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OPTIMAL-ORDER NONNESTED MULTIGRID METHODS FOR SOLVING FINITE ELEMENT EQUATIONS II: ON NON-QUASI-UNIFORM MESHES

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ABSTRACT. Nonnested multigrid methods are proved to be optimal-order solvers for finite element equations arising from elliptic problems in the presence of singularities caused by re-entrant corners and abrupt changes in the boundary conditions, where the multilevel grids are appropriately refined near singularities and are not necessarily nested. Therefore, optimal and realistic finer grids (compared with nested local refinements) could be used because of the freedom in generating nonnested multilevel grids.

1. INTRODUCTION

Multigrid methods provide optimal-order solvers for linear systems of finite element equations arising from elliptic boundary value problems (cf. references in [7] and [5]). The convergence of multigrid methods was proved by many authors (cf., e.g., [8], [2] and [6]) under the assumptions of quasi-uniformity of grids and $H^{1+\alpha}$-regularity. However, quasi-uniform multigrid methods are not well suited for problems with singularities because finite elements do not capture singularities well on such meshes, and the convergence rate would deteriorate owing to singularities ($Cm^{-\alpha/2}$ for $H^{1+\alpha}$-regularity, cf. [2]).

Yserentant proved the convergence of multigrid methods on non-quasi-uniform grids (cf. [10]). However, it is not quite clear how to implement the methods proposed by Yserentant in practical situations. It is difficult to construct, say, four or five nested locally-refined grids which reasonably satisfy the mesh refinement condition proposed by Babuška, Kellogg and Pitkäranta in [1] (see (1.3a–b) below). Couplings of triangles, the minimal-angle condition (see (1.2) below) and the control of the growth of total grid points, give rise to many difficulties in local, nested grid refinements. To avoid these difficulties, it is natural to use nonnested multigrid methods (cf. [11] or [12]). The nonnestedness of multilevel grids is well illustrated by the example shown in Figure 1, where the triangulations on the triangle OAB are generated by simply refining each triangle on the coarser level into four subtriangles and then moving all grid points appropriately to the corner O. This paper is devoted to proving the

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convergence of the nonnested multigrid methods on such non-quasi-uniform meshes (satisfying (1.3a–b)).

Although a mesh could have a quite arbitrary relation with its lower level in nonnested multigrid methods, each mesh must be related to the coarser one in such a way that the intergrid transfer operators (I_k, depending only on the locations of the fine-grid points on the coarse-level triangles) can be easily generated by computer codes. Also, we should construct meshes such that more information can be passed by intergrid transfer operators. A subdividing-moving technique (described above) was used by Scott and the author for generating nonnested, tetrahedral meshes in their general 3-d nonnested multigrid code NMGTM (cf. [9]). Because of many variations in generating nonnested meshes, we leave discussions on grid generations and numerical experiments to [13]. For example, a mapping technique (maps nested, quasi-uniform meshes to nonnested, non-quasi-uniform ones) is also discussed in [13]. Nonnested multigrid methods on quasi-uniform meshes are also analyzed by Bramble, Pasciak and Xu within their framework in [3].

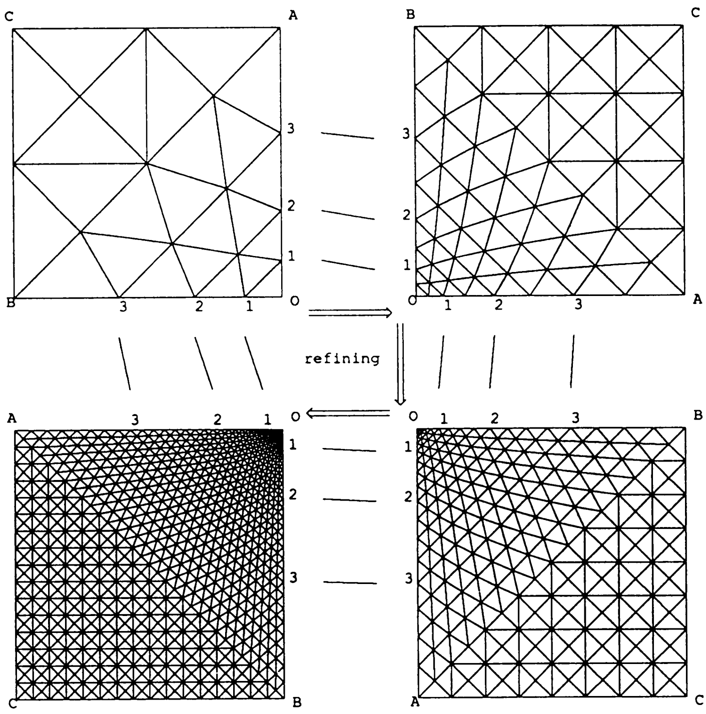

FIGURE 1

Nonnested refinements on $\triangle OAB$. The square $OACB$ (rotated above) is a part of the domain. $\triangle ABC$ has standard multigrid refinements, but a subdividing-moving method is applied in $\triangle OAB$. Here, the grid points on OA satisfy $|x_i - x_{i-1}| = 2^{-k}|x_i|^{1/2}$ (see (1.3a–3b)).

