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A PRECISE CALCULATION OF THE FEIGENBAUM CONSTANTS

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ABSTRACT. The Feigenbaum constants arise in the theory of iteration of real functions. We calculate here to high precision the constants α and δ associated with period-doubling bifurcations for maps with a single maximum of order z, for $2 \le z \le 12$. Multiple-precision floating-point techniques are used to find a solution of Feigenbaum's functional equation, and hence the constants.

1. HISTORY

Consider the iteration of the function

$$f_{\mu,z}(x) = 1 - \mu|x|^z, \qquad z > 0; \tag{1}$$

that is, the sequence

$$x_{i+1} = f_{\mu,z}(x_i), \quad i = 1, 2, \ldots; \qquad x_0 = 0. \tag{2}$$

In 1979 Feigenbaum [8] observed that there exist bifurcations in the set of limit points of (2) (that is, in the set of all points which are the limit of some infinite subsequence) as the parameter μ is increased for fixed z. Roughly speaking, if the sequence (2) is asymptotically periodic with period p for a particular parameter value μ (that is, there exists a stable p-cycle), then as μ is increased, the period will be observed to double, so that a stable $2p$-cycle appears. We denote the critical μ-value at which the 2^j cycle first appears by μ_j.

Feigenbaum also conjectured that there exist certain "universal" scaling constants associated with these bifurcations. Specifically,

$$\delta_z = \lim_{j\to\infty} \frac{\mu_j - \mu_{j-1}}{\mu_{j+1} - \mu_j} \tag{3}$$

exists, and δ_2 is about 4.669. Similarly, if d_j is the value of the nearest cycle element to 0 in the 2^j cycle, then

$$\alpha_z = \lim_{j\to\infty} \frac{d_j}{d_{j+1}} \tag{4}$$

exists, and α_2 is about -2.503.

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The conjecture for the case $z = 2$ was proven by Lanford in 1982 [11], and for $z < 14$ by Epstein in 1985 [7].

Some numerical results in the literature are given in Table 1. (Note that most authors quote $|\alpha|$.)

TABLE 1

Reference	z	α	δ
[8]	2	−2.502907876	4.6692
[9]	2	2.50290787509589284	4.6692016091029909
[4]	2	2.502907875095892822283902873	4.669201609102990671853203 8
[13]	2	2.5029078	4.6692016
	4	1.690	7.29
	6	1.467	9.30
	8	1.358	10.948
	10	1.292	12.37
[5]	2	2.50	4.67
	3	1.93	6.08
	4	1.69	7.29
	5	1.56	8.35
	7	1.41	10.2
	10	1.29	12.3

Some examples of physical systems in which α and δ are relevant are described in [2]. Despite the theoretical and applied interest of these numbers, little is known about them, for example whether they satisfy any simple algebraic relations. On the question of the limits as z tends to ∞ of α_z and δ_z, see [6].

We propose here to evaluate α and δ to high precision for various z, in order to provide data for testing conjectures concerning these numbers.

2. METHOD

Calculating δ directly from the definition is impractical because it would involve finding high iterates of f, which are subject to accumulation of roundoff error, making it difficult to locate the bifurcation values μ_j accurately.

A practical algorithm for δ was described in [3]. However, this is suitable for low precision only, owing to its slow convergence. Another method for δ was proposed in [12], but has the same drawback. It seems that the original method of Feigenbaum [8] is still the best when high precision is desired. We briefly describe this method, which is justified in [8].

One defines an operator T, acting on functions $g: \mathbb{R} \to \mathbb{R}$, by

$$[Tg](x) = \{g(g(g(1)x))\}/g(1). \tag{5}$$

If we find an even real analytic function invariant under T, with $g(0) = 1$, then α is determined by $\alpha = 1/g(1)$.

