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NUMERICAL APPROXIMATIONS OF ALGEBRAIC RICCATI EQUATIONS FOR ABSTRACT SYSTEMS MODELLED BY ANALYTIC SEMIGROUPS, AND APPLICATIONS

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ABSTRACT. This paper provides a numerical approximation theory of algebraic Riccati operator equations with unbounded coefficient operators A and B, such as arise in the study of optimal quadratic cost problems over the time interval $[0, \infty]$ for the abstract dynamics $\dot{y} = Ay + Bu$. Here, A is the generator of a strongly continuous analytic semigroup, and B is an unbounded operator with any degree of unboundedness less than that of A. Convergence results are provided for the Riccati operators, as well as for all the other relevant quantities which enter into the dynamic optimization problem. The present numerical theory is the counterpart of a known continuous theory. Several examples of partial differential equations with boundary/point control, where all the required assumptions are verified, illustrate the theory. They include parabolic equations with L_2-Dirichlet control, as well as plate equations with a strong degree of damping and point control.

1. INTRODUCTION: CONTINUOUS AND DISCRETE OPTIMAL CONTROL PROBLEMS; MAIN RESULTS; LITERATURE

1.1. **Statement of the continuous problem: Assumptions and main results.** Consider the following optimal control problem: Given the dynamical system,

$$y_t = Ay + Bu; \qquad y(0) = y_0 \in H, \tag{1.1}$$

minimize the quadratic functional

$$J(u, y) = \int_0^\infty [\|Ry(t)\|_Z^2 + \|u(t)\|_U^2]\, dt \tag{1.2}$$

over all $u \in L_2(0, \infty; U)$, with y a solution of (1.1) with control function u.

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Because of the paper's length, most of its technical proofs are given in the Supplement section of this issue. Should this hinder the reading, the original manuscript—which incorporates the proofs in the body of the paper in a consequential manner—is available by the authors upon request.

We shall make the following assumptions on (1.1), (1.2):

(i) H, U, and Z are Hilbert spaces.

(ii) $A: H \supset \mathscr{D}(A) \to H$ is the generator of a strongly continuous (s.c.) analytic semigroup e^{At} on H, $t > 0$, generally unstable on H, i.e., with $\omega_0 = \lim[(\ln\|\exp(At)\|)/t] > 0$ as $t \to +\infty$ in the uniform norm $\mathscr{L}(H)$, so that $\|e^{At}\| \leq Me^{(\omega_0+\varepsilon)t}$ for all $\varepsilon > 0$, $t \geq 0$, and M depending on $\omega_0 + \varepsilon$; we then consider throughout the translation $\widehat{A} = -A + \omega I$, $\omega =$ fixed $> \omega_0$, so that $\widehat{A}$ has well-defined fractional powers on H and $-\widehat{A}$ is the generator of an s.c. analytic semigroup $e^{-\widehat{A}t}$ on H satisfying $\|e^{-\widehat{A}t}\| \leq \widehat{M}e^{-\hat{\omega}t}$, $t \geq 0$; $\hat{\omega} = \omega - \omega_0 - \varepsilon > 0$; it will be used without further explicit note that $[\mathscr{D}(\widehat{A}), H]_{1-\theta} = \mathscr{D}(\widehat{A}^{\theta})$, $0 < \theta < 1$, e.g., [28, Theorem 1.25.3, p. 103].

(iii) $B: U \supset \mathscr{D}(B) \to [\mathscr{D}(A^*)]'$, the dual of $\mathscr{D}(A^*)$ with respect to the H-topology, A^* being the H-adjoint of A; more precisely, it is assumed that B^* is $\widehat{A}^{*\gamma}$-bounded, or equivalently,

$$(\widehat{A})^{-\gamma}B \in \mathscr{L}(U; H) \quad \text{for some constant } \gamma,\ 0 \leq \gamma < 1. \tag{1.3}$$

(iv) The operator R is bounded:

$$R \in \mathscr{L}(H, Z). \tag{1.4}$$

Hypotheses (i)–(iv) are assumed to be in force throughout the paper and shall not be repeated.

The next assumption guarantees existence of a unique optimal pair $\{u^0, y^0\}$ of the optimal control problem (1.1), (1.2):

(v) *Stabilizability Condition* (*S.C.*):

$$\begin{cases} \text{there exists } F \in \mathscr{L}(H; U) \text{ such that the s.c. analytic semigroup} \\ e^{(A+BF)t} \text{ (as guaranteed by (1.3), see below) is exponentially} \\ \text{stable on } H, \text{ i.e., } \|e^{(A+BF)t}\|_{\mathscr{L}(H)} \leq M_F e^{-\omega_F t} \text{ for some } \omega_F > 0. \end{cases} \tag{1.5}$$

(Equation (1.3) says that F^*B^* is $((\widehat{A})^*)^{\gamma}$-bounded; thus, since $\gamma < 1$, $A^* + F^*B^*$ is the generator of an s.c. analytic semigroup on H, and the same holds for $A + BF$.)

Finally, we shall make an assumption which guarantees uniqueness of the solution of the corresponding Algebraic Riccati Equation.

(vi) *Detectability Condition* (*D.C.*):

$$\begin{cases} \text{there exists } K \in \mathscr{L}(Z; H) \text{ such that the s.c. analytic} \\ \text{semigroup } e^{(A+KR)t} \text{ is exponentially stable on } H, \text{ i.e.,} \\ \|e^{(A+KR)t}\|_{\mathscr{L}(H)} \leq M_K e^{-\omega_K t} \text{ for some } \omega_k > 0. \end{cases} \tag{1.6}$$

The following main result for problem (1.1), (1.2) has been established in the literature either directly [9, 8] (from the Riccati equation to the control

problem), or through a variational argument [17] (from the control problem to the Riccati equation).

Theorem 1.0 [9, 17, 8]. (1) *Under the stabilizability condition* $(S.C.) = (1.5)$, *there is a unique solution* $\{u^0, y^0\}$ *of the optimal control problem* (1.1), (1.2).

(2) *Under the additional detectability condition* $(D.C.) = (1.6)$, *there is a unique nonnegative operator* $P = P^* \in \mathscr{L}(H)$ *such that, with* $u^0(t) = u^0(t; y_0)$ *and* $y^0(t) = y^0(t; y_0)$, $y_0 \in H$, *we have*

$$u^0(t) = -B^* P y^0(t), \qquad 0 < t < \infty, \tag{1.7}$$

where $(Bu, v)_H = (u, B^* v)_U$, *and* P *satisfies the following Algebraic Riccati Equation* (*A.R.E.*):

$$\begin{aligned}(A^* Px, y)_H + (PAx, y)_H + (R^* Rx, y)_H - (B^* Px, B^* Py)_U \\ = 0 \quad \forall x, y \in \mathscr{D}(\widehat{A}^\varepsilon), \ \text{any } \varepsilon > 0.\end{aligned} \tag{1.8}$$

Moreover,

(3)

$$(\widehat{A}^*)^{1-\varepsilon} P \in \mathscr{L}(H) \quad \forall \varepsilon > 0 \tag{1.9}$$

($\varepsilon = 0$, *if* A *is self-adjoint or normal, or similar to a normal operator*);

(4)

$$J(u^0, y^0) = (Py_0, y_0)_H; \tag{1.10}$$

(5)

$$B^* P \in \mathscr{L}(H, U); \tag{1.11}$$

(6) *the s.c. analytic semigroup* $\Phi(t) = e^{A_P t} = e^{(A - BB^* P)t}$ *generated by* $A_P = A - BB^* P$ *is exponentially stable,*

$$\|e^{(A-BB^*P)t}\|_{\mathscr{L}(H)} \le M_P e^{-\omega_P t} \quad \textit{for some } \omega_P > 0. \tag{1.12}$$

Further properties are collected in §2.1; see in particular identity (2.10) for P.

1.2. **Approximation of dynamics and related properties.** The main goal of this paper is to provide a numerical algorithm for the computation of the solution to the Algebraic Riccati Equation (A.R.E.) and to prove the desired convergence results.

1.2.1. *Approximation assumptions.*

Approximating subspaces. We introduce a family of approximating subspaces $V_h \subset H \cap \mathscr{D}(B^*)$, where h is a parameter of discretization which tends to zero, $0 < h \le h_0$. Let Π_h be the H-orthogonal projection of H onto V_h with the usual approximating property

$$\|\Pi_h x - x\|_H \to 0 \quad \text{for all } x \in H. \tag{1.13}$$

Approximation of A. Let $A_h: V_h \to V_h$ be an approximation of A which satisfies the following requirements (A.1) and (A.2):

(A.1) uniform analyticity; formulation in t-domain:

$$\|A_h^\theta e^{A_h t}\|_{\mathscr{L}(H)} \le \frac{c_\theta e^{(\omega_0+\varepsilon)t}}{t^\theta}, \qquad t > 0,\ 0 \le \theta \le 1 \tag{1.14a}$$

(the cases $0 < \theta < 1$ follow by interpolation from the endpoint cases $\theta = 0$, $\theta = 1$), with constant c_θ independent of h;

equivalent formulation in λ-domain: for $a > \omega_0$, there exists $\Sigma_{\text{app}}(A) = \Sigma_{\text{app}}(A; a; \theta_a)$, a closed triangular sector containing the axis $[-\infty, a]$ and delimited by the two rays $a + \rho e^{\pm i\theta_a}$ for some $\pi/2 < \theta_a \le \theta_0 < 2\pi$,

associated with the analytic semigroup e^{At}, and there exists h_a such that, if Σ^c denotes the complement of Σ in $\mathbb{C}$, then for all $0 < h \le h_a$ we have

$$\sigma(A_h) = \text{spectrum of } A_h \subset \Sigma_{\text{app}}(A),$$

$$\|R(\lambda, A_h)A_h^\theta\|_{\mathscr{L}(H)} \le \frac{C}{|\lambda - a|^\theta} \quad \forall \lambda \in \Sigma^c_{\text{app}}(A),\ 0 \le \theta \le 1 \tag{1.14b}$$

(the cases $0 < \theta < 1$ follow by interpolation from $\theta = 0$ and $\theta = 1$), $R(\lambda, \cdot)$ being the resolvent operator.

