SUMS OF CUBES IN POLYNOMIAL RINGS

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Abstract. For any associative ring $A$ with 1 of prime characteristic $\neq 0$, 2, 3, every element of $A$ is the sum of three cubes in $A$.

For any ring $A$, let $w_3(A)$ denote the least integer $s \geq 0$ such that every sum of cubes in $A$ is a sum of $s$ cubes in $A$. If no such $s$ exists, $w_3(A) = \infty$ by definition.

For example, when $A = \mathbb{Z}$, the integers, it is known [2] that $4 \leq w_3(\mathbb{Z}) \leq 5$.

In this paper, we study $w_3(A)$ for $A = F[x]$, the polynomial ring in one variable $x$ with coefficients in a field $F$. It is easy to see [4] that every polynomial in $F[x]$ is a sum of cubes, if and only if $\text{char}(F) \neq 3$ and $\text{card}(F) \neq 2, 4$. If this is the case, $w_3(F[x])$ coincides with the least $s$ such that $x$ is the sum of $s$ cubes in $F[x]$. Moreover, in this case, every element of any associative $F$-algebra $A$ with 1 is the sum of $w_3(F[x])$ cubes in $A$; in particular, $w_3(A) \leq w_3(F[x])$.

The formula

$$(x + 1)^3 - 2x^3 + (x - 1)^3 = 6x$$

shows that $w_3(A) \leq 4$ for any associative ring $A = 6A$ with 1. In particular, $w_3(F[x]) \leq 4$ for any field $F$ with $\text{char}(F) \neq 2, 3$. By [4], $w_3(F[x]) \leq 4$ also in the case when $\text{char}(F) = 2$ and $\text{card}(F) \neq 2, 4$. When $\text{card}(F) = 2, 4$, formulas on p. 63 of [3] show that $w_3(F[x]) \leq 5$. These formulas together with the formulas

$$(xy + s)^3 + (xy + s + 1)^3 + (x + sy^2)^3 + (x + (1 + s)y^2)^3 = x(y + y^4),$$

$$(x^2y^2 + x(y + y^4) + 1 + s + y + y^8 + y^{10})^3$$

$$+ (x^2(1 + y^2) + x(y + y^4) + s + y + y^2 + y^{10})^3$$

$$+ (x^2y^2 + x(y + y^4) + s + y + y^4 + y^8 + y^{10})^3$$

$$+ (x^2(1 + y^2) + x(y + y^4) + s + y + y^2 + y^4 + y^{10})^3$$

$$= x(y + y^4) + y + y^2,$$

where $2 = 0$ and $s^2 + s + 1 = 0$, show that $w_3(F[x]) \leq 4$ in the case when $\text{card}(F) = 4$.

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On the other hand, it is clear that \( w_3(F[x]) = 1 \) when \( \text{char}(F) = 3 \), and that \( w_3(F[x]) \geq 3 \) when \( \text{char}(F) \neq 3 \).

Our first easy result (which was obtained by the author in July of 1987 and was known to J.-P. Serre since April of 1982) is the following

**Lemma 1.** If \( \text{char}(F) \neq 3 \) and there are nonzero \( \alpha, \beta, \gamma \in F \) such that \( \alpha^3 + \beta^3 + \gamma^3 = 0 \), then \( w_3(F[x]) = 3 \).

**Proof.** The formula

\[
(\alpha x + \beta^2)^3 + (\beta x - \alpha^2)^3 + (\gamma x)^3 = \beta^6 - \alpha^6 - 3\alpha\beta\gamma^3 x,
\]

where \( x \) can be replaced by an arbitrary polynomial in \( F[x] \), shows that \( w_3(F[x]) \leq 3 \). \( \square \)

**Corollary 2.** If \( \text{char}(F) \neq 0, 3 \) and \( \text{card}(F) \neq 2, 4, 7, 13, 16 \), then \( w_3(F[x]) = 3 \).

By more complicated computations, we will prove the following

**Proposition 3.** If \( \text{char}(F) \neq 3 \), then the condition of Lemma 1 is equivalent to the following: \( x \) is the sum of cubes of three polynomials in \( F[x] \) of degree \( \leq 4 \).

