# SHAPE-PRESERVING $C^2$ CUBIC POLYNOMIAL INTERPOLATING SPLINES

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ABSTRACT. In this paper we propose a method to construct shape-preserving  $C^2$  cubic polynomial splines interpolating convex and/or monotonic data. For such given data, the existence or nonexistence of such interpolating splines can be expressed in terms of existence or nonexistence of solutions for a system of linear inequalities in two unknowns.

## **0. INTRODUCTION**

In many interpolation problems it is important that the solution preserves some shape properties such as convexity or monotonicity. Classical methods (the polynomial spline functions being the most widely used) usually ignore these kinds of conditions and thus yield solutions exhibiting undesirable inflections or oscillations. This is the reason why many investigations during the last years have been directed towards interpolation by means of shape-preserving polynomial spline functions.

In [6] McAllister and Roulier, and in [13] Schumaker, have studied quadratic splines which preserve monotonicity and convexity. In [5] Fritsch and Carlson have studied cubic splines that preserve monotonicity. In [1, 2] Costantini and Morandi have studied cubic splines which preserve both convexity and monotonicity. All of these splines are  $C^1$ .

Other authors (Neuman [10, 11] and Mettke [9]) have imposed additional conditions on the monotone, convex data, which yield a solution that belongs to a subspace of polynomial splines. Moreover, McAllister and Roulier [7], and Passow and Roulier [12] have shown that it may be impossible to construct monotonic and convex splines of given degree and deficiency. In [8] Medina gives a survey on shape-preserving interpolation by means of polynomial or of other classes of splines.

In this paper we propose a method to construct  $C^2$  cubic polynomial interpolation splines. Functions of this kind, of course, do not always exist for arbitrary convex monotone data sets. For convex increasing data—this, as we

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shall see later, does not limit generality—the existence or nonexistence of such a  $C^2$  cubic polynomial interpolation spline can be expressed in terms of the existence or nonexistence of solutions for a system of linear inequalities in two unknowns.

## 1. NOTATION AND DEFINITIONS

Let  $J = \{0, 1, ..., n-1\}$  and  $K = \{1, 2, ..., n-1\}$ . Suppose  $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$  is a partition of [a, b], and suppose that  $y_0, y_1, ..., y_n$  are n+1 real numbers. Let  $I_i = [x_i, x_{i+1}]$  for  $i \in J$ ,  $\Delta_i = (y_{i+1} - y_i)/(x_{i+1} - x_i)$  for  $i \in J$ ,  $h_i = x_{i+1} - x_i$  for  $i \in J$ , and  $x_{i+1/3} = x_i + h_i/3$ ,  $x_{i+2/3} = x_i + 2h_i/3$  for  $i \in J$ .

Given the set of points  $M_i$ , with  $M_i = (x_i, y_i)$  for i = 0, 1, ..., n, we say that:

(a) the data are increasing (resp. decreasing) if and only if

$$y_0 \le y_1 \le y_2 \le \cdots \le y_{n-1} \le y_n \quad (\text{resp. } y_0 \ge y_1 \ge \cdots \ge y_n);$$

(b) the data are convex (resp. concave) if and only if

$$\Delta_0 \leq \Delta_1 \leq \cdots \leq \Delta_{n-2} \leq \Delta_{n-1} \quad (\text{resp. } \Delta_0 \geq \Delta_1 \geq \cdots \geq \Delta_{n-1}) \,.$$

## 2. The problem

For clarity we shall assume that the data are convex and increasing, and we shall try to find a degree-3 polynomial spline function, denoted by s, of class  $C^{2}[a, b]$ , interpolating the points  $M_{i}$ , i = 0, 1, ..., n, and preserving the shape of the data.

These conditions are expressed as follows:

- (i) for  $i \in J$  and  $x \in I_i$ ,  $s(x) = P_i(x)$ , where  $P_i$  is a degree-3 polynomial; (ii)  $s(x_0) = P_0(x_0) = y_0$ ; for  $i \in K$ ,  $s(x_i) = P_{i-1}(x_i) = P_i(x_i) = y_i$ ;
- (ii)  $s(x_0) = P_0(x_0) = y_0$ ; for  $i \in K$ ,  $s(x_i) = P_{i-1}(x_i) = P_i(x_i) = y_i$ ;  $s(x_n) = P_{n-1}(x_n) = y_n$ ;
- (iii) for  $i \in K$ ,  $P'_{i-1}(x_i) = P'_i(x_i)$ ,  $P''_{i-1}(x_i) = P''_i(x_i)$ ;
- (iv) for  $i \in J$  and  $x \in I_i$ ,  $P'_i(x) \ge 0$  and  $P''_i(x) \ge 0$ .

