general version of Romberg's algorithm (see [13, p. 292]). The validity of this approach is demonstrated using the known series (equations 14.2.21 and

17.2.16, respectively, in Hansen [10])

$$\sum_{k=0}^{\infty} \frac{\sin(2k+1)\alpha}{(2k+1)^3} = \pi\alpha(\pi - |\alpha|)/8 \quad \text{for } -\pi \le \alpha \le \pi, \tag{1a$'$}$$

$$\sum_{k=0}^{\infty} \frac{\cos(2k+1)\alpha}{(2k+1)^2} = \pi(\pi - 2|\alpha|)/8 \quad \text{for } -\pi \le \alpha \le \pi. \tag{1b$'$}$$

Both sets of series in (1a,b) and (1a$'$,b$'$) are incorporated in Plana's summation formula by considering, for $n = 2$ and $n = 3$,

$$f_n(z) = \frac{e^{i(2z+1)\alpha}}{(2z+1)^n} \qquad (0 \le \alpha \le \pi), \tag{5}$$

which can be shown to satisfy conditions I–III of the theorem. Applying the theorem and equating real and imaginary parts in (3) for $m = 0$ gives

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\sin(2k+1)\alpha}{(2k+1)^n} = \frac{1}{2}\sin\alpha + \int_0^{\infty} \frac{\sin(2x+1)\alpha}{(2x+1)^n}\,dx \\ -2\int_0^{\infty} \frac{q_{s,n}(0,y)}{e^{2\pi y}-1}\,dy,\end{aligned} \tag{6a}$$

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\cos(2k+1)\alpha}{(2k+1)^n} = \frac{1}{2}\cos\alpha + \int_0^{\infty} \frac{\cos(2x+1)\alpha}{(2x+1)^n}\,dx \\ -2\int_0^{\infty} \frac{q_{c,n}(0,y)}{e^{2\pi y}-1}\,dy,\end{aligned} \tag{6b}$$

where

$$q_{s,n} = (C_n \sin\alpha \cosh 2\alpha y - S_n \cos\alpha \sinh 2\alpha y)/(1+4y^2)^n, \tag{7a}$$

$$q_{c,n} = (C_n \cos\alpha \cosh 2\alpha y + S_n \sin\alpha \sinh 2\alpha y)/(1+4y^2)^n, \tag{7b}$$

and in which

$$C_2 = -4y, \qquad S_2 = 4y^2 - 1, \tag{8a}$$

$$C_3 = 8y^3 - 6y, \qquad S_3 = 12y^2 - 1. \tag{8b}$$

The first integral in each of (6a) and (6b) may be written as

$$\begin{aligned}&\int_0^{\infty} \frac{\sin(2x+1)\alpha}{(2x+1)^n}\,dx \\ &= \begin{cases} [\sin\alpha - \alpha\,\mathrm{ci}(\alpha)]/2 & \text{for } n = 2, \\ [\sin\alpha + \alpha\cos\alpha + \alpha^2\mathrm{si}(\alpha)]/4 & \text{for } n = 3, \end{cases}\end{aligned} \tag{9a}$$

$$\begin{aligned}&\int_0^{\infty} \frac{\cos(2x+1)\alpha}{(2x+1)^n}\,dx \\ &= \begin{cases} [\cos\alpha + \alpha\,\mathrm{si}(\alpha)]/2 & \text{for } n = 2, \\ [\cos\alpha - \alpha\sin\alpha + \alpha^2\mathrm{ci}(\alpha)]/4 & \text{for } n = 3, \end{cases}\end{aligned} \tag{9b}$$

respectively, using integration by parts and the following definitions from §8.23 in [8]:

$$\mathrm{si}(\alpha) = -\pi/2 + \int_0^\alpha \frac{\sin t}{t}\, dt, \tag{10a}$$

$$\mathrm{ci}(\alpha) = \gamma + \log \alpha + \int_0^\alpha \frac{\cos t - 1}{t}\, dt. \tag{10b}$$

In (10b), γ is Euler's constant ($\gamma = 0.577215664901532\ldots$).