J.-P. Serre knew this for polynomials of degree \( \leq 2 \) since April of 1982. The case of degree \( \leq 3 \) was done by August 1987 independently by the author and David Hayes. In his letter of August 11, 1987, to Serre, Hayes wrote that the case of degree \( \leq 4 \) defeated him. The author obtained Proposition 3 with \( 2F \neq F \) in September of 1987. He thanks Serre for providing copies of relevant letters and useful suggestions.

This result leads one to wonder whether the converse of Lemma 1 is true. However J.-P. Serre wrote to the author on September 9, 1987, that he did not dare to conjecture anything himself even for \( F = \mathbb{Q} \), the rational numbers, or \( F = \mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/13\mathbb{Z} \).

Computations with polynomials of degree \( \geq 5 \) are very complicated, and it is only after many hours of computations with computers, that the author obtained the following result.

**Theorem 4.** If \( \text{char}(F) \neq 0, 3 \) and \( \text{card}(F) \neq 2, 4, 16 \), then \( w_3(F[x]) = 3 \). So every element of every associative \( F \)-algebra \( A \) is the sum of three cubes in \( A \).

**Proof.** By Corollary 2, we have only the cases \( \text{char}(F) = 7, 13 \) to deal with. Modulo 13, we have

\[
(-4x + 5x^2 + 6x^3 + x^4)^3 + (1 + 4x + x^2 - 5x^3 - 3x^4 + 5x^5)^3
+ (-1 + 5x - 5x^2 + 5x^3 + 3x^4 - 5x^5)^3 = x.
\]

Modulo 7, we have

\[
(x - 3x^3 - x^5)^3 + (-1 - x - x^3 + 3x^4 - x^6)^3 + (1 - x - x^3 - 3x^4 + x^6)^3 = x. \quad \square
\]
**Remarks.** The cases $F = \mathbb{Q}$ and card($F$) = 16 remain unresolved. The above equalities were found using *Mathematica* on Macintosh IIx. We work in one of the following cases: $F = \mathbb{Q}$, $\mathbb{Z}/7\mathbb{Z}$, or $\mathbb{Z}/13\mathbb{Z}$. In degree 5, we proceed as follows. First, we set $d = a^3 + b^3 + c^3$ with

$$
a = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5,
\quad b = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5,
\quad c = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5.
$$

Note that $d = d_0 + d_1 x + \cdots + d_{15} x^{15}$, where $d_j$ are polynomials in $a_j, b_j, c_j$. We want $d_j = 0$ for $i \geq 2$ and $d \neq 0$. (For proving Theorem 4, we want $d_1 \neq 0$.) We assume that char($F$) $\neq 2, 3$. By Proposition 3, we can assume that one of the leading coefficients $a_5, b_5, c_5$ is not 0. Since the condition of Lemma 1 does not hold, one of these coefficients must be 0. Say, $a_5 \neq 0$ and $c_5 = 0$. Then $b_5 = -a_5$, because $d_{15} = 0$. The condition $d \neq 0$ forces $c_4 \neq 0$. By a linear invertible change of the variable $x$ and multiplying both sides of $d = a^3 + b^3 + c^3$ by a nonzero constant, we are reduced to the case when $a_5 = 1 = -b_5, a_4 = 0, c_4 = 1$. Now $d_{15} = 0 \neq d$.

The polynomial equations $d_i = 0$ ($7 \leq i \leq 15$) allow us to exclude all unknown coefficients but $a_3, c_0, c_1, c_2, c_3$. The conditions $d_6 = 0, d_5 = 0, d_4 = 0, d_3 = 0, d_2 = 0$ give a system of five polynomial equations for five unknowns $a_3, c_0, c_1, c_2, c_3$. Namely,