In §3 a solution satisfying conditions (i)-(iv) is constructed. The construction is simplified when the data are only increasing (resp. only convex), since in this case, in (iv), only the condition of positivity of the first (resp. the second) derivative is required. Of course, by symmetry the case of concave and decreasing (resp. only decreasing or only concave) data is treated in the same way. One simply changes the sign in (iv).

## 3. The proposed solution

3.1. Let  $y_{i+1/3}$ ,  $y_{i+2/3}$ ,  $i \in J$ , be 2n real numbers, and

$$P_{i}(x) = \frac{1}{(h_{i})^{3}} \{ y_{i}(x_{i+1} - x)^{3} + 3y_{i+1/3}(x_{i+1} - x)^{2}(x - x_{i}) + 3y_{i+2/3}(x_{i+1} - x)(x - x_{i})^{2} + y_{i+1}(x - x_{i})^{3} \} \text{ for } x \in [x_{i}, x_{i+1}].$$

By construction, s satisfies (i) and (ii). For further use, we give the following first and second derivatives:

$$P'_{i}(x) = \frac{3}{h_{i}^{3}} \{ (y_{i+1/3} - y_{i})(x_{i+1} - x)^{2} + 2(y_{i+2/3} - y_{i+1/3})(x_{i+1} - x)(x - x_{i}) + (y_{i+1} - y_{i+2/3})(x - x_{i})^{2} \},$$

$$P''_{i}(x) = \frac{6}{(h_{i})^{3}} \{ (y_{i+2/3} - 2y_{i+1/3} + y_{i})(x_{i+1} - x) + (y_{i+1} - 2y_{i+2/3} + y_{i+1/3})(x - x_{i}) \}$$

**Theorem 1.** The function s, defined by

 $s(x) = P_i(x)$  for  $i \in J$  and  $x \in I_i$ ,

satisfies (iv) if and only if the following conditions hold:

(1) 
$$y_i \le y_{i+1/3} \le y_{i+2/3} \le y_{i+1}, \quad i \in J,$$

(2) 
$$\frac{y_{i+1/3} - y_i}{x_{i+1/3} - x_i} \le \frac{y_{i+2/3} - y_{i+1/3}}{x_{i+2/3} - x_{i+1/3}} \le \frac{y_{i+1} - y_{i+2/3}}{x_{i+1} - x_{i+2/3}}, \qquad i \in J$$

*Proof.* Suppose that conditions (1) and (2) hold. Obviously,  $P_i(x_i) = y_i$  and  $P_i(x_{i+1}) = y_{i+1}$ . It is furthermore known (Davis [3, pp. 114–115]), or can easily be verified using the expressions of  $P'_i$  and  $P''_i$  given in 3.3, that  $P_i$  satisfies the following properties: for all  $x \in I_i$ ,  $P'_i(x) \ge 0$  (by (1)) and  $P''_i(x) \ge 0$  (by (2)).

Conversely, assume  $P_i$  convex and increasing for  $i \in J$ . This implies, in particular, that the first and second derivatives of  $P_i$  at  $x_i$  and  $x_{i+1}$  are positive for  $i \in J$ . Conditions  $P''_i(x_i) \ge 0$  for  $i \in J$ , and  $P''_i(x_{i+1}) \ge 0$  for  $i \in J$  read:

(a) 
$$y_{i+2/3} - 2y_{i+1/3} + y_i \ge 0$$
 and  $y_{i+1} - 2y_{i+2/3} + y_{i+1/3} \ge 0$  for  $i \in J$ .

Similarly, conditions  $P'_i(x_i) \ge 0$  for  $i \in J$ , and  $P'_i(x_{i+1}) \ge 0$  for  $i \in J$  read:

(
$$\beta$$
)  $y_{i+1/3} - y_i \ge 0$  and  $y_{i+1} - y_{i+2/3} \ge 0$  for  $i \in J$ 

One can easily check that conditions  $(\alpha)$  and  $(\beta)$  imply (1) and (2).  $\Box$ 

Thus, solving our problem is equivalent to finding 2n real numbers  $y_{i+1/3}$ ,  $y_{i+2/3}$  satisfying (1) and (2) and ensuring, in addition, the continuity of the first and second derivatives at the nodes  $x_i$ ,  $i \in K$ .