For simplicity, we consider the model problem

$$-\Delta u + u = f \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \Gamma_D,$$
$$\partial u/\partial n = 0 \quad \text{on} \quad \Gamma_N,$$

where Ω is a bounded polygonal domain in $\mathbf{R}^2$ with the boundary subdivided into two parts Γ_D and Γ_N. We assume Γ_D has a positive measure. Finite element approximation problems are defined as follows: Find $u_k \in V_k$ such that

$$(1.1) \qquad a(u_k, v) = F(v) \quad \forall v \in V_k, \ k = 1, 2, \dots,$$

where $a(u, v) = \int_\Omega (\partial_1 u \partial_1 v + \partial_2 u \partial_2 v + uv)\, dx$, $F(v) = \int_\Omega fv\, dx$, and V_k is the space of continuous, piecewise linear (on a grid $\mathcal{T}_k$) functions which vanish on Γ_D. Here, $\{\mathcal{T}_k\ ,\ k = 1, 2, \dots\}$ is a family of nondegenerate triangulations on Ω: there exists an $\alpha_0 > 0$ such that

$$(1.2) \qquad \min_{K \in \mathcal{T}_k} \min\{\text{ the three angles of } K\} > \alpha_0 \quad \forall k.$$

But we can have $\mathcal{T}_k \not\subset \mathcal{T}_{k+1}$. Let $H_D^1(\Omega)$ be the space of H^1 functions which vanish on Γ_D in the sense of traces; then $V_k \subset H_D^1(\Omega)$.

Now we state the essential assumption on the triangulations (cf. [1]): Let the points $x_1, \dots, x_n$ be the vertices of the domain Ω, including the points of abrupt changes from Dirichlet to natural boundary conditions. For each vertex x_i we define $\kappa_i = 1$ if the two sides having the endpoint x_i belong either both to Γ_D or both to Γ_N, and $\kappa_i = 1/2$ otherwise. Let $0 < \theta_i \le 2\pi$ be the interior angle of Ω at the point x_i; because of possible changes in the boundary conditions, the case $\theta_i = \pi$ is permitted. Let $\alpha_i = \min\{1, \kappa_i\pi/\theta_i\}$. Then $1/4 \le \alpha_i \le 1$ holds. We choose β_i with $1 - \alpha_i < \beta_i < 1$ and $\beta_i = 0$ if $\alpha_i = 1$. Let $\phi_\beta(x) = \prod_{i=1}^n |x - x_i|^{\beta_i}$ for $\beta = (\beta_1, \dots, \beta_n)$, where $|\cdot|$ denotes the Euclidean distance. We assume that the triangulations $\{\mathcal{T}_k\}$ satisfy the condition: Let $K \in \mathcal{T}_k$ be a (closed) triangle; then

$$(1.3a) \quad \underline{c} h_k \max_{x \in K} \phi_\beta(x) \le h_K \le \bar{c} h_k \min_{x \in K} \phi_\beta(x) \quad \text{if } x_i \notin K \text{ for all } i,$$

$$(1.3b) \quad \underline{c} h_k \max_{x \in K} \phi_\beta(x) \le h_K \le \bar{c} h_k \max_{x \in K} \phi_\beta(x) \quad \text{if } x_i \in K \text{ for some } i.$$

Here, $\underline{c}$ and $\bar{c}$ are some positive constants independent of K and k, and h_K is the diameter of K. The h_k in (1.3a–b) is a global "mesh size" of $\mathcal{T}_k$. For convenience, we simply let $h_k = 2^{-k}$ in this paper. It was shown by Babuška et al. in [1] that $\dim V_k \le Ch_k^{-2}$. We assume that $\dim V_k \simeq Ch_k^{-2} \simeq (\dim V_{k+1})/4$ $\forall k$. The meshes generated by the subdividing-moving method mentioned above will satisfy this condition naturally. Further, we assume that the meshes are locally comparable:

$$(1.4)\qquad \begin{aligned} &\sup_{K\in\mathcal{T}_k}\{\text{cardinality}(\{K'\in\mathcal{T}_{k\pm1}\mid K'\cap K\neq\varnothing\})\}\le\beta_0,\\ &\sup_{K\in\mathcal{T}_k}\{h_K/\min\{h_{K'}\mid K'\in\mathcal{T}_{k\pm1},\,K'\cap K\neq\varnothing\}\}\le\beta_0,\end{aligned}\qquad \forall k$$

for some $\beta_0>0$. In fact, (1.4) should be consequences of (1.2) and (1.3a–b).