$$
d_6 = -1/108 + 5a_3^2 + 3a_3 c_0 + 3c_0^2/4 - 18a_3^2 c_2 - 3c_0^2 c_2 + 21a_3^2 c_2^2
- 8c_2^3 - 12a_3^2 c_1 c_3 + 15c_1^2 c_2 c_3 + 63a_3^2 c_3^2 + 3c_0 c_2 c_3 - 159a_3 c_2 c_3^2
+ 99c_0^2 c_3^2 - 3c_1 c_3^2/2 + 12a_3^2 c_3^4 - 21c_2 c_3^4 - 37c_3^6/4 = 0;
\quad d_5 = 9a_3^2 c_3^2/2 + 3c_0 c_1^2/2 - 12a_3^2 c_1 c_2 + 15c_1^2 c_2^2/2 - c_3^2/12
- 30a_3^2 c_3 - 3a_3 c_0 c_3 + 3c_1 c_3^2/4 + 117a_3^2 c_2 c_3 + 6c_0 c_2 c_3
- 150a_3^2 c_3^2 + 63c_2 c_3 - 6a_3 c_1 c_3 + 9c_2 c_3^2/2
+ 57a_3^2 c_3^2/2 + 15c_0 c_3^2/2 - 51a_3 c_2 c_3^2 + 33c_2 c_3^3/2
- 15c_1^2 c_3^3/2 + 57a_3^2 c_3^3/2 - 165c_2 c_3^5/2 - 69c_3^7/4 = 0;
\quad d_4 = a_3^2/18 + 15a_3^4/4 + 3a_3^2 c_0^2/2 + 3c_0^2/4 - c_2^2/12
- 12a_3^2 c_2 + 3c_1^2 c_2/4 + 21a_3^2 c_2^2 - 3c_0 c_2^2/2
- 9c_2^4/4 - 6a_3^2 c_1 c_3 + 3c_0 c_1 c_3 + 2c_1 c_2 c_3
- c_3^2/3 + 42a_3^2 c_3 - 159a_3 c_2 c_3^2/2
+ 9c_0 c_2 c_3^2/2 + 81c_2 c_3^2/2 - 3c_1 c_2 c_3^3
+ 6a_3^2 c_3^4 + 3c_0 c_3^4/2 - 69c_2 c_3^4/4 - 3c_1 c_3^5/2 - 51c_2 c_3^6/4 - 3c_3^8/2 = 0;
$$
\[ d_3 = 21a_3^3c_1/2 + 9a_3c_0c_1/2 + c_1^3 - 33a_3^2c_1c_2 - 3c_0c_1c_2 \\
+ 63a_3^2c_2/2 - 9c_1c_2^3 + a_3c_3/3 - 135a_3^4c_3/4 \\
- 6a_3^2c_0c_3 + 3c_0^2c_3/4 - 18a_3^2c_1c_3 - c_2c_3/2 \\
+ 129a_3^3c_2c_3 + 9a_3c_0c_2c_3 + 45c_1^2c_2c_3/2 \\
- 321a_3^2c_2^2c_3/2 + 63a_3^3c_3c_3 + 9c_0^4c_3/4 \\
+ 219a_3^2c_1^2c_3/2 + 9c_0c_1c_2^2/2 - 549a_3c_1c_2c_3^2/2 \\
+ 333c_1^2c_3^2/2 - 3c_3^3/4 - 209a_3^3c_3^2/2 \\
+ 3a_3c_0c_3^3/2 - 9c_1c_3^3/4 + 393a_3^2c_2c_3^3 \\
+ 9c_0c_2c_3^3/2 - 933a_3c_2^2c_3^3/2 + 171c_2c_3^3 \\
+ 12a_3c_1c_3^4 - 63a_1c_2c_3^4/2 + 42a_3^2c_3^5/2 - 87a_3c_2c_3^5/2 \\
- 81c_2c_3^5/4 - 27c_1c_3^6/2 + 6a_3c_3^7 - 45c_2c_3^7/2 - 9c_3^9/4 = 0; \]