3.2. To simplify, set for  $i \in J$ ,

$$d_{i} = \frac{y_{i+1/3} - y_{i}}{x_{i+1/3} - x_{i}} = 3\frac{y_{i+1/3} - y_{i}}{h_{i}},$$
  

$$d_{i+1/3} = \frac{y_{i+2/3} - y_{i+1/3}}{x_{i+2/3} - x_{i+1/3}} = 3\frac{y_{i+2/3} - y_{i+1/3}}{h_{i}},$$
  

$$d_{i+2/3} = \frac{y_{i+1} - y_{i+2/3}}{x_{i+1} - x_{i+2/3}} = 3\frac{y_{i+1} - y_{i+2/3}}{h_{i}}.$$

Then we have

(3) 
$$d_i + d_{i+1/3} + d_{i+2/3} = 3\Delta_i, \quad i \in J.$$

We now rewrite conditions (1) and (2), using  $d_i$ ,  $d_{i+1/3}$ ,  $d_{i+2/3}$ ,  $i \in J$ . Then (1) is equivalent to the following system of 3n inequalities in 3n unknowns:

(1)' 
$$d_i \ge 0, \quad d_{i+1/3} \ge 0, \quad d_{i+2/3} \ge 0 \text{ for } i \in J$$

and (2) is equivalent to the following system of 2n inequalities in 3n unknowns:

(2)' 
$$d_i \le d_{i+1/3} \le d_{i+2/3}$$
 for  $i \in J$ .

Thus, our problem is equivalent to determining  $\{d_i, d_{i+1/3}, d_{i+2/3}\}, i \in J$ , satisfying (1)', (2)', and (3) in such a manner that the continuity of the first and second derivatives at the nodes is ensured.

3.3. From the expressions of  $P'_i$  and  $P''_i$  (see 3.1) we obtain

$$P'_{i}(x_{i}) = d_{i}, \qquad P'_{i}(x_{i+1}) = d_{i+2/3},$$
$$P''_{i}(x_{i}) = \frac{2}{h_{i}}(d_{i+1/3} - d_{i}), \qquad P''_{i}(x_{i+1}) = \frac{2}{h_{i}}(d_{i+2/3} - d_{i+1/3}).$$

So, to satisfy the continuity of the first derivative at the nodes, it is necessary and sufficient to have

(4) 
$$d_i = d_{i-1+2/3} \text{ for } i \in K$$

Similarly, to satisfy the continuity of the second derivative at the nodes, it is necessary and sufficient to have

(5) 
$$\frac{d_{i-1+2/3} - d_{i-1+1/3}}{h_{i-1}} = \frac{d_{i+1/3} - d_i}{h_i} \quad \text{for } i \in K.$$

In view of (4), it is natural to set  $d_n = d_{n-1+2/3}$ . We have obtained the following theorem.

**Theorem 2.** To solve the problem of  $\S2$ , it is necessary and sufficient to determine  $\{d_i, d_{i+1/3}, d_{i+2/3}\}, i \in J$ , satisfying the five conditions (1)', (2)', (3), (4), and (5).

These conditions lead to the following system of linear equations and inequalities in 3n unknowns:

(S) 
$$\begin{cases} d_i \ge 0, \ d_{i+1/3} \ge 0, \ d_{i+2/3} \ge 0 & \text{for } i \in J, \ (1)' \\ d_i \le d_{i+1/3} \le d_{i+2/3} & \text{for } i \in J, \ (2)' \\ d_i + d_{i+1/3} + d_{i+2/3} = 3\Delta_i & \text{for } i \in J, \ (3) \\ d_i = d_{i-1+2/3} & \text{for } i \in K, \ (4) \\ \frac{d_{i+1/3} - d_i}{h_i} = \frac{d_{i-1+2/3} - d_{i-1+1/3}}{h_{i-1}} & \text{for } i \in K. \end{cases}$$

In what follows we shall denote  $h_i h_{i+1} / (h_i + h_{i+1})$  by  $H_i$ .

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3.4. Let us rewrite this system, modifying (2)', (3), and (5) according to (4):

$$\begin{array}{ll} f & d_i \geq 0, \ d_{i+1/3} \geq 0, \ d_{i+2/3} \geq 0 & \text{for } i \in J, \\ d_i \leq d_{i+1/3} \leq d_{i+1} & \text{for } i \in J, \\ d_i + d_{i+1/3} + d_{i+1} = 3\Delta_i & \text{for } i \in J, \\ d_i = d_{i-1+2/3}, & \text{for } i \in K, \\ d_i = H_{i-1} \left( \frac{d_{i+1/3}}{h_i} + \frac{d_{i-1+1/3}}{h_{i-1}} \right) & \text{for } i \in K. \end{array}$$

In view of (4), the inequalities (1)', (2)'' together are equivalent to

(6) 
$$0 \le d_0 \le d_{0+1/3} \le d_1 \le d_{1+1/3} \le d_2 \le \dots \le d_{n-1} \le d_{n-1+1/3} \le d_n$$
.