We define some weighted Sobolev norms associated with the function $\phi_\beta(x)$ defined above. Let

$$(1.5)\qquad (v\,,\,w)_\beta=\int_\Omega\phi_\beta^2(x)v(x)\,w(x)\,dx$$

be a weighted (weight ϕ_β^2) L_2-inner product and $\|\cdot\|_{0,\beta}$ the norm induced by $(\cdot\,,\,\cdot)_\beta$. Let $H^{0,\beta}(\Omega)$ be the space of all Lebesgue measurable functions having finite $\|\cdot\|_{0,\beta}$-norm. Replacing the ϕ_β^2 by ϕ_β^{-2} in (1.5), we define $(v\,,\,w)_{-\beta}$ and $H^{0,-\beta}(\Omega)$ similarly. Let $\|\cdot\|_{2,\beta}^2=\|\cdot\|_{H^1(\Omega)}^2+\sum_{|\alpha|=2}\|D^\alpha\cdot\|_{0,\beta}^2$ and $H^{2,\beta}(\Omega)$ be the corresponding space. Here, $H^1(\Omega)$ is the usual Sobolev space. It was shown in [1] that $H^{2,\beta}(\Omega)\subset C(\Omega)$ and that $H^1(\Omega)\subset H^{0,\beta}(\Omega)$. Therefore, nodal-value interpolation operators are well defined for functions in $H^{2,\beta}(\Omega)$, and $V_k\subset H^{0,\beta}(\Omega)\ \forall k$.

By the Riesz representation theorem, the bilinear form $a(\cdot\,,\,\cdot)$ defines a linear, symmetric, positive definite operator A_k in $(V_k\,,\,(\cdot\,,\,\cdot)_{-\beta})$: $a(v\,,\,w)=(A_kv\,,\,w)_{-\beta}\ \ \forall v\,,\,w\in V_k$. We define a family of discrete norms on V_k:

$$(1.6)\qquad |||v|||_{s,k}^2=(v\,,\,A_k^sv)_{-\beta}\quad\forall v\in V_k\,,\ \ 0\le s\le 2.$$

We note that $|||\cdot|||_{0,k}=\|\cdot\|_{0,-\beta}$ and that $|||\cdot|||_{1,k}=\|\cdot\|_{H^1(\Omega)}$. For simplicity, we let $|||\cdot|||=|||\cdot|||_{1,k}$. Let $I_k\colon C(\Omega)\cap H_D^1(\Omega)\ \to\ V_k$ be the usual nodal-value interpolation operator associated with the triangulation $\mathcal{T}_k$.

We now define a nonsymmetric (cf. Definition 4.1 in §4), nonnested multigrid scheme (cf. [2]). The full multigrid method has two iterative processes. The overall process involves solving problems (1.1) sequentially for $k=1\,,\,2\,,\,\dots$ to get $\tilde u_k$. To solve (1.1) on the kth level, we interpolate the $\tilde u_{k-1}$ as an initial guess and then repeat several times the second, recursive process:

Definition 1.1. (The kth level nonnested multigrid scheme.)

(1) For $k=1$, (1.1) or the following (1.8) is solved by any method.

(2) For $k>1$, a final guess w_{m+1} will be generated from an initial guess w_0 as follows. First, m presmoothings will be performed:

$$(1.7)\qquad \begin{aligned}(w_l-w_{l-1}\,,\,v)_{0,-\beta}=\lambda_k^{-1}(F(v)-a(w_{l-1}\,,\,v))&\\ \forall v\in V_k\,,\quad l=1\,,\,2\,,\,\dots\,,\,m\,,&\end{aligned}$$

where λ_k is the maximal eigenvalue of A_k and $F(\cdot)$ is either the $F(\cdot)$ in (1.1) or the $\tilde F(\cdot)$ in (1.8) below. To define w_{m+1}, we need to construct a $(k-1)$st level residual problem: Find $\bar{q} \in V_{k-1}$ such that

$$a(\bar{q}, v) = F(I_k v) - a(w_m, I_k v) \overset{\text{def}}{=} \tilde{F}(v) \quad \forall v \in V_{k-1}. \tag{1.8}$$