(A.2)

$$\|\Pi_h \widehat{A}^{-1} - \widehat{A}_h^{-1}\Pi_h\|_{\mathscr{L}(H)} \le Ch^s \quad \text{for some } s > 0 \text{ independent of } h. \tag{1.15}$$

Approximation of B. We shall assume that the operators $B: U \to [\mathscr{D}(A^*)]'$ and $B_h: U \to V_h$ satisfy the following approximating properties, where γ and s are defined by (1.3) and (1.15), respectively:

(A.3) ('inverse approximation property')

$$\|B^* x_h\|_U + \|B_h^* x_h\|_U \le Ch^{-\gamma s}\|x_h\|_H \quad \forall x_h \in V_h. \tag{1.16}$$

(A.4)

$$\|B^*(\Pi_h - I)x\|_U \le Ch^{s(1-\gamma)}\|x\|_{\mathscr{D}(A^*)}, \qquad x \in \mathscr{D}(A^*). \tag{1.17}$$

(A.5)

$$\|B^* x - B_h^*\Pi_h x\|_U \le Ch^{s(1-\gamma)}\|x\|_{\mathscr{D}(A^*)}, \qquad x \in \mathscr{D}(A^*). \tag{1.18}$$

(If, in particular, we take $B_h = \Pi_h B$, then (A.5) is contained in (A.4).)

(A.6)

$$\|B^*\Pi_h x\|_U \le C\|(\widehat{A}^*)^\gamma x\|_H, \qquad x \in \mathscr{D}((\widehat{A}^*)^\gamma). \tag{1.19}$$

Remark 1.1. Notice that assumptions (A.2)–(A.6) are standard approximation properties, where, moreover, in the case of spline approximations, s is the order of the differential operator A. They are consistent with the regularity of the

original operators A and B. Moreover, they are satisfied by typical schemes (finite elements, finite differences, mixed methods, spectral approximations). The property of uniform analyticity (A.1) is not a standard assumption and needs to be verified in each case. However, to our knowledge, it is satisfied for most of the schemes and examples which arise from analytic semigroup problems. For instance, a sufficient condition for (A.1) to hold true is the uniform coercivity of the bilinear form associated with A_h (see Lemma 4.2 in [14]). There are, however, a number of significant physical examples (e.g., structurally damped elastic systems), where the bilinear form is not coercive, while the underlying semigroups $e^{A_h t}$ are uniformly analytic (see §6).

1.2.2. *Consequences of approximating assumptions on* A. From (A.1) and (A.2), the following "rough" data estimates follow (see [14, Appendix] and [3]) (in a form to be used later):

$$\text{(1.20)} \quad \text{(i)} \quad \|e^{A_h t}\Pi_h - e^{At}\|_{\mathscr{L}(H)} = \|e^{A_h^* t}\Pi_h - \Pi_h e^{A^* t}\|_{\mathscr{L}(H)} \le C\frac{h^{s\theta}e^{(\omega_0+\varepsilon)t}}{t^\theta}, \quad t>0,\ 0\le\theta\le 1,\ \forall\varepsilon>0;$$

$$\text{(1.21)} \quad \text{(ii)} \quad \|R(\lambda, A) - R(\lambda, A_h)\Pi_h\|_{\mathscr{L}(H)} \le Ch^s, \qquad s>0,$$

uniformly in $\lambda \in \Sigma^c_{\text{app}}(A)$ (see definition of $\Sigma^c_{\text{app}}(A)$ below (1.14a));

$$\text{(1.22)} \qquad \text{(iii)} \quad \|e^{A_h^* t}\Pi_h - \Pi_h e^{A^* t}\|_{\mathscr{L}(\mathscr{D}(A^*);H)} \le Ch^s$$

uniformly in $t>0$ on compact intervals.

1.3. **Approximation of dynamics and of control problem. Related Riccati equation.** We now introduce an approximation of the control problem and of the corresponding Algebraic Riccati Equation.

Control problem. Given the approximating dynamics $y_h(t) \subset V_h$ satisfying

$$\text{(1.23)} \qquad \dot{y}_h(t) = A_h y_h(t) + B_h u(t), \qquad y_h(0) = \Pi_h y_0,$$

minimize over all $u \in L_2(0, \infty; U)$ the cost

$$\text{(1.24)} \qquad J(u, y_h) = \int_0^\infty [\|Ry_h(t)\|_Z^2 + \|u(t)\|_U^2]\,dt.$$

It will be shown in §§4.1 and 4.2 that the approximating dynamics (1.23) is stabilizable and detectable, in fact, uniformly in h. Thus, it is a standard finite-dimensional result (on V_h)—which is, in fact, contained in Theorem 1.0 when specialized to V_h—that there exists a unique Riccati approximating operator P_h, nonnegative self-adjoint, associated with (1.23), (1.24), solution of the following Algebraic Riccati Equation (A.R.E.)$_h$:

$$\text{(1.25)} \qquad \begin{aligned}(A_h^* P_h x_h, y_h)_H + (P_h A_h x_h, y_h)_H + (Rx_h, Ry_h)_Z \\ = (B_h^* P_h x_h, B_h^* P_h y_h)_U \quad \forall x_h, y_h \in V_h.\end{aligned}$$

Equation (1.25) leads to a standard matrix Riccati equation for the numerical solution of which there exists a vast literature (see, e.g., [12]). Further properties of the approximating problem will be collected in §2.2; see in particular the identity (2.20) for P_h.

1.4. Main results of approximating schemes: Theorems 1.1 and 1.2.

Theorem 1.1. *Assume*

I. *the continuous hypotheses* (1.3), $(S.C.) = (1.5)$, $(D.C.) = (1.6)$, *and*, *in addition,*

(1.26a) (a) *either* $R > 0$,

(1.26b) (b) *or else* $\widehat{A}^{-1}KR\colon H \to H$ *compact*;

(1.27a) (a) *either* $B^*\widehat{A}^{*-1}\colon H \to U$ *compact,*

(1.27b) (b) *or else* $F\colon H \to U$ *compact*;

II. *the approximation properties* (1.3) *and* $(A.1) = (1.14)$–$(A.6) = (1.19)$.

Then there exists $h_0 > 0$ *such that for all* $h < h_0$ *the solution* P_h *to* $(A.R.E.)_h$ *in* (1.25) *exists, is unique, and the following uniform bounds and convergence properties hold true*:

(i)

$$\|e^{A_{h,P_h}t}\|_{\mathscr{L}(H)} \le C_P e^{-\overline{\omega}_P t}, \qquad \overline{\omega}_P > 0 \text{ uniformly in } h \tag{1.28}$$

(*see Theorem* 4.6, *equation* (4.27)), *where*

$$A_{h,P_h} = A_h - B_h B_h^* P_h. \tag{1.29}$$

(ii)

$$\|(\widehat{A}_h^*)^{1-\varepsilon}P_h\|_{\mathscr{L}(H)} + \|(\widehat{A}_h^*)^{1/2-\varepsilon}P_h\widehat{A}_h^{1/2-\varepsilon}\|_{\mathscr{L}(H)} \le C_\varepsilon \quad \forall\varepsilon > 0, \text{ uniformly in } h \tag{1.30}$$

(*see Theorem* 4.7).

(iii)

$$\|P_h\Pi_h - P\|_{\mathscr{L}(H)} \le Ch^{\varepsilon_0} \to 0 \quad \text{as } h \downarrow 0, \ \forall\varepsilon_0 < s(1-\gamma) \tag{1.31}$$

(*see Theorem* 5.1, *equation* (5.1)).

(iv)

$$\|B_h^* P_h \Pi_h - B^* P\|_{\mathscr{L}(H;U)} \to 0 \quad \text{as } h \downarrow 0 \tag{1.32}$$

(*see Theorem* 5.3, *equation* (5.5)).

(v) *For all* $\varepsilon_0 < s(1-\gamma)$,

$$\sup_{t\ge 0} e^{\overline{\omega}_P t}\|u_h^0(t, \Pi_h x) - u^0(t, x)\|_{\mathscr{L}(H;U)} \le Ch^{\varepsilon_0} \to 0 \quad \text{as } h \downarrow 0, \ x \in H \tag{1.33}$$

(*see Corollary* 5.5, *equation* (5.9)).

(vi) *For all* $\varepsilon_0 < s(1-\gamma)$,

$$\|y_h^0(\cdot, \Pi_h x) - y^0(\cdot, \Pi x)\|_{\mathscr{L}(H; L_2(0,\infty; H))} \le Ch^{\varepsilon_0} \to 0 \quad \text{as } h \downarrow 0 \tag{1.34}$$

(*see Lemma* 5.4, *equation* (5.7)).

(vii)

$$\|y_h^0(\cdot, \Pi_h x) - y^0(\cdot, \Pi x)\|_{C([0,\infty]; U)} \to 0 \quad \text{as } h \downarrow 0, \ x \in H \tag{1.35}$$

(*see Lemma* 5.4, *equation* (5.8)).

(viii) *For all* $\varepsilon_0 < s(1-\gamma)$ *and for all* $\varepsilon > 0$,

$$\sup_{t \ge 0} t^{\varepsilon} e^{\overline{\omega}_P t} \|y_h^0(t, x) - y^0(t, x)\|_{\mathscr{L}(H)} \le Ch^{\varepsilon_0 \varepsilon} \to 0 \quad \text{as } h \downarrow 0 \tag{1.36}$$

(*see Theorem* 5.2, *equation* (5.4)).