Given that the quantities  $d_{i+1/3}$  and  $d_{i-1+1/3}$  are positive and, by (5)',  $d_i$  is a weighted average of them,  $d_i$  is necessarily included between these two numbers, and inequalities (6) can be written more simply as

(I) 
$$0 \le d_0 \le d_{0+1/3} \le d_{1+1/3} \le \dots \le d_{n-2+1/3} \le d_{n-1+1/3} \le d_n$$
.

System (S) now reads as follows:

$$\begin{pmatrix} 0 \le d_0 \le d_{0+1/3} \le d_{1+2/3} \le \dots \le d_{n-2+1/3} \le d_{n-1+1/3} \le d_n, & (I) \\ d_i + d_{i+1/3} + d_{i+1} = 3\Delta_i & \text{for } i \in J, \\ \end{pmatrix}$$

$$d_i = d_{i-1+2/3}$$
 for  $i \in K$ , (4)

$$d_i = H_{i-1} \left( \frac{d_{i+1/3}}{h_i} + \frac{d_{i-1+1/3}}{h_{i-1}} \right) \quad \text{for } i \in K.$$
 (5)'

Using (5)', we express (3)' in the following way:

(E) 
$$\begin{cases} (h_0 + H_0) \frac{d_{0+1/3}}{h_0} + H_0 \frac{d_{1+1/3}}{h_1} = 3\Delta_0 - d_0, \\ H_{i-1} \frac{d_{i-1+1/3}}{h_{i-1}} + (h_i + H_{i-1} + H_i) \frac{d_{i+1/3}}{h_i} + H_i \frac{d_{i+1+1/3}}{h_{i+1}} = 3\Delta_i \\ \text{for } i = 1, 2, \dots, n-2, \\ H_{n-2} \frac{d_{n-2+1/3}}{h_{n-2}} + (h_{n-1} + H_{n-2}) \frac{d_{n-1+1/3}}{h_{n-1}} = 3\Delta_{n-1} - d_n. \end{cases}$$

Finally, (S) is equivalent to the system of linear equations and inequalities (I), (E), (4), (5)'.

3.5. We now study system (E) of *n* linear equations in *n* unknowns  $d_{i+1/3}$ ,  $i \in J$ , where  $d_0$  and  $d_n$  are parameters in the right-hand side. In terms of the unknowns  $D_i = d_{i+1/3}/h_i$ , the corresponding matrix, denoted by *H*, can

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be written as follows:

$$\begin{pmatrix} h_0 + H_0 & H_0 & & \\ H_0 & h_1 + H_0 + H_1 & H_1 & & \\ & \ddots & \ddots & \ddots & & \\ & & H_{n-3} & h_{n-2} + H_{n-3} + H_{n-2} & H_{n-2} \\ & & & H_{n-2} & h_{n-1} + H_{n-2} \end{pmatrix}.$$

This is a diagonally dominant, tridiagonal, symmetric, hence invertible matrix; thus, (E) has a unique solution. The unknowns  $D_i$ ,  $i \in J$ , hence  $d_{i+1/3}$ , will be written as linear combinations of  $d_0$  and  $d_n$ . Once  $d_{i+1/3}$ ,  $i \in J$ , are determined, we obtain  $d_i$  and  $d_{i-1+2/3}$  for  $i \in K$ , using (5)' and (4), respectively.

There remains to verify inequalities (I):

$$d_0 \ge 0, d_{0+1/3} - d_0 \ge 0, d_{1+1/3} - d_{0+1/3} \ge 0, \dots,$$
  
 $d_{n-1+1/3} - d_{n-2+1/3} \ge 0, d_n - d_{n-1+1/3} \ge 0.$ 

3.6. The system (E) can be written symbolically in the following manner:  $H\tilde{d} = \tilde{\Delta} - d_0 e_1 - d_n e_n$ , where  $\tilde{d}$ ,  $\tilde{\Delta}$ ,  $e_1$ ,  $e_n$  are respectively the vectors

$$\tilde{d} = (D_0, D_1, \dots, D_{n-1})^t, \qquad \tilde{\Delta} = (3\Delta_0, 3\Delta_1, \dots, 3\Delta_{n-1})^t, e_1 = (1, 0, \dots, 0)^t, \qquad e_n = (0, 0, \dots, 1)^t.$$

Therefore,  $\tilde{d} = H^{-1} \tilde{\Delta} - d_0 H^{-1} e_1 - d_n H^{-1} e_n$ .