Let $q \in V_{k-1}$ be the approximation of $\bar{q}$ obtained by applying p iterations of the $(k-1)$st level scheme to (1.8) starting with initial guess zero; then we let

$$w_{m+1} = w_m + I_k q. \tag{1.9}$$

If meshes are nested, the I_k in (1.8) and (1.9) is an identity. However, an intergrid transfer operator, "I_k", is necessary in computation, no matter whether the meshes are nested or not. If we remove the I_k in (1.8), $\bar{q}$ would be the $a(\cdot, \cdot)$-projection of an iterative error. Correspondingly, we define two operators Q_{k-1}, $P_{k-1} : H^1_D(\Omega) \to V_{k-1}$ by

$$\begin{aligned} a(Q_{k-1}v, w) &= a(v, I_k w), \\ a(P_{k-1}v, w) &= a(v, w), \end{aligned} \quad \forall w \in V_{k-1}. \tag{1.10}$$

2. Preliminary results

In this section, we will prove some lemmas concerning I_k and introduce some results of Babuška, Kellogg and Pitkäranta in [1]. In the rest of this paper, "$\lesssim$" stands for "$\le C$" with the constant C independent of the functions being estimated and the level number k.

Lemma 2.1. *Let* (1.2) *and* (1.4) *hold*; *then*

$$|||w - I_k v||| \lesssim |||w - v||| \quad \forall v \in V_{k-1}, \ \forall w \in V_k.$$

Proof. See Proposition 2.1 of [12]. The equivalence of $|\cdot|_{H^1(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)} = |||\cdot|||$ in $H^1_D(\Omega)$ follows from a general Poincaré inequality in [4, Theorem 1.2.1]. □

Lemma 2.2. *Let* (1.2)–(1.4) *hold*; *then*

$$\|v - I_k v\|_{0,-\beta} \lesssim h_k |||v||| \quad \forall v \in V_{k-1}.$$

To prove the lemma, we use a lemma in [1].

Lemma 2.3 (Lemma 4.3 in [1]). *Let K be a triangle with one vertex at 0, $\alpha \neq 0$, and let u be defined on K with weak first derivatives which satisfy $\int_K |x|^\alpha |\nabla u|^2 dx < \infty$. Then there is a constant A depending on u and a constant $C(\alpha, \alpha_K)$, independent of u but depending on α and the minimal angle α_K of K, such that*

$$\int_K |x|^{\alpha-2} |u - A|^2 \le C \int_K |x|^\alpha |\nabla u|^2 dx,$$

and

$$|A| \le C \int_T |x|^\alpha |\nabla u| dx. \quad \square$$

Proof of Lemma 2.2. For any $K \in \mathcal{T}_k$, let $S_K = \bigcup\{K' \in \mathcal{T}_{k-1} \mid K' \cap K \neq \emptyset\}$; then $K \subset S_K$. Let $v \in V_{k-1}$; then $v - I_k v$ is piecewise linear on K and vanishes at the three vertices of K. We consider two cases: $x_i \notin K$ $\forall i$ or some vertex x_i of Ω is a vertex K. In the first case, we use (1.3a) and (1.4) to derive

$$\begin{aligned}\int_K \phi_\beta^{-2}(v - I_k v)^2\,dx &\le \bar{c}^2 h_k^2 h_K^{-2} \int_K \left(h_K \max_{x\in K}\{2|\nabla v|\}\right)^2 dx \\ &= 4\bar{c}^2 h_k^2 \int_K \max_{x\in S_K} |\nabla v|^2\,dx \lesssim h_k^2 \int_{S_K} |\nabla v|^2\,dx.\end{aligned} \tag{2.1}$$

If $x_i \in K$, without loss of generality, we may assume $x_i = 0$. Applying Lemma 2.3 and the Schwarz inequality, we obtain

$$\begin{aligned}\int_K \phi_\beta^{-2}|v - I_k v|^2\,dx &\lesssim \int_K |x|^{-2\beta_i}|v - I_k v|^2\,dx \\ &\lesssim 2\int_K |x|^{-2\beta_i}|(v - I_k v) - A|^2\,dx + 2A^2 \int_K |x|^{-2\beta_i}\,dx \\ &\lesssim \int_K |x|^{2-2\beta_i}|\nabla(v - I_k v)|^2\,dx \\ &\quad + \left(\int_K |x|^{2-2\beta_i}|\nabla(v - I_k v)|\,dx\right)^2 \int_K \phi_\beta^{-2}\,dx \\ &\lesssim \int_K |x|^{2-2\beta_i}|\nabla(v - I_k v)|^2\,dx \left(1 + \int_K |x|^{2-2\beta_i}\,dx \int_\Omega \phi_\beta^{-2}\,dx\right).\end{aligned}$$