(ix) *For all* $\varepsilon_0 < s(1-\gamma)$,

$$|J(u_h^0(\cdot, \Pi_h x), y_h^0(\cdot, \Pi_h x)) - J(u^0(\cdot, x), y^0(\cdot, x))| \le Ch^{\varepsilon_0} \to 0 \quad \text{as } h \downarrow 0 \tag{1.37}$$

(*consequence of property* (iii) *in* (1.31) *and of* (1.10) *and* (2.23)).

(x) *Moreover, if in addition, for some* $0 < \theta < 1$, $V_h \subset \mathscr{D}((\widehat{A}^*)^\theta)$ *and*

$$\|(\widehat{A}^*)^\theta x_h\|_H \le C_\theta \|(\widehat{A}_h^*)^\theta x_h\|_H \quad \text{or} \quad (\widehat{A}^*)^\theta (\widehat{A}_h^{*-1})^\theta \in \mathscr{L}(V_h, H), \tag{1.38}$$

then (*see Proposition* 5.6, *equations* (5.11), (5.12)),

$$\|(\widehat{A}^*)^\theta (P_h \Pi_h - P)x\|_H \to 0 \quad \text{as } h \downarrow 0, \ x \in H, \ 0 \le \theta < 1; \tag{1.39 (x$_1$)}$$

$$\|(\widehat{A}^*)^\theta (P_h \Pi_h - P)\widehat{A}^\theta x\|_H \to 0 \quad \text{as } h \downarrow 0, \ x \in H, \ 0 \le \theta < \tfrac{1}{2}. \tag{1.40 (x$_2$)}$$

Remark 1.2. Assumption (1.38) typically holds true with $\theta = \frac{1}{2}$. This is certainly the case when A is coercive and A_h is a standard Galerkin approximation of A, i.e., $(A_h x_h, y_h)_H = (Ax_h, y_h)_H$.

Remark 1.3. If A is self-adjoint (or, more generally, if $A = A_1 + A_2$, with A_1 self-adjoint and $A_2 \colon H \supset \mathscr{D}((-A_1)^{1-\varepsilon}) \to H$ is bounded), one can take $\theta = \frac{1}{2}$ in (1.40).

Theorem 1.2. (i) *The following uniform exponential stability holds true*:

$$\|e^{(A - BB_h^* P_h)t}\|_{\mathscr{L}(H)} \le \widehat{C} e^{-\hat{\omega}_P t}, \qquad \hat{\omega}_P > 0, \tag{1.41}$$

under the same assumptions as in Theorem 1.1.

(ii) *Moreover,*

$$\sup_{t \ge 0} e^{\hat{\omega}_P t} \|e^{(A - BB_h^* P_h)t} - e^{(A - BB^* P)t}\|_{\mathscr{L}(H)} \to 0 \quad \text{as } h \downarrow 0. \tag{1.42}$$

Theorem 1.1 provides the basic convergence results (with rates) for the optimal solutions of the approximating problem (1.23), (1.24), the corresponding

Riccati operators, and gain operators, to the same quantities of the original problem (1.1), (1.2).

The advantage of Theorem 1.2 is this: It states that the *original* system, once acted upon by the discrete feedback control law given by $u_h^*(t, \Pi_h x) = -B^* P_h y_h^*(t, \Pi_h x)$, yields (uniformly) exponentially stable solutions (see also [18, §4.3]).

Remark 1.4. Instead of the original inner product $(x_h, y_h)_H$, one can introduce an equivalent inner product $(x_h, y_h)_{H_h}$, where $c_1\|x_h\|_H \leq \|x_h\|_{H_h} \leq c_2\|x_h\|_H$. In some situations, it is more convenient to work with a discrete inner product $(\ ,\ )_{H_h}$ so as to simplify the computations for the adjoint operators for the discrete problem.

Remark 1.5. The literature on approximating schemes of optimal problems and related Riccati equations generally assumes (see [11])

(i) convergence properties of the "open loop" solutions, i.e., of the maps $u \to y$ of the continuous problem;
(ii) "uniform stabilizability/detectability" hypotheses for the approximating problems.

In contrast, our basic assumptions are:

(a) stabilizability/detectability hypotheses (S.C.)/(D.C.) of the continuous system;
(b) a "uniform analyticity" hypothesis (A.1) on the approximations.

Starting from (a) and (b), we then derive both the convergence properties of the open loop and the uniform stabilizability/detectability hypotheses—(i) and (ii) above—which are taken as assumptions in other treatments. Thus, the theory presented here is "optimal," in the sense that it assumes only what is strictly needed. Indeed, it can be shown that assumptions (A.1) and (S.C.)/(D.C.) are not only sufficient, but also necessary, for the main theorems presented here. These considerations are an important aspect of the entire theory, since, in the case where B is an unbounded operator, the requirement, corresponding to (i) above in other treatments, of convergence $L_h \to L$ of the open loop solutions (see (2.1), (2.11) below) is a very strong assumption. Generally, even when L is bounded and the scheme is consistent, it may well happen that the scheme is not even stable, i.e., L_h may not be uniformly bounded in h. The properties of the composition $e^{At}B$ may not be retained in the approximation $e^{A_h t}B_h$. Special care must be exercised in approximating B.

1.5. **Literature.** Within the literature concerned with approximation schemes for Algebraic Riccati Equations (A.R.E.) in infinite-dimensional spaces, we shall refer here only to works which focus on the case where the original free dynamics is modelled by an analytic semigroup e^{At}, as in the present paper. Approximation results for parabolic problems with distributed controls, i.e., with the operator B bounded ($\gamma = 0$ in (1.3)), are given in [1]. Next, [18] analyzed

the case of a parabolic problem defined on a bounded domain $\Omega \subset R^n$ with Dirichlet boundary control, via an abstract semigroup approach, where then $\gamma = \frac{3}{4} + \varepsilon$ in (1.3), i.e., the operator $\widehat{A}^{-(3/4+\varepsilon)}B$ is bounded for all $\varepsilon > 0$. This case may be viewed as a canonical illustration of the purely abstract situation where one has $\widehat{A}^{-\gamma}B$ bounded for $\gamma < 1$, and A has compact resolvent. Thus, *the treatment in* [18] *works equally well, mutatis mutandis, in the abstract case of an analytic semigroup generator* A *with compact resolvent, and with* $\widehat{A}^{-\gamma}B$ *bounded,* $0 \leq \gamma < 1$. There is a natural "cutting line" in the range of values of γ, which crucially bears on the degree of technical difficulties present in the treatment of the optimal control problem and its algebraic Riccati approximation: this is given by the special value $\gamma = \frac{1}{2}$.

Indeed, if $\widehat{A}^{-\gamma}B$ is bounded with $\gamma < \frac{1}{2}$, then the corresponding input $\rightarrow$ solution operator L is a priori continuous into $C([0, T]; H)$, so that all the trajectories of the continuous dynamical system are a priori pointwise continuous in time, and the operator B^*P is then a priori a bounded operator. Thus, in the case $\gamma < \frac{1}{2}$, a derivation of the A.R.E. may be given which closely parallels the pattern where B is a bounded operator. (The same applies to the case $\gamma = \frac{1}{2}$ if A is self-adjoint, or, more generally, it has a Riesz basis property on H.)

Instead, if $\widehat{A}^{-\gamma}B$ is bounded, with $\frac{1}{2} < \gamma$, the operator L is not continuous into $C([0, T]; H)$, i.e., the open loop trajectories are generally not pointwise continuous in time. Here, a main technical difficulty is therefore to show that, nevertheless, the gain operator B^*P is bounded. This is done by carefully analyzing the properties of the *optimal solutions* $y^0(t)$ (as distinguished from ordinary solutions $y(t)$) and by eventually showing via a boot strap argument that the *optimal* solutions $y^0(t)$ are pointwise continuous in time (unlike ordinary solutions $y(t)$ which are only, say, in $L_2(0, T; H)$).

The strategy outlined above for the case $\gamma > \frac{1}{2}$—which was successfully implemented in [18] in the canonical case of a parabolic equation with Dirichlet boundary control, where $\gamma = \frac{3}{4} + \varepsilon$, and A has compact resolvent—is also followed in the present abstract treatment, which moreover dispenses altogether with the assumption that A has compact resolvent. This will cover, in a unified framework, some physically significant examples (see §6) of damped elastic systems, where, in fact, A does not have compact resolvent. Thus, although much of the conceptual and technical developments of the present paper are a natural generalization of the arguments in [18], there are, however, also points of departure from [18] which require a different analysis, because of the now missing property that A have compact resolvent (see Remark 5.1), which was naturally built in the parabolic problem [18]. Like [18], our treatment here uses, as a starting point, two sources: on the one hand, the properties of the continuous optimal control problem and related Algebraic Riccati Equation following the variational approach of [17]; and, on the other hand, the approximation results for analytic semigroups (see [14, 15, 3]).

The importance of having a theory of approximation valid for $\gamma > \frac{1}{2}$ is fully justified by important physical problems, which are not solved by the direct, straightforward generalization from the case of B bounded to the case of $\widehat{A}^{-\gamma}B$ bounded with $\gamma \leq \frac{1}{2}$. Relevant examples where $\gamma > \frac{1}{2}$ include, in addition to parabolic problems with Dirichlet boundary control, also structurally damped elastic equations (see §6).

It was suggested from various sources that it would serve a useful purpose in the area to write a fully abstract explicit treatment of the general case $\gamma < 1$ modelled after [18]. This is done in the present paper. Generally, we shall rely again on a combination of ideas and techniques of the continuous problem [17], together with general approximating properties of analytic semigroups [14, 15].