The numerical method proceeds by approximating g by the form

$$g(x) = 1 + \sum_{i=1}^{n} g_i |x|^{zi}. \tag{6}$$

An approximate fixed point of T can then be found by a collocation method. We require (5) to be satisfied at n points x_j in the interval $(0, 1]$, and solve the resulting n nonlinear equations by an n-dimensional Newton iteration. Thus we require (for $j = 1, \ldots, n$)

$$\begin{aligned}&\left(1 + \sum_{i=1}^{n} g_i\right)\left(1 + \sum_{i=1}^{n} g_i |x_j|^{zi}\right)\\ &\qquad - 1 - \sum_{i=1}^{n} g_i \left| 1 + \sum_{i=1}^{n} g_i \left| \left(1 + \sum_{i=1}^{n} g_i\right) x_j \right|^{zi} \right|^{zi} = 0.\end{aligned} \tag{7}$$

If we call the left side of this equation f_j, the Newton iteration requires the inversion of the Jacobian matrix $\partial f_j / \partial g_i$. This is the major part of the computational task. For the smaller z-values, it was found that the initial approximation to the g-coefficients was not critical, but for the larger z-values some trial and error was necessary before convergence was obtained. For z greater than 12, all initial approximations tried produced divergence of the Newton iteration. However, it is probable that a solution to (7) does exist for all z.

Feigenbaum has shown [8] that the constant δ_z is the largest eigenvalue of the local linearization of T about the fixed point function g found above. A simple calculation shows that this operator L is given by

$$[Lf](x) = -\alpha f(g(x/\alpha)) - \alpha g'(g(x/\alpha)) f(x/\alpha). \tag{8}$$

Once the approximate fixed point g has been found, one may construct a finite-dimensional matrix approximating L by a method similar to that used above. That is, one evaluates the right side of (8) at the n nodes x_i. The largest eigenvalue of the matrix can then easily be found by the power method [10, §7.3].

3. Results

We have implemented the above scheme with arbitrary-precision floating-point arithmetic, using the methods described by Brent [1]. The choice of the points x_i was found to be not critical, linear spacing being adequate for small z. However, the nonlinear spacing $x_i = (i/n)^{1/z}$ $(i = 1, \ldots, n)$ produced more stable results for the higher z-values, and was therefore used for all the results quoted. It is observed that the g-coefficients decrease rapidly in magnitude; for example for $z = 2$, $|g_i|$ is about 10^{-i}. This gives a guide to the value of n needed; since α is $1/g(1)$, we must set n about equal to the number of decimals desired for α, and preferably greater. We first found α and δ for $z = 2, \dots, 12$ with $n = 75$ and a working precision of 150 decimal places. We then repeated all calculations with $n = 100$ and a working precision of 200 decimal places. The results given in Table 2 show as many digits as agree between the two calculations. Thus, it is probable that all quoted digits are correct.

TABLE 2

Feigenbaum's α and δ for $z = 2, 3, \dots, 12$

α_2	−2.5029078750 9589282228 3902873218 2157863812 7137672714 9977336192 0567792354 6317959020 6703299649 7464338341 29595232
δ_2	4.6692016091 0299067185 3203820466 2016172581 8557747576 8632745651 3430041343 3021131473 71387
α_3	−1.9276909638 4764084494 9994352966 3190518926 5896703673 2620743579 6727408667 7490009
δ_3	5.9679687037 7745104099 4193019979 6723235126 0291982742 3948393172 0
α_4	−1.6903029714 0524485334 3780150324 1613482282 7805970956 1966682423 263
δ_4	7.2846862170 7334336430 8930567995 5530694780 4661979979 065907212
α_5	−1.5557712501 9651840213 2978629657 4844101923 2289917422 9329
δ_5	8.3494991320 6696352110 9747401811 2355832574 76
α_6	−1.4677424503 1990094445 3834315108 9737463687 971293967
δ_6	9.2962468327 7137008283 4476566367 4575503066 88756
α_7	−1.4051107883 1683179942 5671289266 7982571940 6757
δ_7	10.2221595288 3488165524 1801329347 44
α_8	−1.3580172791 3805034548 7376333106 2614006580 6
δ_8	10.9486242659 4159042553 4207900712 234803
α_9	−1.3211857598 0525276678 2332645011 12163344
δ_9	11.7683336395 5408532268 157502
α_{10}	−1.2915168672 6234456962 5592342901 483728
δ_{10}	12.3414090453 4929383969 7630423331
α_{11}	−1.2670614079 0247246329 0059733681 36867
δ_{11}	13.0765458056 5116239270 558
α_{12}	−1.2465277517 2074929543 9806587251 9
δ_{12}	13.5350756661 7027649570 05633538

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