For computational convenience, the second integral in each of (6a) and (6b) may be expressed as follows:

$$\int_0^\infty \frac{q(0, y)}{e^{2\pi y} - 1}\, dy = \int_0^1 \frac{q(0, y)}{e^{2\pi y} - 1}\, dy + \int_0^1 \frac{q(0, 1/x)}{(e^{2\pi/x} - 1)} \frac{1}{x^2}\, dx. \tag{11}$$

The slowly convergent series in (2a,b) were not directly encountered in the analysis of unilateral contact problems [3–5] but did in fact cause the slow convergence of the following hyperbolic series:

$$\sum_{m=1,3,\ldots}^{\infty} \frac{\cosh(mz)}{m^2 \cosh(mb)}, \quad \sum_{m=1,3,\ldots}^{\infty} \frac{\sinh(mz)}{m^3 \cosh(mb)} \quad (0 \le z \le b). \tag{12a,b}$$

By letting $B = e^{-2b}$, $A = e^{-(b-|z|)}$, $G = e^{-(b+|z|)}$, and $H = e^{-(3b-|z|)}$, (12a,b) become

$$\sum_{m=1,3,\ldots}^{\infty} \frac{\cosh(mz)}{m^2 \cosh(mb)} = \sum_{m=1,3\ldots}^{\infty} \frac{A^m}{m^2} + \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^2} \left[\frac{G^m - H^m}{1 + B^m} \right], \tag{13a}$$

$$\sum_{m=1,3,\ldots}^{\infty} \frac{\sinh(mz)}{m^3 \cosh(mb)} = \mathrm{sgn}(z) \left\{ \sum_{m=1,3,\ldots}^{\infty} \frac{A^m}{m^3} - \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^3} \left[\frac{G^m + H^m}{1 + B^m} \right] \right\}. \tag{13b}$$

The problem is that the series in (13a,b) converge slowly when z approaches b. It is difficult to improve the convergence of the related series $\sum_{m=1,3,\ldots}^{\infty} A^m/m^3$ and $\sum_{m=1,3,\ldots}^{\infty} A^m/m^2$. For instance, when the variable A in the series $\sum_{m=1,3,\ldots}^{\infty} A^m/m^2$ tends to 1, by direct summation it would take as many as 500,000 terms to give results accurate to six decimal places.

By using Plana's summation formula, the slowly convergent series in (13a,b) may be expressed as

$$\sum_{k=0}^{\infty} \frac{A^{2k+1}}{(2k+1)^n} = \frac{1}{2} A + \int_0^\infty \frac{A^{2x+1}}{(2x+1)^n}\, dx - 2 \int_0^\infty \frac{q_n(0, y)}{e^{2\pi y} - 1}\, dy, \tag{14}$$

where

(15) $$q_n(0, y) = A[C_n \cos(2y \log A) - S_n \sin(2y \log A)]/(1 + 4y^2)^n,$$

in which C_n and S_n $(n = 2, 3)$ are defined in (8a,b).

The first integral in (14) can be transformed to the following finite integrals (the second integral can be modified as in (11)):

(16) $$\int_0^\infty \frac{A^{2x+1}}{(2x+1)^n} dx = \frac{1}{2} \int_0^1 v^{n-2} A^{1/v} dv.$$

3. Application of Plana's summation formula

The finite integrals remaining in (10), (11), and (16) can now be readily evaluated by repeated Simpson's rule and the more general Romberg's method, since in both methods the error of integration can be predetermined [2, 13]. Romberg's method is one of the most widely used methods, since, among other advantages, it gives a simple strategy for the automatic determination of a suitable step size. Simpson integration is a special case of Romberg integration.