2. Background material

2.1. **Continuous problem.** In order to prove the main results, Theorems 1.1 and 1.2, we shall use explicit representation formulas (in operator form) which describe the optimal solution pair $\{u^0, y^0\}$ and the Riccati operator. The purpose of this section is to provide these representation formulas. To this end, we introduce the solution operator L of problem (1.1) when $y_0 = 0$:

$$(Lu)(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau: \begin{cases} \text{continuous } L_2(0, T; U) \\ \quad \to L_2(0, T; H); \\ \text{continuous } L_\infty(0, T; U) \\ \quad \to C([0, T]; \mathscr{D}(\widehat{A}^{1-\gamma})). \end{cases} \tag{2.1, 2.2}$$

Its L_2-adjoint L^*: $(Lu, v)_{L_2(0,T;H)} = (u, L^*v)_{L_2(0,T;U)}$ is given by

$$(L^*v)(t) = B^* \int_t^T e^{A^*(\tau-t)}v(\tau)\,d\tau. \tag{2.3}$$

We shall similarly introduce the corresponding operators related to the generator $-\widehat{A} = A - \omega I$, rather than to the generator A:

$$(\widehat{L}u)(t) = \int_0^t e^{-\widehat{A}(t-\tau)}Bu(\tau)\,d\tau: \text{continuous } L_2(0, \infty; U) \to L_2(0, \infty; H); \tag{2.4}$$

$$(\widehat{L}^*v)(t) = B^* \int_t^\infty e^{-\widehat{A}^*(\tau-t)}v(\tau)\,d\tau: \text{continuous } L_2(0, \infty; H) \to L_2(0, \infty; U). \tag{2.5}$$

With ω fixed once and for all, as in the Introduction below (1.2) in the standing assumption (ii), we introduce the notation

$$\hat{u}^0(t, y_0) = e^{-\omega t}u^0(t, y_0), \qquad \hat{y}^0(t, y_0) = e^{-\omega t}y^0(t, y_0), \tag{2.6}$$

where $u^0(t, y_0), y^0(t, y_0)$ is the optimal pair of problem (1.1), (1.2), which originates at the point y_0 at time $t = 0$. We set

$$\widehat{\Phi}(t)x = \hat{y}^0(t, x) = e^{-\omega t}y^0(t, x) = e^{-\omega t}\Phi(t)x, \qquad x \in H. \tag{2.7}$$

Then, the optimal control and the corresponding optimal trajectory are given by the following explicit formulas [17, Theorem 2.4] (where the operator $\widehat{A}$ in [17] corresponds to $-\widehat{A}$ in the present paper):

$$\hat{y}^0(\cdot, y_0) = \widehat{\Phi}(\cdot)y_0 = [I + \widehat{L}\widehat{L}^*(R^*R + 2\omega P)]^{-1}\{e^{-\widehat{A}\cdot}y_0\} \in L_2(0, \infty; H); \tag{2.8}$$

$$\begin{aligned} -\hat{u}^0(\cdot, y_0) &= [I + \widehat{L}^*(R^*R + 2\omega P)\widehat{L}]^{-1}\widehat{L}^*[R^*R + 2\omega P]\{e^{-\widehat{A}\cdot}y_0\} \\ &\in L_2(0, \infty; U) \qquad \text{(2.9a)} \\ &= \{\widehat{L}^*[R^*R + 2\omega P]\hat{y}^0(\cdot, x)\}(t), \qquad \text{(2.9b)} \end{aligned}$$

with inverses well defined in $L_2(0, \infty; \cdot)$, $\cdot = H$ or U. The solution P to the A.R.E. in (1.8) satisfies the relation for $x \in H$ [17, Theorem 2.8] (where $\widehat{A}$ in [17] corresponds to $-\widehat{A}$ now),

$$Px = \int_0^\infty e^{-\widehat{A}^* t}[R^*R + 2\omega P]\widehat{\Phi}(t)x\,dt. \tag{2.10}$$

2.2. **Discrete problem.** In order to describe the solution to the discrete problem (1.23), (1.24), we similarly introduce the operators,

$$(L_h u)(t) = \int_0^t e^{A_h(t-\tau)}B_h u(\tau)\,d\tau\text{: continuous } L_2(0, T; U) \to L_2(0, T; V_h), \tag{2.11}$$

$$(L_h^* v)(t) = B_h^* \int_t^T e^{A_h^*(\tau - t)}\Pi_h v(\tau)\,d\tau, \tag{2.12}$$

with L_h^* being the L_2-adjoint of L_h in the sense of (2.2), and finally,

$$(\widehat{L}_h u)(t) = \int_0^t e^{-\widehat{A}_h(t-\tau)}B_h u(\tau)\,d\tau\text{: continuous } L_2(0, \infty; U) \to L_2(0, \infty; V_h), \tag{2.13}$$

$$(\widehat{L}_h^* v)(t) = B_h^* \int_t^\infty e^{-\widehat{A}_h^*(\tau - t)}\Pi_h v(\tau)\,d\tau\text{: continuous } L_2(0, \infty; V_h) \to L_2(0, \infty; U), \tag{2.14}$$

where we have defined, consistently with the continuous case below (1.2),

$$\widehat{A}_h = -A_h + \omega I. \tag{2.15}$$

Now let $\{u_h^0(t, x), y_h^0(t, x)\}$ be the optimal pair of the discrete optimal problem (1.23), (1.24), originating at the point $x \in V_h$ at the time $t = 0$, and set

$$\hat{u}_h^0(t, x) = e^{-\omega t}u_h^0(t, x), \qquad \hat{y}_h^0(t, x) = e^{-\omega t}y_h^0(t, x), \tag{2.16}$$

$$\widehat{\Phi}_h(t)x = \hat{y}_h^0(t, x) = e^{-\omega t}\Phi_h(t)x, \qquad x \in V_h, \tag{2.17}$$

consistently with (2.6), (2.7). Then, the discrete optimal pair of problem (1.23), (1.24), is given by the following explicit formulas with $y_{0h} = \Pi_h y_0 \in V_h$, which are the counterpart of formulas (2.8), (2.9) in the continuous case [18, p. 192]:

$$\hat{y}_h^0(\cdot, y_{0h}) = \hat{\Phi}_h(\cdot) y_{0h} = [I + \hat{L}_h \hat{L}_h^* \Pi_h (R^* R + 2\omega P_h)]^{-1} \{e^{-\hat{A}_h \cdot} y_{0h}\} \in L_2(0, \infty; V_h), \tag{2.18}$$

$$-\hat{u}_h^0(\cdot, y_{0h}) = [I + \hat{L}_h^* \Pi_h (R^* R + 2\omega P_h) \hat{L}_h]^{-1} \hat{L}_h^* \Pi_h [R^* R + 2\omega P_h] \{e^{-\hat{A}_h \cdot} y_{0h}\} \in L_2(0, \infty; U). \tag{2.19}$$

The corresponding Riccati operator P_h satisfies

$$P_h x = \int_0^\infty e^{-\hat{A}_h^* t} \mathrm{i}\Pi_h [R^* R + 2\omega P_h] \hat{\Phi}_h(t) x \, dt, \qquad x \in V_h. \tag{2.20}$$

The proofs of our main convergence results are based on a careful analysis of the convergence properties of the basic operators $\hat{L}_h$ and $\hat{L}_h^*$ to be given in §3. A few more formulas to be invoked later are:

$$\hat{y}_h^0(t, x) = e^{-\hat{A}_h t} x + \{\hat{L}_h \hat{u}_h^0(\cdot, x)\}(t), \qquad x \in V_h, \tag{2.21}$$

$$-\hat{u}_h^0(t, x) = \{\hat{L}_h^* [R^* R + 2\omega P_h] \hat{y}_h^0(\cdot, x)\}(t), \qquad x \in V_h, \tag{2.22}$$

counterparts of (2.8) and (2.9b), and finally, the counterpart of (1.10),

$$J(u_h^0(\cdot, x), y_h^0(\cdot, x)) = (P_h x, x), \qquad x \in V_h. \tag{2.23}$$

3. Convergence properties of the operators L_h and L_h^*; $\hat{L}_h$ and $\hat{L}_h^*$

Lemma 3.1. *Let assumptions* (A.1) = (1.14) *through* (A.6) = (1.19) *hold true. Then we have for all* $0 < h \le h_0$, *with constants independent of* h:

$$\text{(i)} \quad \|B_h^* e^{A_h^* t} \Pi_h - B^* e^{A^* t}\|_{\mathscr{L}(H; U)} \le C \frac{h^{s(1-\gamma)}}{t} e^{(\omega_0 + \varepsilon)t}, \qquad t > 0; \tag{3.1}$$

$$\text{(ii)} \quad \|B_h^* e^{A_h^* t} \Pi_h - B^* e^{A^* t}\|_{\mathscr{L}(\mathscr{D}(\hat{A}^*); U)} \le C h^{s(1-\gamma)} e^{(\omega_0 + \varepsilon)t}; \tag{3.2}$$

and by interpolation between (3.1) *and* (3.2), *with* $0 \le \theta \le 1$,

$$\text{(iii)} \quad \|B_h^* e^{A_h^* t} \Pi_h - B^* e^{A^* t}\|_{\mathscr{L}(\mathscr{D}((\hat{A}^*)^\theta); U)} \le C_\theta \frac{h^{s(1-\gamma)}}{t^{1-\theta}} e^{(\omega_0 + \varepsilon)t}, \qquad t > 0. \tag{3.3}$$

Moreover,

$$\text{(iv)} \quad \|B_h^* e^{A_h^* t} \Pi_h\|_{\mathscr{L}(H; U)} \le C \frac{e^{(\omega_0 + \varepsilon)t}}{t^\gamma}, \qquad t > 0; \tag{3.4}$$

$$\text{(v)} \quad \|B_h^* e^{A_h^* t} \Pi_h - B^* e^{A^* t}\|_{\mathscr{L}(H; U)} \le C \frac{h^{s(1-\gamma)\theta}}{t^{\theta + (1-\theta)\gamma}} e^{(\omega_0 + \varepsilon)t}, \qquad t > 0, \ 0 \le \theta \le 1; \tag{3.5}$$

and by interpolation between (3.5) *and* (3.2), *with* $0 \le r \le 1$,

$$\text{(3.6) (vi)} \quad \|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathscr{L}(\mathscr{D}((\widehat{A}^*)^r); U)} \le \frac{Ch^{s(1-\gamma)[r+(1-r)\theta]}}{t^{(1-r)[\theta+\gamma(1-\theta)]}} e^{(\omega_0+\varepsilon)t}, \quad t > 0, \ 0 \le \theta \le 1.$$

Proof. In the Supplement section. □

The proofs of the next two main results of this section are given in the Supplement.