\[ d_2 = -a_3^3/9 + 9a_3^2/2 + 9a_3c_0 + 9a_3c_0^2/2 + 27a_3^2c_1^2/4 + 3c_0c_1^2 \\
+ a_3c_2/3 - 99a_3^2c_2/2 - 63a_3c_0c_2/2 - 15c_0^2c_2/4 - 18a_3^2c_2 \\
- c_2^2/4 + 54a_3^2c_2^2 + 36a_3c_0c_2^2 + 45c_1^2c_2^2/4 - 117a_3^2c_2^2/2 \\
- 27c_0c_2^2/2 + 63a_3c_2^2/2 - 27c_2^4/4 - 135a_3c_1c_3/2 - 27a_3c_0c_1c_3 \\
+ 252a_3^2c_1c_2c_3 - 27c_0c_3^2/2 - 4a_3c_1c_2c_3 + 36c_0c_1c_2c_3 - 621a_3c_1c_2c_3/2 + 126c_1c_2c_3 \\
+ 5a_3c_1^2/6 + 549a_3^2c_3^2/4 + 108a_3c_2c_3^2 + 9c_0^2c_3^2/2 + 9a_3c_1^2c_3 \\
- 5c_3_2^2/4 - 639a_3^2c_2c_3 - 270a_3c_0c_2c_3 - 63c_2^2c_3^3/4 \\
+ 1107a_3^2c_2^3c_3^2 + 333c_0c_2c_3^2/2 - 846a_3c_2^3 + 963c_2^4c_3^2/4 \\
- 153a_3^2c_1c_3^2/2 + 9c_0c_1c_3^2/2 + 228a_3c_1c_2c_3^3 - 333c_1^2c_3^3/2 \\
- c_3^4 + 357a_3c_3^4/2 + 12a_3c_0c_3^4 - 9c_1c_3^4 - 720a_3^2c_2c_3^4 \\
- 27c_0c_2c_3^4 + 951a_3^2c_3^4 - 1647c_2c_3^4/4 + 165a_3c_1c_3^5/2 \\
- 108c_1c_2c_3^5 - 519a_3c_3^6/4 - 27c_0c_3^6/2 + 294a_3c_2c_3^6 - 639c_2c_3^6/4 \\
- 27c_1c_3^7/2 + 21a_3c_8^3/2 - 9c_2c_3^3/4 + 9c_3^3/4 = 0. \]

Further exclusion of variables required too much computer memory. So a complete search was used instead for \( F = \mathbb{Z}/7\mathbb{Z} \) and \( F = \mathbb{Z}/13\mathbb{Z} \). The author thanks A. Ocneanu for help with programming. The computer search showed that there was no solution for this system of five polynomial equations in five variables in the case \( F = \mathbb{Z}/7\mathbb{Z} \). Thus, the equation \( d = a + b + c \) has no solutions in \( (\mathbb{Z}/7\mathbb{Z})[x] \) such that \( 5 \geq \deg(a) \geq \deg(b) \geq \deg(c) \), \( \deg(a) \neq 0 \), and \( \deg(d) = 0 \) or 1. In the case \( F = \mathbb{Z}/13\mathbb{Z} \), the computer took 57128 seconds to try all 135 possible solutions and found all 12 solutions. The first of them is

\[(1+3x+10x^2+x^5)^3-(12+12x+6x^2+x^5)^3+(2+4x+8x^2+9x^3+x^4)^3 = 10+8x.\]
It was written above in the proof of Theorem 4 after a linear change of variable. Here are five more solutions (mod 13):

\[(3 + x + 10x^2 + x^5)^3 - (10 + 4x + 6x^2 + x^5)^3 + (5 + 4x + 11x^2 + 3x^3 + x^4)^3 = 10 + 11x;\]
\[(9 + 9x + 10x^2 + x^5)^3 - (4 + 10x + 6x^2 + x^5)^3 + (6 + 4x + 7x^2 + x^3 + x^4)^3 = 10 + 7x;\]
\[(5 + 7x + 7x^2 + 7x^3 + x^5)^3 + (7 + 3x + 7x^2 + 3x^3 + x^4)^3 - (12 + 10x + 3x^2 + 7x^3 + x^5)^3 = 1;\]
\[(2 + 11x + 7x^2 + 8x^3 + x^5)^3 - (10 + 12x + 3x^2 + 8x^3 + x^5)^3 + (11 + 3x + 7x^2 + x^3 + x^4)^3 = 1;\]
\[(5 + 7x + 7x^2 + 7x^3 + x^5)^3 - (4 + 4x + 3x^2 + 1lx^3 + x^5)^3 + (8 + 3x + 1lx^2 + 9x^3 + x^4)^3 = 1.\]

The other six equalities can be obtained from the above six solutions by switching a and b and replacing x by -x.