Tables 1–3 present the accuracy and convergence for

$$\sum_{k=0}^{\infty} \frac{\sin(2k+1)\alpha}{(2k+1)^n} \qquad (n = 2, 3)$$

by direct summation and P.S.F. (ε_N denotes the difference between an exact result and either direct summation or P.S.F.; $N = (b-a)/2h$, where N is the number of divisions in Simpson's or Romberg's rule). When α equals zero, direct summation converges extremely slowly (for $n = 2$), yet P.S.F. performs extremely well, especially if Romberg's method is used. Tables 4–6 present the accuracy and convergence for

$$\sum_{k=0}^{\infty} \frac{\cos(2k+1)\alpha}{(2k+1)^n} \qquad (n = 2, 3)$$

by direct summation and P.S.F. For $\alpha = 0$ and $\alpha = \pi/3$, the sum in (1b) is known to be given by $\lambda(3) = 1.05179\ 97902\ 64645$ and $\frac{7}{18}\zeta(3) = 0.46746\ 65734\ 50953$, respectively (see [1, 23.2.18, 23.2.20, and Table 23.3] as well as [15, p. 146]).

Table 1

Direct summation for $\sum_0^\infty \sin(2k+1)\alpha/(2k+1)^n$

α	N	$\varepsilon_N : n = 2$	α	N	$\varepsilon_N : n = 3$ [eq. (1a′)]
$\pi/2$	100	1.24×10^{-5}	$\pi/4$	10000	6.76×10^{-14}
	1000	1.24×10^{-7}	$\pi/3$	10000	1.15×10^{-13}
	10000	1.25×10^{-9}	$\pi/2$	10000	4.01×10^{-15}

TABLE 2
Plana's summation formula and $\sum_0^\infty \sin(2k+1)\alpha/(2k+1)^n$

α	N	ε_N (Simpson's Rule)		α	N	ε_N (Romberg's Method)	
$\pi/2$		$n=2$	$n=3$ [Eq. (1a′)]	$\pi/2$		$n=2$	$n=3$ [eq. (1a′)]
	4	1.22×10^{-4}	5.64×10^{-4}		4	6.47×10^{-5}	2.35×10^{-4}
	16	3.96×10^{-7}	1.62×10^{-6}		16	1.04×10^{-8}	9.25×10^{-8}
	64	1.53×10^{-9}	6.25×10^{-9}		64	3.77×10^{-15}	6.21×10^{-14}

TABLE 3
$\alpha=\pi/4$ *and* $\alpha=\pi/6$ *for* $\sum_0^\infty \sin(2k+1)\alpha/(2k+1)^2$

α	$\pi/4$	$\pi/6$
$\sum_1^{100000}$	.75288 07524 87526	.61064 37294 63285
Simpson's Rule ($N=512$)	.75288 07525 05710	.61064 37294 51354
Romberg's Method ($N=64$)	.75288 07525 05896	.61064 37294 51477

TABLE 4
Direct summation for $\sum_0^\infty \cos(2k+1)\alpha/(2k+1)^n$

α	N	$\varepsilon_N : n=3$	α	N	$\varepsilon_N : n=3$	α	N	$\varepsilon_N : n=2$ [eq. (1b′)]
0	100	6.24×10^{-6}	$\pi/3$	100	6.27×10^{-8}	$\pi/4$	10000	2.44×10^{-13}
	1000	6.25×10^{-8}		1000	6.25×10^{-11}	$\pi/3$	10000	1.25×10^{-9}
	10000	6.26×10^{-10}		10000	4.99×10^{-15}	$\pi/2$	10000	1.37×10^{-15}

By making comparisons between Table 2 and Table 5, the integrals in equation (6a) are seen to converge much faster than those in equation (6b). Since crude summation for $\sum_0^\infty \sin(2k+1)\alpha/(2k+1)^2$ converges very slowly, the advantages of Plana's summation formula are obvious.