The proposed method can be described by the following

Algorithm. (a) Express  $d_{0+1/3}, d_{1+1/3}, \ldots, d_{n-1+1/3}$  as functions of the two free parameters  $(d_0, d_n)$  by solving three systems of linear equations with the same tridiagonal matrix.

(b) Substitute these expressions in (I), which becomes a system of n+2 linear inequalities in the two unknowns  $(d_0, d_n)$ .

(c) Find a  $(d_0, d_n)$  in the polygon defined by (I).

(d) IF (I) is empty, THEN our problem has no solution: STOP.

ELSE calculate numerically  $d_{0+1/3}$ ,  $d_{1+1/3}$ , ...,  $d_{n-1+1/3}$ , then  $P_i$  for  $i \in J$ : STOP.

*Remarks.* (R<sub>1</sub>) Because of the uniqueness of the solution of (E), if the interpolation data are points chosen in the graph of a convex monotone polynomial of degree  $\leq 3$ , and if we choose  $d_0$  (resp.  $d_n$ ) equal to the value of the derivative of the interpolating function at the first node (resp. at the last node), the solution is the function itself.

 $(R_2)$  This method can be extended to splines of higher degrees and smoothness (for instance degree 4 and smoothness class  $C^3$ ). The number of parameters will be larger, thus giving rise to a larger linear system of inequalities.

 $(R_3)$  Dupin and Freville [4], for a uniform mesh, give some sufficient conditions for the existence of such shape-preserving  $C^2$  cubic polynomial interpolating splines and give a corresponding algorithm.

 $(R_4)$  For solving (I), there exist algorithms requiring a number of iterations which is bounded by a polynomial in terms of the size of the problem (Karmarkar's and Kachyan's algorithm).

Of course, the simplex method, or even a graphical method, can also be used. 3.7. Uniform subdivision. In this case,  $h_0 = h_1 = h_2 = \cdots = h_{n-2} = h_{n-1}$ , and (E) becomes

$$\int \frac{3}{2}d_{0+1/3} + \frac{1}{2}d_{1+1/3} = 3\Delta_0 - d_0,$$
  

$$\frac{1}{2}d_{i-1+1/3} + 2d_{i+1/3} + \frac{1}{2}d_{i+1+1/3} = 3\Delta_i \quad \text{for } i = 1, 2, \dots, n-2,$$
  

$$\int \frac{1}{2}d_{n-2+1/3} + \frac{3}{2}d_{n-1+1/3} = 3\Delta_{n-1} - d_n.$$

The corresponding matrix H is now also symmetric relative to its center. So  $(z_0, z_1, \ldots, z_{n-1})^t = H^{-1}e_n$  is obtained from  $(t_0, t_1, \ldots, t_{n-1})^t = H^{-1}e_1$  by  $z_i = t_{n-1-i}$  for  $i \in J$ .

3.8. **Example.** We interpolate ten points  $M_i(x_i, z_i)$  of the graph of the function f(t) = (-9t+2)/(4t+5) (cf. [8, p. 50]). This function is decreasing and convex on the interval [-1, 8]. We use a uniform subdivision  $h_i = 1$  for  $i = 0, 1, \ldots, 8$ .

	t <sub>i</sub>	-1	0	1	2	3	4	5	6	7	8
ſ	z <sub>i</sub>	11	0.4	-0.77777	-1.23077	-1.47059	-1.6190	-1.72	-1.79310	-1.84848	-1.89189

Taking  $d_0 = -27$  and  $d_9 = -0.03$ , we obtain a solution satisfying inequalities (I). Its graph is shown below.



#### Conclusions

A direct, inexpensive, constructive method for interpolating convex, monotone data with shape-preserving  $C^2$  cubic polynomial splines is proposed. Whenever the corresponding polyhedron in  $\mathbf{R}^2$  is nonempty, it determines the two degrees of freedom that occur in the classical cubic spline interpolation problem in such a way as to ensure the shape conditions. The technique seems promising for higher degrees and smoothness and from the point of view of accuracy.

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