The degree-6 case was treated similarly. The following lemma was used to restrict possible values for the coefficients of a, b, c.

**Lemma 5.** Let F be a field such that there is no α, β, γ in F with \(α^3 + β^3 + γ^3 = 0 \neq αβγ\). Let \(x = a^3 + b^3 + c^3\) with \(a, b, c \in F[x]\), \(N = \deg(a) ≥ \deg(b) ≥ \deg(c) ≥ 1\), \(a = \sum a_i x^i, b = \sum b_i x^i, c = \sum c_i x^i\), and \(a_i, b_i, c_i \in F\). Then \(c_0 = 0\) and \(a_0/a_N\) is a cube in F.

**Proof.** By the condition, \(c_N = 0\), hence \((-b_N/a_N)^3 = 1\). Replacing b by \(bb_N/a_N\), we can assume that \(b_N = -aN \neq 0\).

Let us show that the assumption that \(c_0 \neq 0\) leads to a contradiction. Indeed, in this case \(a_0b_0 = 0\). Say, \(b_0 = 0\). Replacing c by \(ca_0/c_0\) we can assume that \(c_0 = a_0\).

Let \(α_i\) be the zeros of \(b + c\) in a field extension of F. Then \(a(α_i)^3 = α_i\) for all i, hence \(a_0/a_N = (-1)^N \prod α_i = ((-1)^N \prod a(α_i))^3\) is a cube in F. Applying this argument to the zeros \(β_j\) of \(a^2 - ab + b^2\), we obtain that \(a_0^2/(3a_N^2) = (\prod c(β_j))^3\) is also a cube in F. So \(9\) is a cube (in fact, a 12th power) in F. In the case \(2F = F\), this leads to a contradiction, because \(9 = 8 + 1\) is also the sum of two cubes.

In the case \(2F = 0\), assume first that F contains an element ε such that \(ε^2 = ε + 1\). Using the zeros of \(a + εb\), we conclude that \(a_0/(a_N(1 + ε))\) is a cube in F. Using the zeros of \(a + εc\), we conclude that \(a_0(1 + ε)/a_N\) is a cube in F. So \((1 + ε)^2 = ε\) is a cube in F. This leads to a contradiction, because \(ε = 1 + 1/ε\) is also the sum of two cubes.

Assume now that \(2F = 0\) and F does not contain any ε as above. Then we consider a field extension \(F[ε]\) with ε as above. Using the zeros of \(a + εb\),
we conclude that $a_0/(a_N(1 + \varepsilon))$ is a cube in $F[\varepsilon]$. Using the zeros of $a + \varepsilon c$, we conclude that $a_0(1 + \varepsilon)/a_N$ is a cube in $F[\varepsilon]$. So $(1 + \varepsilon)^2 = \varepsilon$ is a cube in $F[\varepsilon]$. We write $\varepsilon = (u + ve)^3$ with $u, v \in F$. Then

$$0 = u^3 + uv^2 + v^3 = (u + v)^3 - u^2v \quad \text{and} \quad 1 = u^2v + uv^2 = uv(u + v),$$

hence $u^5v^4 = 1$. Combining this with

$$0 = (1 + u^2v + uv^2)^4 = 1 + u^8v^4 + u^4v^8,$$

we obtain that $1 + u^3 + u^{-6} = 0$, which contradicts the condition of the lemma.

Thus, $c_0 = 0$ in all cases. Now we use the zeros $\gamma_k$ of $a + c$ to conclude that $a_0/a_N = ((-1)^N \prod b(\gamma_k))^3$ is a cube in $F$. \hfill $\square$

**Proof of Proposition 3.** It was shown above that the condition of Lemma 1 implies that $x$ is the sum of cubes of three polynomials of degree 1.