Tables 7–10 present the accuracy and convergence obtained using direct summation and P.S.F. for

$$\sum_0^\infty \frac{A^{2k+1}}{(2k+1)^n} \qquad (n=2,3).$$

For A equal to unity in (14), the sums are known [1] and are given by 1.23370 05501 36170 and 1.05179 97902 64645 for $n=2$ and 3, respectively. These specific check cases are used in Tables 7–10. Observe from Tables 7, 8 and 9, 10 that direct summation actually converges faster in (14) when A is not close to unity; when A is close to or equal to unity, P.S.F. (using Romberg's method of integration) is vastly preferable.

TABLE 5

Plana's summation formula and $\sum_0^\infty \cos(2k+1)\alpha/(2k+1)^n$

α	N	ε_N (Simpson's Rule)		α	N	ε_N (Simpson's Rule)	
0		$n=2$ [eq. (1b′)]	$n=3$	$\pi/3$		$n=2$ [eq. (1b′)]	$n=3$
	4	1.04×10^{-3}	3.59×10^{-4}		4	1.00×10^{-3}	3.59×10^{-4}
	16	2.59×10^{-6}	1.03×10^{-6}		16	2.45×10^{-6}	1.03×10^{-6}
	64	9.94×10^{-9}	3.98×10^{-9}		64	9.39×10^{-9}	3.99×10^{-9}
α	N	ε_N (Romberg's Method)		α	N	ε_N (Romberg's Method)	
0		$n=2$ eq. (1b′)]	$n=3$	$\pi/3$		$n=2$ [eq. (1b′)]	$n=3$
	4	2.15×10^{-4}	1.50×10^{-4}		4	1.92×10^{-4}	1.50×10^{-4}
	16	2.81×10^{-7}	5.89×10^{-8}		16	2.78×10^{-7}	5.89×10^{-8}
	64	1.27×10^{-13}	3.66×10^{-14}		64	1.24×10^{-13}	3.90×10^{-14}

TABLE 6

$\alpha=\pi/4$ *and* $\alpha=\pi/6$ *for* $\sum_0^\infty \cos(2k+1)\alpha/(2k+1)^3$

α	$\pi/4$	$\pi/6$
$\sum_1^{10000}$	.67767 73183 78142	.85739 98075 96532
Simpson's Rule ($N=512$)	.67767 73183 75865	.85739 98075 94300
Simpson's Rule ($N=2048$)	.67767 73183 78244	.85739 98075 96723
Romberg's Method ($N=128$)	.67767 73183 78214	.85739 98075 96690

TABLE 7

Direct summation for $\sum_0^\infty A^{2k+1}/(2k+1)^2$

A	N	ε_N	A	N	$\sum_1^N$
1	100	2.49×10^{-3}	$\pi/4$	10	.85738 78296 97219
	1000	2.49×10^{-4}		30	.85741 75390 62508
	10000	2.50×10^{-5}		50	.85741 75393 17405
				70	.85741 75393 17411

TABLE 8

Plana's summation formula and $\sum_0^\infty A^{2k+1}/(2k+1)^2$

A	N	ε_N (Romberg's Method)	A	N	Romberg's Method
1	4	1.49×10^{-4}	$\pi/4$	64	.85741 75532 06681
	16	5.88×10^{-8}		128	.85741 75389 31794
	64	3.66×10^{-14}		256	.85741 75393 22545
				512	.85741 75393 17399

TABLE 9

Direct summation for $\sum_0^\infty A^{2k+1}/(2k+1)^3$

A	N	ε_N	A	N	$\sum_1^N$
1	100	6.24×10^{-6}	$\pi/4$	10	.80651 09541 19534
	1000	6.25×10^{-8}		30	.80651 22514 82909
	10000	6.26×10^{-10}		50	.80651 22514 86918

TABLE 10

Plana's summation formula and $\sum_0^\infty A^{2k+1}/(2k+1)^3$

A	N	ε_N (Romberg's Method)	A	N	Romberg's Method
1	4	2.15×10^{-4}	$\pi/4$	64	.80651 22538 34243
	16	2.81×10^{-7}		128	.80651 22514 54534
	64	1.27×10^{-13}		256	.80651 22514 87145
				512	.80651 22514 86922