Here, we used the fact that $|x|^{-2\beta} \sim \phi_{\beta_i}^{-2}$ on K. By (1.3b), it follows that $h_K^2 \le \bar{c}^2 h_k^2 \max\{\phi_\beta^2\} \lesssim h_k^2|x|^{2\beta_i} \lesssim h_k^2 h_K^{2\beta_i}$. Therefore,

$$\begin{aligned}\int_K \phi_\beta^{-2}|v - I_k v|^2\,dx &\lesssim h_K^{2-2\beta_i} \int_K |\nabla(v - I_k v)|^2\,dx \\ &\lesssim h_k^2 \int_K \max_{x\in K}\{4|\nabla v|^2\}\,dx \lesssim h_k^2 \int_{S_K} |\nabla v|^2\,dx.\end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we conclude, by (1.4), that

$$\begin{aligned}\|v - I_k v\|_{0,-\beta}^2 &= \sum_{K\in\mathcal{T}_k} \int_K \phi_\beta^{-2}|v - I_k v|^2\,dx \lesssim h_k^2 \sum_{K\in\mathcal{T}_k} \int_{S_K} |\nabla v|^2\,dx \\ &\lesssim h_k^2 \sum_{K'\in\mathcal{T}_{k-1}} \int_{K'} |\nabla v|^2\,dx = C h_k^2 |v|_{H^1(\Omega)}^2 \lesssim h_k^2 |||v|||^2. \quad \square\end{aligned}$$

Theorem 2.4 (Theorem 3.2 in [1]). *Let* $f \in H^{0,\beta}(\Omega)$ *and* $u \in H_D^1(\Omega)$ *satisfy* $a(u, v) = F(v)$ $\forall v \in H_D^1(\Omega)$. *Then* $u \in H^{2,\beta}(\Omega)$ *and*

$$\|u\|_{2,\beta} \le C\|f\|_{0,\beta}. \quad \square$$

Lemma 2.5 (Lemma 4.5 in [1]). *Let* (1.2) *and* (1.3a–b) *hold*; *then*

$$\|u - I_k u\|_{H^1(\Omega)} \le C h_k \|u\|_{2,\beta} \quad \forall u \in H^{2,\beta}(\Omega) \cap H^1_D(\Omega). \quad \square$$

Theorem 2.6 (Theorem 5.2 in [1]). *Let* (1.2) *and* (1.3a–b) *hold*; *then*

$$\|u - P_k u\|_{0,-\beta} \le C h_k \|u - P_k u\|_{H^1(\Omega)} \quad \forall u \in H^1_D(\Omega). \quad \square$$

3. Optimal computational order of W-cycle methods

In this section, we will show some lemmas concerning coarse-level corrections and coarse-level projections. We then prove an inverse inequality. Finally, the two-level and multilevel W-cycle nonnested multigrid methods will be shown to have a constant rate of convergence. The proof given here follows a route of Bank and Dupont in [2] and is almost identical to a proof given by the author in [12] for nonnested multigrid methods on quasi-uniform meshes.

Lemma 3.1. *Let* (1.2)–(1.4) *hold*; *then*

$$|||v - P_{k-1}v||| \lesssim h_k |||v|||_{2,k} \quad \forall v \in V_k.$$

Proof. For any $v \in V_k$, let $\tilde{v} = P_{k-1}v$. By the Schwarz inequality, we have

$$|||v - \tilde{v}|||^2 = a(v, v - \tilde{v}) = a(v, v - P_k\tilde{v}) \le |||v|||_{2,k} |||v - P_k\tilde{v}|||_{0,k}. \tag{3.1}$$

We then use a duality argument to estimate the second term: Let $u \in H^1_D(\Omega)$ solve

$$a(u, w) = (\phi_\beta^{-2}(v - P_k\tilde{v}), w) \quad \forall w \in H^1_D(\Omega).$$

By Theorem 2.4, $u \in H^{2,\beta}(\Omega)$ and $\|u\|_{2,\beta} \lesssim \|\phi_\beta^{-2}(v - P_k\tilde{v})\|_{0,\beta} = \|v - P_k\tilde{v}\|_{0,-\beta}$. Therefore, by Lemma 2.5, we can derive

$$\begin{aligned} |||v - P_k\tilde{v}|||^2_{0,k} &= (v - P_k\tilde{v}, v - P_k\tilde{v})_{-\beta} = a(u, v - P_k\tilde{v}) \\ &= a(u, v - \tilde{v}) + a(u, \tilde{v} - P_k\tilde{v}) \\ &\le |||u - I_{k-1}u|||\,|||v - \tilde{v}||| + |||u - I_k u|||\,|||\tilde{v} - P_k\tilde{v}||| \\ &\lesssim \|u\|_{2,\beta}(h_{k-1}|||v - \tilde{v}||| + h_k|||\tilde{v} - P_k\tilde{v}|||) \\ &\lesssim h_k\|u\|_{2,\beta}(3|||v - \tilde{v}||| + |||P_k(v - \tilde{v})|||) \\ &\lesssim h_k|||v - P_k\tilde{v}|||_{0,k}|||v - \tilde{v}|||. \end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), the lemma is proved. $\square$