Theorem 3.2. *With reference to the operators defined in* (2.1), (2.3), (2.11), (2.12), *we have the following results under the assumptions of Lemma* 3.1, *where* $0 < \theta < 1$ *is arbitrary and* $0 < h \le h_0$:

$$\text{(3.7)} \qquad \text{(i)} \quad \|L_h - L\|_{L_2} = \|L_h^* - L^*\|_{L_2} \le C_T h^{s(1-\gamma)\theta},$$

where the first norm is in $\mathscr{L}(L_2(0, T; U); L_2(0, T; H))$ *and similarly for the second norm with* U *and* H *interchanged*;

$$\text{(3.8)} \qquad \text{(ii)} \quad \|L_h - L\|_{\mathscr{L}(L_\infty(0,T;U); C([0,T];H))} \le C_T h^{s(1-\gamma)\theta}.$$

Theorem 3.3. *With reference to the operators defined in* (2.4), (2.5) *and* (2.13), (2.14), *the following convergence results hold true under the assumptions of Lemma* 3.1, *where* $0 \le \theta < 1$ *is arbitrary*:

$$\text{(3.9a)} \qquad \text{(i)} \quad \|\widehat{L}_h - \widehat{L}\|_{L_2} = \|\widehat{L}_h^* - \widehat{L}^*\|_{L_2} \le C_\theta h^{s(1-\gamma)\theta} \to 0 \quad \text{as } h \downarrow 0,$$

where the first norm is in $\mathscr{L}(L_2(0, \infty; U); L_2(0, \infty; H))$, *while the second norm is similar with* U *and* H *interchanged*;

$$\text{(3.9b)} \qquad \text{(ii)} \quad \|\widehat{L}_h - \widehat{L}\|_{\infty, C} = \|\widehat{L}_h^* - \widehat{L}^*\|_{\infty, C} \le Ch^{s(1-\gamma)\theta} \to 0 \quad \text{as } h \downarrow 0,$$

where the first norm is in $\mathscr{L}(L_\infty(0, \infty; U); C([0, \infty]; H))$, *while the second norm is similar with* U *and* H *interchanged.*

4. Perturbation results

The goal of the first two subsections is to show that the properties of analyticity and exponential stability of the semigroup $e^{A_F t}$ are preserved, uniformly in h, by its approximations.

4.1. Uniform analyticity and uniform exponential stability of the operators $A_{h,F_h} = A_h + B_h F_h$ and $A_{F_h} = A + BF_h$.

4.1.1. *Uniform analyticity.* Throughout this subsection, we let $F \in \mathscr{L}(H; U)$, and we consider the operator

$$\text{(4.1)} \qquad A_F \equiv A + BF : H \supset \mathscr{D}(A_F) \to H$$

which, in view of the standing assumption (1.3), generates likewise an s.c., analytic semigroup $e^{A_F t}$ on H, $t \ge 0$ (as justified below (1.5)). In later sections,

but not in this subsection, we shall consider the case where F is a stabilizing feedback operator. We let

$$\begin{aligned}\Sigma(A_F) &= \Sigma(A_F\,;\,a_F\,;\,\theta_F)\\ &= \text{(closed) triangular sector containing the axis}\\ &\quad [-\infty,\,a_F] \text{ and delimited by the two rays } a_F+\rho e^{\pm i\theta_F},\\ &\quad 0\le\rho<\infty,\ \text{ for some } \theta_F,\ \pi/2<\theta_F<2\pi,\end{aligned}\tag{4.2}$$

such that the spectrum $\sigma(A_F)\subset\Sigma(A_F)$. For a stabilizing F as in the assumption (S.C.) = (1.5), we have $a_F=-\omega_F<0$ in the notation of (1.5). In comparing $\Sigma(A_F)$ with the sector $\Sigma_{\text{app}}(A)\supset\sigma(A)$ in §1.2.2, we may say that we can always assume without loss of generality that one sector is contained in the other: If $a_F\le a$, then we can choose $\theta_a<\theta_F$, and then $\Sigma(F)\subset\Sigma_{\text{app}}(A)$; instead, if $a_F>a$, we can choose $\theta_a>\theta_F$, and then $\Sigma(F)\supset\Sigma_{\text{app}}(A)$. The first instance with $a_F=-\omega_F<a$ occurs if F is a stabilizing feedback operator. For the sake of definiteness, in the lemma below we *shall assume that* $a_F\le a$, *and so* $\Sigma(F)\subset\Sigma_{\text{app}}(A)$, the case which arises with F a stabilizing operator. Next, we consider the approximation of A_F defined by

$$A_{h,F_h}\equiv A_h+B_hF_h\colon V_h\to V_h\tag{4.3}$$

with $F_h\in\mathscr{L}(V_h\,;\,U)$ and A_h, B_h as in §1.2.1.

The next result shows that if $\{\|F_h\|\}$ is uniformly bounded in h, then the operators A_{h,F_h} defined by (4.3) generate "uniformly" analytic semigroups on H. With the stipulation above (4.3), we have the following lemma, whose proof is in the Supplement section.

Lemma 4.1. *Let* $\|F_h\|_{\mathscr{L}(V_h\,;\,U)}\le\text{const}$, *uniformly in* h. *Then, given* $1>\delta>0$, *there exist* $r_\delta>0$ *and* $h_\delta>0$ *such that for a suitable* $\Sigma^c_{\text{app}}(A)=\Sigma^c_{\text{app}}(A\,;\,a,\,\theta_a)$

$$\text{(i)}\quad \|R(\lambda,\,A_{h,F_h})\|_{\mathscr{L}(V_h)}\le\frac{1}{1-\delta}\|R(\lambda,\,A_h)\|_{\mathscr{L}(V_h)}\quad \forall\lambda\in\Sigma^c_{\text{app}}(A),\ |\lambda|\ge r_\delta,\ \forall h,\ 0<h<h_\delta\le h_a;\tag{4.4}$$

$$\text{(ii)}\quad \|R(\lambda,\,A_{h,F_h})\|_{\mathscr{L}(V_h)}\le\frac{C}{1-\delta}\,\frac{1}{|\lambda-a|},\tag{4.5}$$

λ *and* h *as in* (4.4);

$$\text{(iii)}\quad \|R(\lambda,\,A_{h,F_h})A^\theta_{h,F_h}\|_{\mathscr{L}(V_h)}\le\frac{C}{1-\delta}\,\frac{1}{|\lambda-a|^{1-\theta}},\qquad 0\le\theta\le1,\tag{4.6}$$

λ *and* h *as in* (4.4).

Remark 4.1. Lemma 4.1 on uniform analyticity holds true also for the operators

$$A_{F_h}\equiv A+BF_h,\qquad \mathscr{D}(A^*_{F_h})=\mathscr{D}((\widehat{A}^*)^\gamma),\tag{4.7}$$

in which case the proof is simpler.

4.1.2. *Uniform exponential stability of* A_{h,F_h} *and* A_{F_h}. In this subsection we assume explicitly the stabilizability condition (S.C.) = (1.5) that $F \in \mathscr{L}(H; U)$ is stabilizing, i.e.,

$$\|e^{A_F t}\|_{\mathscr{L}(H)} \subseteq M_F e^{-\omega_F t}, \qquad t \geq 0, \ \omega_F > 0. \tag{4.8}$$

We shall prove in the Supplement section that $e^{A_{h,F_h} t}$ and $e^{A_{F_h} t}$ are uniformly exponentially stable.

Theorem 4.2. *Assume* (4.8) *and, moreover, that with* $\|F_h\| \leq \text{const}$, *we have*

$$\|R(\lambda_0, A)B(F_h\Pi_h - F)\|_{\mathscr{L}(H)} \to 0 \quad \text{as } h \downarrow 0, \text{ for some } \lambda_0 \in \rho(A). \tag{4.9}$$

Then, given $\varepsilon > 0$, *there exists* $h_\varepsilon > 0$ *and a three-sided sector* $\Sigma_{\text{app}}(A_F)$, *which may be taken to be*

$$\Sigma_{\text{app}}(A_F) = \Sigma_{\text{app}}(A) \cap \{\operatorname{Re}\lambda \leq -\omega_F + \varepsilon\}, \tag{4.10}$$

$\Sigma_{\text{app}}(A) = \Sigma_{\text{app}}(A; a; \theta_a)$ *for some* $a > \omega_0$, *such that for all* $0 < h \leq h_\varepsilon \leq h_a$, *the operators in* (4.3) *satisfy*

$$\text{(i)} \quad \sigma(A_{h,F_h}) \subset \Sigma_{\text{app}}(A_F); \tag{4.11}$$

$$\text{(ii)} \quad \|e^{A_{h,F_h} t}\|_{\mathscr{L}(V_h)} \leq M_{1\varepsilon} e^{(-\omega_F + \varepsilon)t}, \qquad t \geq 0; \tag{4.12}$$

$$\text{(iii)} \quad \|R(\lambda, A_{h,F_h})A_{h,F_h}^{\beta}\|_{\mathscr{L}(V_h)} \leq \frac{C}{|\lambda + \omega_F - \varepsilon|^{1-\beta}}, \quad \lambda \in \Sigma_{\text{app}}^c(A_F), \ 0 \leq \beta \leq 1, \tag{4.13}$$

uniformly in h. *Thus,* $e^{A_{h,F_h} t}$ *is uniformly exponentially stable and, a fortiori from Lemma* 4.1, *uniformly analytic.*

Remark 4.2. We note explicitly that either one of the following conditions is sufficient for assumption (4.9) to hold true:

(i) either $F_h^* \to F^*$ strongly, and $B^* R(\lambda_0, A^*)$ compact $H \to U$ (as assumed in (1.27a)) (see [13, p. 151]);
(ii) or else $F_h \Pi_h \to F$ in the uniform operator norm $\mathscr{L}(H; U)$.