Assume now that there is no $\alpha, \beta, \gamma$ in $F = 6F$ such that $\alpha^3 + \beta^3 + \gamma^3 = 0 \neq \alpha\beta\gamma$. We want to prove that $d_{cg}(d) / 1$ for any

$$d = a^3 + b^3 + c^3,$$

where $a, b, c \in F[x]$, provided that $a, b, c$ are all of degree $\leq 4$.

Let $\deg(a) \geq \deg(b) \geq \deg(c)$. Assume that $\deg(d) = 1$. Then it is clear that all three polynomials $a, b, c$ cannot have the same degree. So $\deg(a) = \deg(b) > \deg(c)$. Dividing both sides of (6) by the cube of the leading coefficient of $a$, we can assume that $a$ and $-b$ are monic, i.e., their leading coefficients are 1. Then the next two coefficients must also be equal. This leads immediately to a contradiction when the degree of $a$ is $\leq 2$.

Consider now the case when $\deg(a) = 3$ or 4. If $2F = F$, we set $u = a + b$ and $v = a - b$. The equation takes the form

$$u(u^2 + 3v^2)/4 + c^3 = d.$$

When $\deg(a) = 3$, we see that $\deg(c) = 2$, $\deg(u) = 0$, and $\deg(v) = 3$. We rewrite the equation as $v^2/4 + c^3/3u = d/3u - u^2/3 = d'$ with $\deg(d') = 1$. We write $v/2 = x^3 + v_2x^2 + v_1x + v_0$ and $c^3/3u = -(x^2 + c_1x + c_0)^3$. Our equation takes the form

$$(x^3 + v_2x^2 + v_1x + v_0)^2 - (x^2 + c_1x + c_0)^3 = d'.$$

Then we replace $x$ by $x - v_2/3$ to make $v_2 = 0$. Looking at terms of degree 5, we conclude that $c_1 = 0$. Looking at terms of degree 3, we conclude that $v_0 = 0$. Looking at terms of degree 1, we obtain a contradiction. (The "abc-theorem" yields that $V^2 - C^3$ cannot be a nonzero constant for nonconstant polynomials $V$ and $C$, if $\text{char}(F) = 0$. It is not difficult to show that the necessary and sufficient condition on a field $F$ for this conclusion to be true is that $6F = F$.)
Let now $\deg(a) = 4$ (and still $2F = F$). Since $\deg(v) = 4$, it is clear that $\deg(u) = 1$ and $\deg(c) = 3$. Now we set $y = 1/u$. After we divide both sides of (6) by $u^9$, it takes the form

$$3/4(v/u^4)^2 + (c/u^3)^3 = -y^6/4 + x/u^9.$$ 

Note that $c/u^3$ is a polynomial of degree 3 in $y$ with a constant term $e' \neq 0$. After dividing the equation by $e'^3$, it takes the form

$$1 + v_1y + v_2y^2 + v_3y^3 + v_4y^4 - y^6/4e'^3 + e''y^8 + e'''y^9,$$

with $e', e'', e'''$ in $F$ and $e'e'' \neq 0$.

Now we see that, modulo $y^6F[y]$,

$$(1 + v_1y + v_2y^2 + v_3y^3 + v_4y^4)^2 \equiv (1 + c_1y + c_2y^2 + c_3y^3)^3,$$

hence

$$1 + v_1y + v_2y^2 + v_3y^3 + v_4y^4 \equiv (1 + f_1y + f_2y^2 + f_3y^3 + f_4y^4 + f_5y^5)^3$$

and

$$1 + c_1y + c_2y^2 + c_3y^3 \equiv (1 + f_1y + f_2y^2 + f_3y^3 + f_4y^4 + f_5y^5)^2$$

for some $f_i$ in $F$. Set $f = 1 + f_1y + f_2y^2 + f_3y^3 + f_4y^4 + f_5y^5$.