Lemma 3.2. *Let* (1.2)–(1.4) *hold*; *then*

$$|||P_{k-1}v - Q_{k-1}v||| \lesssim h_k |||v|||_{2,k} \quad \forall v \in V_k.$$

Proof. For any $v \in V_k$, by (1.10), it follows that

$$a(P_{k-1}v - Q_{k-1}v, w) = a(v, w) - a(v, I_k w) \quad \forall w \in V_{k-1}.$$

By Lemma 2.2 and Theorem 2.6, we can complete the proof as follows:

$$\begin{aligned} |||P_{k-1}v - Q_{k-1}v||| &= \sup_{w \in V_{k-1}, |||w|||=1} a(v, w - I_k w) \\ &= \sup_w a(v, P_k w - I_k w) \le \sup_w |||v|||_{2,k} |||P_k w - I_k w|||_{0,k} \qquad (3.3) \\ &\le |||v|||_{2,k} \sup_w (\|w - I_k w\|_{0,\beta} + \|w - P_k w\|_{0,\beta}) \lesssim h_k |||v|||_{2,k}. \quad \square \end{aligned}$$

We remark that the P_k in (3.1) and (3.3) are introduced only for the purpose of applying the Schwarz inequality.

Lemma 3.3. *Let* (1.2)–(1.3b) *hold; then*

$$|||v||| \lesssim h_k^{-1} |||v|||_{0,k} \quad \forall v \in V_k. \tag{3.4}$$

Proof. Let $v \in V_k$ and $K \in \mathcal{T}_k$; then v is linear on K. By (1.2)–(1.3b), we get

$$\int_K |\nabla v|^2 \, dx \lesssim \max_{x \in K} v^2(x) = C \int_K h_K^{-2} v^2 \, dx \lesssim h_k^{-2} \int_K \phi_\beta^{-2}(x) v^2 \, dx.$$

Summing over all triangles in $\mathcal{T}_k$, (3.4) is proved. $\square$

Theorem 3.4 (Two-level methods). *Let* (1.2)–(1.4) *hold and* $q = \bar{q}$ *in* (1.9). *For any* $0 < \gamma < 1$, *there exists an integer* m *independent of the level number* k *such that*

$$|||u_k - w_{m+1}||| \le \gamma |||u_k - w_0|||,$$

where u_k, w_i, q *and* $\bar{q}$ *are defined in* (1.1) *and* (1.7)–(1.9).

Proof. Let the iterative errors be denoted by $e_l = u_k - w_l$ for $0 \le l \le m+1$. Our goal is to estimate $e_{m+1} = e_m - I_k \bar{q}$. Noting that $\bar{q} = Q_{k-1} e_m$, we can apply Lemmas 2.1, 3.1 and 3.2 sequentially to get

$$|||e_{m+1}||| \lesssim |||e_m - \bar{q}||| \lesssim |||e_m - P_{k-1}e_m||| + |||P_{k-1}e_m - \bar{q}||| \lesssim h^k |||e_m|||_{2,k}. \tag{3.5}$$

By (3.4), it follows that the maximal eigenvalue of A_k, λ_k, is bounded by Ch_k^{-2}. Therefore, the following estimates for the smoothing (1.7) hold (see, e.g., [2]),

$$|||e_m||| \le |||e_0||| \quad \text{and} \quad |||e_m|||_{2,k} \lesssim h_k^{-1} m^{-1/2} |||e_0|||. \tag{3.6}$$

We conclude that $|||e_{m+1}||| \le Cm^{-1/2}|||e_0|||$. Let $m \ge C^2/\gamma^2$; then the theorem is proved. $\square$

It is standard to derive the constant rate of convergence of W-cycle methods from the constant rate of convergence of two-level methods.

Theorem 3.5 (W-cycle methods). *Let* (1.2)–(1.4) *hold. For any* $0 < \gamma < 1$, *there exists a* $\hat{\gamma} \leq \gamma$ *and an integer* m, *all independent of the level number* k, *such that, if*

$$|||q - \bar{q}||| \leq \hat{\gamma}^p |||\bar{q}||| \tag{3.7}$$

for some $p > 1$, *then* $|||u_k - w_{m+1}||| \leq \hat{\gamma}|||u_k - w_0|||$.