Remark 4.3. Theorem 4.2 holds true also for the operators $A_{F_h} = A + BF_h$ in (4.7), where, in fact, a simpler proof applies.

Remark 4.4. The role between assumption (4.8) and conclusion (4.12) can be reversed. The same proof yields the following result. Let

$$\|e^{A_{h,F_h} t}\|_{\mathscr{L}(H)} \leq M e^{-\delta t}, \qquad \delta > 0$$

(instead of (4.8)), and assume the convergence property (4.9) as before. Then, we obtain the conclusions corresponding to (4.11), (4.12), (4.13), with A_{h,F_h} replaced by A_F; in particular, we obtain

$$\|e^{A_F t}\| \leq \overline{M} e^{-\overline{\omega}_F t}, \qquad \overline{\omega}_F > 0.$$

4.2. **Uniform detectability of the generators $A_{h,K} = A_h + \Pi_h K R \Pi_h$; L_2-stability of $L_{h,K} B_h$ and $L_{h,K} \Pi_h K$.** Throughout this subsection, we consider the operator

$$A_K = A + KR, \qquad H \supset \mathscr{D}(A_K) \to H, \tag{4.14}$$

with $K \in \mathscr{L}(Z; H)$ satisfying the detectability condition (D.C.) = (1.6), so that

$$\|e^{A_K t}\|_{\mathscr{L}(H)} \le M_K e^{-\omega_K t} \quad \text{for some } \omega_K > 0. \tag{4.15}$$

Moreover, we assume throughout that $R(\lambda_0, A)KR$: compact $H \to H$, which is hypothesis (1.26b) of Theorem 1.1. We take the following approximations,

$$A_{h,K} = A_h + \Pi_h K R \Pi_h : V_h \to V_h, \tag{4.16}$$

i.e., $R_h = R\Pi_h \to R$ strongly, $K_h = \Pi_h K \to K$ strongly. Then in view of the compactness assumption above, we have $R(\lambda_0, A)KR(\Pi_h - I) \to 0$ in the uniform norm $\mathscr{L}(H)$, and from Theorem 4.2 and Lemma 4.1, we obtain at once

Proposition 4.3 (Uniform detectability). *The semigroups $e^{A_{h,K} t}$ are uniformly analytic and, moreover, uniformly exponentially stable in h. Given $\varepsilon > 0$, there exist $h_\varepsilon > 0$ and a three-sided sector*

$$\Sigma_{\text{app}}(A_K) \equiv \Sigma_{\text{app}}(A) \cap \{\operatorname{Re}\lambda \le -\omega_K + \varepsilon\} \tag{4.17}$$

such that

$$\sigma(A_{k,K}) \subset \Sigma_{\text{app}}(A_k) \tag{4.18}$$

and

$$\|e^{A_{h,K} t}\|_{\mathscr{L}(H)} \le M_{K,\varepsilon} e^{(-\omega_K + \varepsilon)t}, \qquad t > 0. \tag{4.19}$$

The following perturbation result will be invoked later.

Lemma 4.4. *For any $1 > \delta > 0$, there is $r_\delta > 0$ such that*

$$\|B_h^* R(\lambda, A_{h,K}^*)\|_{\mathscr{L}(H;U)} \le \frac{\text{const}}{(1-\delta) r_\delta^{1-\gamma}}, \qquad \lambda \in \Sigma_{\text{app}}^c(A_K), \ |\lambda| \ge r_\delta. \tag{4.20}$$

Proof. Since, as we say in (S.4.4) of the Supplement,

$$R(\lambda, A_{h,K}) = [I - R(\lambda, A_h)\Pi_h K R \Pi_h]^{-1} R(\lambda, A_h),$$

an estimate like (S.4.5) holds true for the inverse term $[\ ]^{-1}$ for $|\lambda| \ge r_\delta$, and thus, taking adjoints,

$$\begin{aligned} \|B_h^* R(\lambda, A_{h,K}^*)\|_{\mathscr{L}(H;U)} &\le \frac{1}{1-\delta} \|B_h^* R(\lambda, A_h^*)\|_{\mathscr{L}(H;U)} \\ &\le \frac{1}{1-\delta} \frac{C}{|\lambda - a|^{1-\gamma}}, \qquad \lambda \text{ as in (4.20)}, \end{aligned} \tag{4.21}$$

recalling (S.4.2) of the Supplement, and (4.20) follows from (4.21). □

As a corollary to Lemma 4.4, we shall derive a stability result involving the approximating operators

$$(L_{h,K}f_h)(t) \equiv \int_0^t e^{A_{h,K}(t-\tau)} f_h(\tau)\,d\tau \colon L_2(0,\infty;V_h) \to \text{itself}. \tag{4.22}$$

Corollary 4.5. *With reference to* (4.22), *we have*

$$\text{(i)} \quad \|L_{h,K}B_h u\|_{L_2(0,\infty;H)} \le C\|u\|_{L_2(0,\infty;U)}, \tag{4.23}$$

$$\text{(ii)} \quad \|L_{h,K}\Pi_h K z\|_{L_2(0,\infty;H)} \le C\|z\|_{L_2(0,\infty;Z)}. \tag{4.24}$$

Proof. (i) By the Parseval equality, it suffices to show that

$$\|R(\lambda, A_{h,K})B_h\hat{u}(\lambda)\|_H \le C\|\hat{u}(\lambda)\|_U \tag{4.25}$$

for all $\{\lambda : \operatorname{Re}\lambda = 0\} \subset \Sigma^c_{\text{app}}(A_K)$. But (4.25) holds true by duality on (4.20). The proof for (ii) is identical. □

4.3. **Uniform stability of the feedback semigroup** $\exp(A_{h,P_h}t)$. Let P be the Riccati operator asserted by Theorem 1.0. With reference to the approximating optimal control problem (1.23), (1.24), we may now say—on the basis of the results of §§4.1 and 4.2—that the approximating dynamics (1.23) is stabilizable and detectable, in fact uniformly in h. Thus, it is a standard result (contained in Theorem 1.0 when interpreted on V_h) that there exists a unique Riccati (approximating) operator P_h associated with (1.23), (1.24), solution of the $(\text{A.R.E.})_h$ in (1.25). The goal of this subsection is to prove (in the Supplement) that the corresponding operator

$$A_{h,P_h} \equiv A_h - B_h B_h^* P_h \tag{4.26}$$

satisfies the uniform exponential stability condition (1.28) of Theorem 1.1.

Theorem 4.6 (Uniform stability of $e^{A_{h,P_h}t}$). *Under the sets of assumptions* I *and* II *of Theorem* 1.1, *there exist* $\overline{\omega}_P > 0$, $\overline{M}_P > 0$ *such that*

$$\|\Phi_h(t)\|_{\mathscr{L}(H)} = \|e^{A_{h,P_h}t}\|_{\mathscr{L}(H)} \le \overline{M}_P e^{-\overline{\omega}_P t}, \qquad t \ge 0, \tag{4.27}$$

thereby proving (1.28) *of Theorem* 1.1. *Recall from* (4.26) *and* (2.16) *that* $\exp(A_{h,P_h})t = \Phi_h(t)$.

4.4. **Uniform regularity of** P_h. We recall that P_h is given, as described at the beginning of §4.3. In this section we shall show (in the Supplement) that P_h is uniformly bounded not only in $\mathscr{L}(H)$, as already claimed by Lemma S.4.3 of the Supplement, equation (S.4.19), but in fact in a stronger norm. In particular, we shall prove statement (1.30) of the main Theorem 1.1.

Theorem 4.7. *We have, uniformly in* $h \downarrow 0$,

$$\text{(i)} \quad \|(\widehat{A}_h^*)^\theta P_h\|_{\mathscr{L}(H)} \le \text{const}_\theta, \qquad 0 \le \theta < 1; \tag{4.28}$$

$$\text{(ii)} \quad \|B_h^* P_h\|_{\mathscr{L}(H;U)} \le \text{const}; \tag{4.29}$$

$$\text{(iii)} \quad \|(\widehat{A}_h^*)^\theta P_h \widehat{A}_h^\theta\|_{\mathscr{L}(H)} \le \text{const}_\theta, \qquad 0 \le \theta < \tfrac{1}{2}; \tag{4.30a}$$

or equivalently,

$$\text{(4.30b)} \qquad \| P_h \|_{\mathscr{L}(\mathscr{D}((\widehat{A}_h^*)^\theta);[\mathscr{D}((\widehat{A}_h^*)^\theta)]')} \leq \text{const}_\theta .$$

Corollary 4.8. *If for some* $\theta < 1$ *we have*

$$\text{(4.31)} \qquad \|(\widehat{A}^*)^\theta x_h\|_h \leq C\|(\widehat{A}_h^*)^\theta x_h\|_H \quad \forall x_h \in V_h ,$$

then

$$\text{(4.32)} \qquad \|(\widehat{A}^*)^\theta P_h\|_{\mathscr{L}(H)} \leq \text{const}_\theta$$

and, consequently,

$$\text{(4.33)} \qquad \|(\widehat{A}^*)^\theta P_h \widehat{A}^\theta\|_{\mathscr{L}(H)} \leq \text{const}_\theta , \qquad 0 \leq \theta < \tfrac{1}{2} .$$

Remark 4.5. Assumption (4.31) holds true with any $\theta \leq \frac{1}{2}$ for Galerkin approximations.