Then, modulo $y^6F[y]$,

$$f^2 \equiv 1 + 2f_1y + (2f_2 + f_1^2)y^2 + (2f_3 + 2f_1f_2)y^3 + (2f_4 + 2f_1f_3 + f_2^2)y^4 + (2f_5 + 2f_1f_4 + 2f_2f_3)y^5$$

and

$$f^3 \equiv 1 + 3f_1y + (3f_2 + 3f_1^2)y^2 + (3f_3 + 6f_1f_2 + f_1^3)y^3 + (3f_4 + 3f_1f_3 + 3f_2^2)y^4 + (3f_5 + 6f_1f_4 + 6f_2f_3 + 3f_1^2f_2 + 3f_1f_3^2)y^5.$$

So

$$0 = c_4 = 2f_4 + 2f_1f_3 + f_2^2,$$

$$0 = c_5 = 2f_5 + 2f_1f_4 + 2f_2f_3,$$

$$0 = v_5 = 3f_5 + 3f_1f_2 + 6f_1f_4 + 6f_2f_3 + 3f_1^2f_3,$$

hence

$$f_4 + f_3^2/2 + f_5f_1 = 0, \quad f_5 - f_3f_1 - f_3f_1^2 = 0, \quad f_3f_2 + f_3^2f_1/2 = 0.$$

We set $C = 1 + 2f_1y + (2f_2 + f_1^2)y^2 + (2f_3 + 2f_1f_2)y^3$,

$$V = 1 + 3f_1y + (3f_2 + 3f_1^2)y^2 + (3f_3 + 6f_1f_2 + f_1^3)y^3 + (3f_4 + 3f_1f_3 + 3f_2^2)y^4 + 6f_1f_4 + 6f_2f_3 + 3f_1^2f_3.$$
and $S = V^2 - C^3$. Then (7) takes the form

$$S = V^2 - C^3 = S_5y^5 + S_6y^6 + S_7y^7 + S_8y^8 + S_9y^9$$

$$= -y^6/4e'^3 + e''y^8 + e'''y^9,$$

where the coefficients $S_i$ of $S$ are polynomials in $f_i$. Substituting $f_4 = -f_2^2/2 - f_3^3 f_1$ and $f_5 = f_2^2 f_1 + f_3^3 f_2^2$ into $S_i$, we obtain:

$$S_9 = -8f_3^3 - 24f_2^2 f_2 f_1 - 8f_2^3 f_1^3 = e'''',$$

$$S_8 = -24f_3^2 f_2^2 - 9f_2^4/4 - 39f_3 f_2 f_1 - 3f_3^2 f_1^2$$
$$- 15f_2^2 f_1^2 - 8f_3 f_2 f_1^3 - 3f_2^2 f_1^4 = e'' 
eq 0,$$

$$S_7 = -15f_3^2 f_2^2 - 6f_2^2 f_1 - 6f_2^2 f_1 - 18f_3 f_2 f_1^2 - 9f_2^2 f_1^3 = 0,$$

$$S_6 = -3f_3 f_2^2 + f_2^3 - 18f_3 f_2 f_1 - 9f_2^2 f_1^2 = -1/4e'^3 
eq 0,$$

$$S_5 = -6f_3 f_2^2 - 3f_2^2 f_1 = 0.$$

Since $S_5 = 0$, we see that either $f_2 = 0$, or $2f_3 + f_2 f_1 = 0$.

In the first case,

$$(S_9, S_8, S_7, S_6) = (-8f_3^3, -3f_3^2 f_2, -6f_2^2 f_1, -3f_2^3),$$

and $S_7 = -6f_2^2 f_1 = 0$ implies that $S_8 = -3f_2^2 f_1^2 = 0$, so this case is impossible.

Assume now that $f_2 
eq 0 = 2f_3 + f_1 f_2$. Then

$$S_9 = -f_2^3 f_1^3, \quad S_8 = 9f_2^4/4 - 3f_2^3 f_1^2/2 - 3f_2^2 f_1^4/4,$$

$$S_7 = 3f_2^3 f_1/2 - 3f_2^2 f_1^3/2, \quad S_6 = f_2^3 - 3f_2^2 f_1^2/4.$$

From $S_7 = 3f_2^3 f_1/2 - 3f_2^2 f_1^3/2 = 0$, we obtain that either $f_1 = 0$, or $f_2 = f_2^3$.