TABLE 11

Grossman's eq. (17) *vs P.S.F. for* $\sum_0^\infty \cos(2n+1)t/(2n+1)^3$

$t/(\pi/18)$	eq. (17)	P.S.F. (Romberg's Method), $N = 128$
0	1.0517 99790 26464	1.0517 99790 26465
1	1.0218 07415 95651	1.0218 07420 53666
2	.95298 32900 92100	.95298 35832 28726
3	.85739 64682 24771	.85739 98075 96690
4	.74160 16553 79955	.74162 04236 49969
5	.61030 31247 96820	.61037 47642 92090
6	.46725 24181 28560	.46746 65734 50957
7	.31562 49850 66774	.31616 60109 56054
8	.15821 34521 77232	.15942 24291 95318
9	-2.4613×10^{-3}	-1.3592×10^{-16}

4. OTHER CONVERGENCE ACCELERATION SCHEMES

A rough draft of the present paper attracted the attention of Glasser [7] and Grossman [9]. Grossman undertook to determine the asymptotic behavior of $\sum_{n=1,2,\ldots}^\infty \cos(tn)/n^r$ and $\sum_{n=1,2,\ldots}^\infty \sin(tn)/n^r$ as $t \to 0^+$ $(r > 0)$, using the Mellin transform and residue theory. The expression found for the latter series with $r = 3$ coincided with the known exact expression. The expression found by Grossman [9] for (1b), on the other hand, was for $t \to 0^+$:

$$\sum_{n=0}^\infty \frac{\cos(2n+1)t}{(2n+1)^3} \sim \frac{7}{8}\zeta(3) - \phi(t) + \phi(2t)/8, \tag{17}$$

where

$$\phi(t) = \frac{t^2}{2}\left[\log\left(\frac{1}{t}\right) + \frac{3}{2}\right] + \sum_{j=2}^\infty \frac{B_{2j-2}}{(2j-2)(2j)!} t^{2j}, \tag{18}$$

in which B_n denotes Bernoulli numbers [1]. The expression in (17) is exact for $t = 0$, accurate to eight decimal places for $t = \pi/18$, but accurate to only three decimal places for $t = 4\pi/9$ (see Table 11).

Glasser suggested that (1a) might be summed appropriately using results published in [6], i.e.,

$$\begin{aligned} \sum_0^\infty \frac{\sin(2k+1)\alpha}{(2k+1)^2} = \frac{1}{2}\sin\alpha \int_0^\infty \frac{x}{\cosh x - \cos\alpha}\,dx \\ - \frac{1}{8}\sin 2\alpha \int_0^\infty \frac{x}{\cosh x - \cos 2\alpha}\,dx. \end{aligned} \tag{19}$$

TABLE 12

Glasser's eq. (19) *vs P.S.F. for* $\sum_0^\infty \sin(2k+1)\alpha/(2k+1)^2$

$\alpha/(\pi/6)$	eq. (17), $N = 32$	P.S.F. (Romberg's Method), $N = 32$
1	$.610\overline{6}2\ 66370$	$.61064\ 3729\overline{4}$
2	$.7528\overline{8}\ 12103$	$.75288\ 0752\overline{5}$
3	$.91596\ 55\overline{9}10$	$.91596\ 5594\overline{1}$

TABLE 13

Levin's method and Aitken's algorithm for $\sum_0^\infty 1/(2k+1)^n$

ε_l (Levin's method)			ε_m (Aitken's algorithm)		
l	$n = 2$	$n = 3$	m	$n = 2$	$n = 3$
1	3.73×10^{-4}	2.49×10^{-5}	1	6.70×10^{-2}	3.76×10^{-3}
2	1.38×10^{-6}	2.49×10^{-7}	3	1.11×10^{-3}	1.51×10^{-3}
3	2.56×10^{-8}	3.62×10^{-9}	5	1.00×10^{-5}	1.05×10^{-5}
4	8.36×10^{-10}	4.67×10^{-11}	7	1.94×10^{-8}	2.48×10^{-7}
5	2.04×10^{-10}	2.69×10^{-11}	9	1.36×10^{-9}	2.64×10^{-9}
6	1.79×10^{-2}	3.19×10^{-4}	11	1.36×10^{-10}	3.00×10^{-11}
7	1.45×10^{-2}	2.13×10^{-4}	13	8.11×10^{-11}	2.47×10^{-9}
			15	2.95×10^{-8}	