Proof. By the induction hypothesis (3.7) and by (3.5), it follows that

$$\begin{aligned}|||e_{m+1}||| &\lesssim |||e_m - \bar{q}||| + |||\bar{q} - q||| \lesssim (1 + \hat{\gamma}^p)|||e_m - \bar{q}||| + \hat{\gamma}^p |||e_m||| \\ &\lesssim h^k |||e_m|||_{2,k} + \hat{\gamma}^p |||e_m|||.\end{aligned}$$

By (3.6), we get $|||e_{m+1}||| \leq (Cm^{-1/2} + C\hat{\gamma}^p)|||e_0|||$. To complete the proof, we first choose $\hat{\gamma}$ small enough such that $C\hat{\gamma}^p \leq \hat{\gamma}/2$, and then choose m large enough such that $Cm^{-1/2} \leq \hat{\gamma}/2$. □

In practice, we replace the $(\cdot, \cdot)_{-\beta}$ in the presmoothing (1.7) by some simpler inner product $b_k(\cdot, \cdot)$, which induces a norm equivalent to the $H^{0,-\beta}$-norm (cf. [2]). In [10], Yserentant gave a good $b_k(\cdot, \cdot)$:

$$b_k(u, v) = h_k^2 \sum_{K \in \mathcal{T}_k} \sum_{n_i \in \{\text{ three vertices of } K\}} u(n_i)v(n_i) \quad \forall u, v \in V_k.$$

After such a modification of (1.7), Theorem 3.4 and 3.5 remain valid. It is then standard to show the linear order of computation of the W-cycle methods (cf., e.g., [2]).

4. Convergence of V-cycle methods

In this section, we will define symmetric nonnested multigrid methods first and then study the rate of convergence of the (variable) V-cycle, symmetric, nonnested multigrid methods.

Definition 4.1 (The kth level symmetric nonnested multigrid scheme).

(1) For $k = 1$, (1.1) or (1.8) is solved exactly or by doing m_1 smoothings of (1.7).

(2) For $k > 1$, w_{2m_k+1} will be generated from w_0 as follows:

(2-a) m_k presmoothings of (1.7) are performed to generate w_{m_k};

(2-b) w_{m_k} is corrected by $I_k q$ (see (1.8)–(1.9)) to generate w_{m_k+1};

(2-c) m_k postsmoothings of (1.7) are performed to generate w_{2m_k+1}.

We note that the number of presmoothings and postsmoothings, m_k, may depend on the level number k. By (1.10), $I_k Q_{k-1}$ is a selfadjoint operator in $(V_k, a(\cdot, \cdot))$:

$$a(I_k Q_{k-1} v, v) = a(Q_{k-1} v, Q_{k-1} v) = a(v, I_k Q_{k-1} v) \quad \forall v \in V_k. \tag{4.1}$$

We can combine Lemmas 2.1 and 3.1–3.3 to get

$$\begin{aligned} |||v - I_k Q_{k-1} v||| &\lesssim |||v - Q_{k-1} v||| \\ &\lesssim |||v - P_{k-1} v||| + |||P_{k-1} v - Q_{k-1} v||| \\ &\lesssim h_k |||v|||_{2,k} = c_0 \lambda_k^{-1/2} |||v|||_{2,k} \quad \forall v \in V_k. \end{aligned} \tag{4.2}$$

Let $\rho(B)$ be the spectral radius of an operator B. In contrast to nested multigrid methods, (i) $I_k Q_{k-1}$ may not be an $a(\cdot, \cdot)$-projection operator; (ii) $\rho(I - I_k Q_{k-1}) \le 1$ may not hold; and (iii)

$$\rho(I_k Q_{k-1}) \le 1 \tag{4.3}$$

may fail. In particular, violation of (4.3) means $(I - I_k Q_{k-1})$ is not a non-negative operator. To study such operators, we need the following lemma. By expanding functions in V_k as Fourier series of the eigenfunctions of A_k, it is straightforward to verify the next lemma.