The latter case is impossible because then the coefficient $S_7$ vanishes. So let $f_1 = 0$.

Then the coefficients $(S_9, S_8, S_7, S_6)$ become $(0, 9f_2^4/4, 0, f_2^3)$. So $-e'^3/4 = f_2^3$, hence 2 is a cube in $F$. Since 2 is also the sum of two cubes, we are done.

The case $2F = 0$, $\deg(a) = 3$ or 4 was done using the computer. When $\deg(a) = 3$, we consider (6) with $a = x^3 + a_2 x^2 + a_1 x + a_0$, $b = a + f$, $c = c_2 x^2 + c_1 x + c_0$, $d = \sum d_i x^i 
eq 0$, $d_i = 0$ for $i \geq 2$.

It is clear that $\deg(f) = 0$. Replacing $x$ by $x + c_2$, we make $c_2 = 0$. From $d_6 = f + c_3^2 = 0$, we get $f = c_2^3$. Substituting this into $d$, we get that $d_5 = c_1 c_2^2 = 0$. If $c_1 = 0$, then $d_3 = c_2^6 = 0$. Thus, $c_2 = 0$, hence $d_2 = c_0 c_1^2$ and $d_1 = c_0^2 c_1^4$. So $d_2 = 0$ implies that $d_1 = 0$.

The reader is spared from longer computations which were done in the case when $2F = 0$ and $\deg(a) = 4$. \Box

**Remark.** The identity

$$x = ((x^3 - 1/27)^3 + (x^2 + x/3)^3 + (-x^3 + x/3 + 1/27)^3)/(x^2 + x/3 + 1/9)^3$$
show that \( w_3(F(x)) \leq 3 \) and \( w_3(F) \leq 3 \) for any field \( F \) with \( \text{char}(F) \neq 3 \); if \( \text{char}(F) = 3 \), then \( w_3(F(x)) = w_3(F) = 1 \).

Remarks. All the solutions of (6) with \( \deg(a) = \deg(b) = 3, \deg(c) = 2, \deg(d) = 0 \) can be obtained from

\[
(1 + 6x^3)^3 + (1 - 6x^3)^3 + (-6x^2)^3 = 2
\]

by a linear change of variable and scaling. There are no such solutions when \( 2F = 0 \).

All the solutions of (6) with \( \deg(a) = \deg(b) = 4, \deg(c) = 3, \deg(d) = 0 \) can be obtained from

\[
(9x^4)^3 + (3x - 9x^4)^3 + (1 - 9x^3)^3 = 1
\]

by a linear change of variable, switching \( a \) and \( b \), and scaling. P. Erdős pointed out to the author that the last two formulas can be found in [1].

All solutions of (6) with \( \deg(a) = \deg(b) = 5, \deg(c) = 4, \deg(d) = 0 \) or 1 were described above when \( \text{char}(F) = 13 \). They do not exist when \( \text{card}(F) = 5, 7 \). When \( \text{char}(F) = 11 \), all such solutions can be obtained from the equalities

\[
(2 + 2x^2 + 3x^3 + x^5)^3 - (2 + 5x + 6x^2 + 3x^3 + x^5)^3
\]

\[
+ (x + 10x^2 + 5x^3 + x^4)^3 = 6x,
\]

\[
(1 + 4x + x^2 + 5x^3 + x^5)^3 - (9 + 6x + 5x^2 + 5x^3 + x^4)^3
\]

\[
+ (8 + x + 2x^3 + x^4)^3 = 4,
\]

\[
(4 + 4x + 3x^2 + 8x^3 + x^5)^3 - (8 + 6x + 7x^2 + 8x^3 + x^5)^3
\]

\[
+ (5 + 4x + 9x^2 + 2x^3 + x^4)^3 = 7
\]

by a linear change of variable, switching \( a \) and \( b \), and scaling.

Bibliography


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