The operators A_{h,P_h} generate a family of semigroups $e^{A_{h,P_h}t}$ which are uniformly stable by Theorem 4.6. They are also uniformly analytic by the next corollary.

Corollary 4.9. *The operators* $A_{h,P_h} = A_h - B_h B_h^* P_h$ *in* (4.26) *generate a uniformly analytic family* $e^{A_{h,P_h}t}$ *of semigroups.*

Proof. We just apply Lemma 4.1 with $F_h = B_h^* P_h$, which is legal by the uniform estimate (4.29) of Theorem 4.7. □

5. Uniform convergence $P_h \Pi_h \to P$

5.1. Uniform convergence $P_h \Pi_h \to P$ of Riccati operators.

Theorem 5.1 (Property (1.31) of main Theorem 1.1). *For any* $\varepsilon_0 < s(1-\gamma)$,

$$\text{(5.1)} \qquad \|P_h \Pi_h - P\|_{\mathscr{L}(H)} \leq C h^{\varepsilon_0} \to 0 \quad \textit{as } h \downarrow 0 .$$

The proof of Theorem 5.1 is given in the Supplement. It is based, among other things, on the following four operators:

$$\text{(5.2)} \qquad A_P = A - BB^* P , \qquad A_{h,P} = A_h - B_h B^* P ,$$

$$\text{(5.3)} \qquad A_{P_h} = A - BB_h^* P_h , \qquad A_{h,P_h} = A_h - B_h B_h^* P_h .$$

The first and the fourth, which were already defined by (1.12) and (4.26), refer to optimal dynamics, continuous and discrete. The second and third are defined here for the first time. They define competitive dynamics.

As a corollary of part of the proof of Theorem 5.1, we obtain (see Supplement)

Theorem 5.2 (Property (1.36) of main Theorem 1.1). *For any* $\varepsilon > 0$ *we have*

$$\begin{aligned} &\sup_{0\le t} e^{\overline{\omega}_P t} t^{\varepsilon} \| e^{A_{h,P_h} t} \Pi_h - e^{A_P t} \|_{\mathscr{L}(H)} \\ &\qquad \le C h^{\varepsilon_0 \varepsilon} \to 0 \quad \text{as } h \downarrow 0, \ \forall \varepsilon_0 < s(1-\gamma), \end{aligned} \tag{5.4}$$

$$\Phi(t)x = e^{A_P t} x = y^0(\cdot, x), \qquad \Phi_h(t)\Pi_h x = e^{A_{h,P_h} t}\Pi_h x = y_h^0(\cdot, \Pi_h x).$$

Remark 5.1. The present proof of Theorem 5.1 is somewhat different from that in [18] and, moreover, it applies to the case of more general approximating assumptions (not necessarily Galerkin) without requiring that A has compact resolvent as in [18]. One may also extend to the present case of B unbounded the original approach of [11] given there for B bounded, but this route—based on the finite time problem—is much longer.

5.2. **Uniform convergence $B_h^* P_h \Pi_h \to B^* P$ of gain operators.**

Theorem 5.3 (Property (1.32) of main Theorem 1.1). *We have*

$$\| B_h^* P_h \Pi_h - B^* P \|_{\mathscr{L}(H;U)} \to 0 \quad \text{as } h \downarrow 0, \tag{5.5}$$

where the rate is h^{ε_0}, $\varepsilon_0 < s(1-\gamma)$ *if* $\gamma < \frac{1}{2}$.

The proof of Theorem 5.3, given in the Supplement, requires the following lemma.

Lemma 5.4 (Properties (1.33), (1.36), (1.37) of main Theorem 1.1). *We have for* $\varepsilon_0 < s(1-\gamma)$:

(i)

$$\| u_h^0(\cdot, \Pi_h x) - u^0(\cdot, x) \|_{\mathscr{L}(H; L_2(0,\infty;U))} \le C h^{\varepsilon_0} \to 0 \quad \text{as } h \downarrow 0, \tag{5.6a}$$

equivalently by (S.4.20) *with reference to* (5.3),

$$\| B_h^* P_h e^{A_{h,P_h} t} \Pi_h - B^* P e^{A_P t} \|_{\mathscr{L}(H; L_2(0,\infty;U))} \le C h^{\varepsilon_0} \quad \text{as } h \downarrow 0; \tag{5.6b}$$

(ii)

$$\| \Phi_h(\cdot)\Pi_h - \Phi(\cdot) \|_{\mathscr{L}(H; L_2(0,\infty;H))} \le C h^{\varepsilon_0} \quad \text{as } h \downarrow 0; \tag{5.7}$$

(iii) *for any finite* $0 < T < \infty$, *the following result, which complements* (5.5), *holds true*:

$$\| [\Phi_h(\cdot)\Pi_h - \Phi(\cdot)]x \|_{C([0,\infty];H)} \to 0 \quad \text{as } h \downarrow 0, \ x \in H, \tag{5.8}$$

where $y_h^0(\cdot, \Pi_h x) = \Phi_h(\cdot)\Pi_h x$, $y^0(\cdot, x) = \Phi(\cdot)x$ (*see* (2.17), (2.7)).

5.3. **Uniform convergence $u_h^0 \to u^0$.**

Corollary 5.5 (Property (1.33) of main Theorem 1.1). *We have* (*see Supplement*)

$$\| u_h^0(\cdot, \Pi_h x) - u^0(\cdot, x) \|_{\mathscr{L}(H; C([0,\infty];U))} \le C h^{\varepsilon_0} e^{-\overline{\omega}_P t} \quad \text{as } h \downarrow 0, \ x \in H. \tag{5.9}$$

5.4. **Convergence** $(\widehat{A}^*)^\theta(P_h\Pi_h - P)x \to 0$. So far, we have shown conclusions (1.28)–(1.37) of the main Theorem 1.1. We now complete the proof of Theorem 1.1 by establishing properties (1.39) and (1.40) as well.

Proposition 5.6 (Properties (1.39) and (1.40) of Theorem 1.1). *Assume the approximating property* (1.38), *rewritten here as*

$$(5.10)\qquad \begin{aligned} &(\widehat{A}^*)^\theta(\widehat{A}_h^{*-1})^\theta \in \mathscr{L}(V_h; H), \quad \textit{or} \\ &\|(\widehat{A}^*)^\theta x_h\|_H \le C_\theta \|(\widehat{A}_h^*)^\theta x_h\|_H, \qquad 0 \le \theta < 1, \ \forall x_h \in V_h. \end{aligned}$$

Then

$$(5.11)(\text{i})\qquad \|(\widehat{A}^*)^\theta(P_h\Pi_h - P)x\|_H \to 0 \quad \textit{as } h \downarrow 0, \ x \in H, \ 0 \le \theta < 1;$$

$$(5.12)(\text{ii})\qquad \|(\widehat{A}^*)^\theta(P_h\Pi_h - P)\widehat{A}^\theta x\|_H \to 0 \quad \textit{as } h \downarrow 0, \ x \in H, \ 0 \le \theta < \tfrac{1}{2}.$$

The proof is given in the Supplement. After this, the proof of the main Theorem 1.1 is complete. □

Completion of the proof of Theorem 1.2 is given in the Supplement.

6. Examples

In this section, we illustrate the applicability of Theorems 1.1 and 1.2 to some partial differential equations problems which exhibit the properties required by the theory. A few canonical cases of diffusion/heat equations with boundary control and strongly damped plate equations with point or boundary controls will be treated. For lack of space we shall concentrate only on the three most representative examples, which exemplify the following situations:

(i) $\gamma > \frac{1}{2}$ (heat equation with Dirichlet boundary conditions);
(ii) $\gamma > \frac{1}{2}$ and noncoercive nature of the problem (damped plate equation with point control);
(iii) $\gamma < \frac{1}{2}$ and $R(\lambda, A)$ noncompact (Kelvin-Voigt plate model with point control).

Other examples, as the ones given in [19], can be treated in the same manner.

6.1. **Heat equation with Dirichlet boundary control.** This problem has been considered in [18]. For the sake of completeness we shall show how it fits into the present theory.

6.1.1. *Continuous problem.* Let $\Omega \subset R^n$ be an open bounded domain with sufficiently smooth boundary Γ. In Ω, we consider the Dirichlet mixed problem for the heat equation in the unknown $y(t, x)$:

$$(6.1\text{a})\qquad y_t = \Delta y + c^2 y \quad \text{in } (0, T] \times \Omega \equiv Q,$$

$$(6.1\text{b})\qquad y(0, \cdot) = y_0 \quad \text{in } \Omega,$$

$$(6.1\text{c})\qquad y|_\Sigma = u \quad \text{in } (0, T] \times \Gamma \equiv \Sigma,$$

with boundary control $u \in L_2(\Sigma)$ and $y_0 \in L_2(\Omega)$. The cost functional which we wish to minimize is

$$J(u, y) = \int_0^\infty \{\|y(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)}\}\, dt. \tag{6.2}$$

Abstract setting (see [17]). To put problem (6.1), (6.2) into the abstract setting of the preceding sections, we introduce the following operators and spaces:

$$Ah = \Delta h + c^2 h; \qquad \mathscr{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \tag{6.3}$$

$$Z = H = L_2(\Omega); \qquad U = L_2(\Gamma), \tag{6.4}$$

$$Bu = -AD_1 u; \qquad R = I, \tag{6.5}$$

where D_1 (Dirichlet map) is defined by

$$h = D_1 g \quad \text{iff} \quad (\Delta + c^2)h = 0 \text{ in } \Omega; \qquad h|_\Gamma = g, \tag{6.6}$$

$$D_1\text{: continuous } L_2(\Gamma) \to H^{1/2}(\Omega) \subset H^{1/2-2\varepsilon}(\Omega) \equiv \mathscr{D}(A_D^{1/4-\varepsilon}) \quad \forall \varepsilon > 0, \tag{6.7}$$

$$A_D h = -\Delta h, \qquad \mathscr{D}(A_D) = H^2(\Omega) \cap H_0^1(\Omega), \tag{6.8}$$

by elliptic theory, and where A in (6.5) is the isomorphic extension of A in (6.3), from, say, $L_2(\Omega) \to [\mathscr{D}(A)]'$.