TABLE 14

Aitken's algorithm vs P.S.F. for $\sum_0^\infty A^{2k+1}/(2k+1)^n$

	Aitken's algorithm	P.S.F. (Romberg's Method)
$A = \pi/4$, $n = 2$	.85741 75393 17400 ($m = 41$)	.85741 75393 17399 ($N = 512$)
$A = \pi/4$, $n = 3$	.80651 22514 86922 ($m = 38$)	.80651 22514 86922 ($N = 512$)

Though the range is infinite, the integrals converge exponentially. By using the Gauss-Laguerre rule, the results are presented in Table 12. The overbar denotes the accurate decimal place. Obviously, the accuracy and convergence are not good, especially for small α.

The work by Glasser [7] and Grossman [9] should be useful to those not requiring highly precise evaluations.

When $A = 1$, the series (2a,b) are logarithmic. It was suggested by a reviewer that Levin's methods might be useful in this case. There are several nonlinear transformations for improving convergence of sequences such as the t-transformation, e-transformation, and u-transformation [14]. Here, the u-transformation is chosen because the u-transformation is very efficient for the series such as $\sum_{m=1}^\infty 1/m^2$ [14]. Table 13 presents the accuracy and convergence found using the u-transformation. In Table 13, for l less than or equal to 5 (l denotes the number of terms summed), the results are good, but not as good as those of P.S.F. (see Tables 7, 8 and 9, 10). For example, for $l = 5$, $\varepsilon_l = 2.04 \times 10^{-10}$ (for $\sum_{k=0}^\infty 1/(2k+1)^2$) and $\varepsilon_l = 2.69 \times 10^{-11}$

(for $\sum_{k=0}^{\infty} 1/(2k+1)^3$) by Levin's method, but by P.S.F., $\varepsilon_N = 3.66 \times 10^{-14}$ and $\varepsilon_N = 1.27 \times 10^{-13}$, respectively. Disconcertingly, Levin's method is also not stable; notice in Table 13 that the results are getting worse for $l > 5$.

When A is different from 1, Aitken's algorithm was suggested. Tables 13 and 14 present the results found using Aitken's algorithm; m is the number of interpolated points in Aitken's algorithm. When A is equal to 1, the accuracy of Aitken's algorithm is not as good as that of P.S.F., but more stable than Levin's method. For A different from 1, the results are in good agreement with those of P.S.F. The convergence is slower than in the case of $A = 1$.

5. Conclusions

Plana's summation formula is presented here for the evaluation of the Fourier series

$$\sum_0^\infty \frac{\sin(2k+1)\alpha}{(2k+1)^2}, \qquad \sum_0^\infty \frac{\cos(2k+1)\alpha}{(2k+1)^3}$$

and the series

$$\sum_0^\infty \frac{A^{2k+1}}{(2k+1)^2}, \qquad \sum_0^\infty \frac{A^{2k+1}}{(2k+1)^3},$$

which are very slowly convergent for $0 \le \alpha \le \pi$ and as $A \to 1^-$, respectively. As shown by the tables, P.S.F. is generally superior for

$$\sum_0^\infty \cos(2k+1)\alpha/(2k+1)^3$$

and works very well for $\sum_0^\infty \sin(2k+1)\alpha/(2k+1)^2$. When A is close to unity, P.S.F. also gives consistent and satisfactory results for

$$\sum_0^\infty \frac{A^{2k+1}}{(2k+1)^2}, \qquad \sum_0^\infty \frac{A^{2k+1}}{(2k+1)^3}.$$

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