Lemma 4.2. *Let* $B: V_k \to V_k$ *be a selfadjoint operator in* $(V_k, a(\cdot, \cdot))$; *then the following are equivalent*:

(1) $|a(Bv, v)| \le \gamma a(v, v) \quad \forall v \in V_k$;
(2) $|||Bv||| \le \gamma |||v||| \quad \forall v \in V_k$;
(3) $\rho(B) \le \gamma$. □

Let B_k be the V-cycle ($p = 1$ in Definition 1.1) multigrid reducing operator associated with the iteration defined in Definition 4.1; i.e., after one cycle of the iteration, the iterative error changes from e to $B_k e$. Then

$$B_k = (I - \lambda_k^{-1} A_k)^{m_k} (I - (I_k - I_k B_{k-1}) Q_{k-1})(I - \lambda_k^{-1} A_k)^{m_k}, \quad k = 2, 3, \dots. \tag{4.4}$$

By (4.1), it follows that all B_k are, inductively, selfadjoint with respect to $a(\cdot, \cdot)$.

Lemma 4.3. *If* $\rho(B_{k-1}) \le \gamma \le 1$, *then* $\rho(B_k) \le \gamma + c_0/\sqrt{2m_k}$.

Proof. Let $v = \sum c_i \phi_{k,i} \in V_k$, where $\phi_{k,i}$ is an eigenfunction of A_k associated with $\lambda_{k,i}$. Let $\tilde{v} = (I - A_k/\lambda_k)^{m_k} v$. By Lemma 4.2, (4.1) and (4.4), we have

$$\begin{aligned} |a(B_k v, v)| &\le |a(\tilde{v} - I_k Q_{k-1} \tilde{v}, \tilde{v})| + |a(B_{k-1} Q_{k-1} \tilde{v}, Q_{k-1} \tilde{v})| \\ &\le |a(\tilde{v} - I_k Q_{k-1} \tilde{v}, \tilde{v})| + \gamma a(I_k Q_{k-1} \tilde{v}, \tilde{v}) \\ &\le (1 + \gamma) |a(\tilde{v} - I_k Q_{k-1} \tilde{v}, \tilde{v})| + \gamma a(\tilde{v}, \tilde{v}) \\ &\le (1 + \gamma) |||\tilde{v} - I_k Q_{k-1} \tilde{v}||| \, |||\tilde{v}||| + \gamma |||\tilde{v}|||^2. \end{aligned}$$

By (4.2), the Schwarz inequality and $\gamma \le 1$, it follows that

$$\begin{aligned}
|a(B_k v, v)| &\le (1+\gamma)c_0\lambda_k^{-1/2}|||\tilde{v}|||_{2,k}|||\tilde{v}||| + \gamma|||\tilde{v}|||^2 \\
&\le (1+\gamma)c_0\left(\frac{\sqrt{2m_k}|||\tilde{v}|||_{2,k}^2}{2\lambda_k} + \frac{|||\tilde{v}|||^2}{2\sqrt{2m_k}}\right) + \gamma|||\tilde{v}|||^2 \\
&= \sum\left(\frac{(1+\gamma)c_0}{2\sqrt{2m_k}}\left(2m_k\frac{\lambda_{k,i}}{\lambda_k}+1\right)+\gamma\right)\left(1-\frac{\lambda_{k,i}}{\lambda_k}\right)^{2m_k} c_i^2\lambda_{k,i} \\
&\le \left(\frac{(1+\gamma)c_0}{2\sqrt{2m_k}}+\gamma\right)\sup_{0\le x\le 1}(2m_k x+1)(1-x)^{2m_k}\sum c_i^2\lambda_{k,i} \\
&\le \left(c_0\Big/\sqrt{2m_k}+\gamma\right)|||v|||^2. \quad \square
\end{aligned}$$

By Lemma 4.3, we can derive trivially the constant rate of convergence of symmetric W-cycle methods (see Theorem 3.5). The following two theorems, too, are direct corollaries of Lemma 4.3. However, we cannot show the optimal order of computations of V-cycle methods from either one. With additional assumptions on the multiple meshes (e.g., assuming (4.3)), it is possible to obtain somewhat better results (cf. [3]). But in view of a numerical example in [11], the number of smoothings, m_k, has to be sufficiently large, depending on couplings of meshes.

Theorem 4.4 (V-cycle methods). *Let* $m_j = m \ge c_0^2/2$ *for* $j = 2, 3, \ldots$; *then*

$$\rho(B_k) \le kc_0\Big/\sqrt{2m}, \qquad 2 \le k \le (\sqrt{2m}/c_0) + 1,$$

provided that $\rho(B_1) \le c_0\Big/\sqrt{2m}$. $\square$

Theorem 4.5 (Variable V-cycle methods). *For any* $0 < \gamma < 1$, *let* $m_j = 2^{2j-1}c_0^2/\gamma^2$ *for* $j = 2, 3, \ldots$; *then*

$$\rho(B_k) \le \gamma, \qquad k = 2, 3, \ldots,$$

provided $\rho(B_1) \le \gamma/2$. $\square$

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