The approximation framework for problem (6.1) and the verification of all required assumptions for both the continuous as well as the discrete problem are given in the Supplement.

6.2. **Structurally damped plates with point control.**

6.2.1. *Continuous problem. The case* $\alpha = \frac{1}{2}$ [4, 5]. Consider the following model of a plate equation in the deflection $w(t, x)$, where $\rho > 0$ is any constant:

$$w_{tt} + \Delta^2 w - \rho \Delta w_t = \delta(x - x^0)u(t) \quad \text{in } (0, T] \times \Omega = Q, \tag{6.9a}$$

$$w(0, \cdot) = w_0; \; w_t(0, \cdot) = w_1 \quad \text{in } \Omega, \tag{6.9b}$$

$$w|_\Sigma \equiv \Delta w|_\Sigma \equiv 0 \quad \text{in } (0, T] \times \Gamma = \Sigma, \tag{6.9c}$$

with load concentrated at the interior point x^0 of an open bounded (smooth) domain Ω of R^n, $n \le 3$. Regularity results for problem (6.9), and other problems of this type, are given in [27]. Consistently, the cost functional we wish to minimize is

$$J(u, w) = \int_0^\infty \{\|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + \|u(t)\|^2_{L_2(\Gamma)}\}\, dt, \tag{6.10}$$

where $\{w_0, w_1\} \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega)$.

Abstract setting. To put problems (6.9), (6.10) into the abstract setting of the preceding sections, we introduce the strictly positive definite operator

$$\mathscr{A} h = \Delta^2 h; \qquad \mathscr{D}(\mathscr{A}) = \{h \in H^4(\Omega) : |h|_\Gamma = \Delta h|_\Gamma = 0\} \tag{6.11}$$

and select the following spaces and operators:

$$(6.12)\quad H \equiv \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega) = [H^2(\Omega) \cap H_0^1(\Omega)] \times L_2(\Omega); \qquad U = \mathbb{R}^1,$$

$$(6.13)\quad A = \begin{bmatrix} 0 & I \\ -\mathscr{A} & -\rho\mathscr{A}^{1/2} \end{bmatrix}; \qquad Bu = \begin{bmatrix} 0 \\ \delta(x - x^0)u \end{bmatrix}; \qquad R = I$$

to obtain the abstract model (1.1), (1.2). Again, approximation framework and verification of all required assumptions are given in the Supplement.

6.3. **Kelvin-Voigt plate equation with point control.**

6.3.1. *Continuous problem. The case* $\alpha = 1$ [4, 5]. The Kelvin-Voigt model for a plate equation in the deflection $w(t, x)$ is

$$\begin{aligned}
&(6.14a) && w_{tt} + \Delta^2 w + \rho\Delta^2 w_t = \delta(x - x^0)u(t) && \text{in } (0, T] \times \Omega = Q,\\
&(6.14b) && w(0, \cdot) = w_0;\ w_t(0, \cdot) = w_1 && \text{in } \Omega,\\
&(6.14c) && \Delta w|_\Sigma + (1 - \mu)B_1 w \equiv 0 && \text{in } (0, T] \times \Gamma = \Sigma,\\
&(6.14d) && \left.\frac{\partial \Delta w}{\partial \nu}\right|_\Sigma + (1 - \mu)B_2 w \equiv 0 && \text{in } \Sigma,
\end{aligned}$$

with $0 < \mu < \frac{1}{2}$ the Poisson modulus and $\rho > 0$ any constant, where again x^0 is an interior point of the open bounded $\Omega \subset R^n$, $n \le 2$. The boundary operators B_1 and B_2 are zero for $n = 1$, and for $n = 2$ [16]:

$$(6.15)\quad \begin{aligned} B_1 w &= 2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx};\\ B_2 w &= \frac{\partial}{\partial \tau}[(\nu_1^2 - \nu_2^2)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})], \end{aligned}$$

where $\partial/\partial\tau$ is the tangential derivative. Regularity results for problem (6.14) are given in [27]. Consistently with these, we take the cost functional to be the same as (6.10) with $\{w_0, w_1\} \in H^2(\Omega) \times L_2(\Omega)$.

Abstract setting. We introduce the nonnegative self-adjoint operator

$$(6.16)\quad \mathscr{A}h = \Delta^2 h,$$

$$\mathscr{D}(\mathscr{A}) = \left\{ h \in H^4(\Omega) : \Delta h + (1 - \mu)B_1 h|_\Gamma = 0;\ \frac{\partial \Delta h}{\partial \nu} + (1 - \mu)B_2 h|_\Gamma = 0 \right\},$$

and select the following spaces and operators:

$$(6.17)\quad H = \mathscr{D}(\mathscr{A}^{1/2}) \times L_2(\Omega) = H^2(\Omega) \times L_2(\Omega); \qquad U = \mathbb{R}^1,$$

$$(6.18)\quad A = \begin{bmatrix} 0 & I \\ -\mathscr{A} & -\rho\mathscr{A} \end{bmatrix}; \qquad Bu = \begin{bmatrix} 0 \\ \delta(x - x^0)u \end{bmatrix}; \qquad R = I$$

to obtain the abstract model (1.1), (1.2). Approximation framework as well as verification of all required assumptions are given in the Supplement.

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Supplement to

NUMERICAL APPROXIMATIONS OF ALGEBRAIC RICCATI EQUATIONS FOR ABSTRACT SYSTEMS MODELLED BY ANALYTIC SEMIGROUPS, AND APPLICATIONS

I. LASIECKA AND R. TRIGGIANI

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Section 3

Proof of Lemma 3.1. (i) We compute, with Π_h the orthogonal projection H onto V_h, after adding and subtracting

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(H;U)} \le \|B_h^*(e^{A_h^* t}\Pi_h - \Pi_h e^{A^* t})\|_{\mathcal{L}(H;U)} + \|B^* e^{A^* t} - B_h^*\Pi_h e^{A^* t}\|_{\mathcal{L}(H;U)} \tag{S.3.1}$$

(using (A.3) = (1.16), the rough data estimate (1.20) for $\Theta = 1$ and $\Pi_h^2 = \Pi_h$ in the first term of (S.3.1); and (A.5) = (1.18) on the second term of (S.3.1))

$$\le C\, h^{-\gamma s}\, \frac{h^s e^{(\omega_0+\varepsilon)t}}{t} + C\, h^{s(1-\gamma)} \|A^* e^{A^* t}\|_{\mathcal{L}(H)}$$

$$\le C\, h^{s(1-\gamma)}\, \frac{e^{(\omega_0+\varepsilon)t}}{t}, \tag{S.3.2}$$

where in the last step we have used the analyticity of $e^{A^* t}$. Thus (3.1) is proved.

(ii) Similarly,

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(\mathcal{D}(A^*);U)} \le \|B_h^*(e^{A_h^* t}\Pi_h - \Pi_h e^{A^* t})\|_{\mathcal{L}(\mathcal{D}(A^*);U)}$$

$$+ \|B^* e^{A^* t} - B_h^*\Pi_h e^{A^* t}\|_{\mathcal{L}(\mathcal{D}(A^*);U)} \tag{S.3.3}$$

(using (A.3) = (1.16) and (1.22) on the first term of (S.3.3); and (A.5) = (1.18) on the second term of (S.3.3)),

$$\le C\, h^{-\gamma s} h^s + C\, h^{s(1-\gamma)}\, e^{(\omega_0+\varepsilon)t}, \tag{S.3.4}$$

and *a fortiori* (3.2) follows from (S.3.4).

(iii) Eq. (3.3) follows from (3.1) and (3.2) by use of the interpolation (moment) inequality [17, p. 19].

(iv) First, from the full assumption (A.3) = (1.16) and uniform analyticity (A.1) = (1.14a), we obtain

$$\|B_h^* e^{A_h^* t}\Pi_h - B^*\Pi_h e^{A^* t}\|_{\mathcal{L}(H;U)} \le C\, h^{-\gamma s}\, e^{(\omega_0+\varepsilon)t}. \tag{S.3.5}$$

Next, we shall obtain

$$\|B_h^* e^{A_h^* t}\Pi_h - B^*\Pi_h e^{A^* t}\|_{\mathcal{L}(H;U)} \le C\, \frac{h^{s(1-\gamma)}}{t}\, e^{(\omega_0+\varepsilon)t} \quad t > 0, \tag{S.3.6}$$

through a computation similar to the ones above. Indeed, adding and subtracting

$$\|B_h^* e^{A_h^* t}\Pi_h - B^*\Pi_h e^{A^* t}\|_{\mathcal{L}(H;U)} \le \|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(H;U)} + \|B^*(I-\Pi_h)e^{A^* t}\|_{\mathcal{L}(H;U)} \tag{S.3.7}$$

(using (3.1) of part (i) on the first term of (S.3.7) and (A.4) = (1.17) on the second term of (S.3.7)),

$$\le C\, \frac{h^{s(1-\gamma)}}{t} \left\{ e^{(\omega_0+\varepsilon)t} + \|A^* e^{A^* t}\|_{\mathcal{L}(H)} \right\}, \tag{S.3.8}$$

and (S.3.6) follows from (S.3.8) by analyticity of $e^{A^* t}$. Next, we raise (S.3.5) to the power $(1-\gamma)$, we raise (S.3.6) to the power γ, and we multiply the resulting expressions together. This way we obtain

$$\|B_h^* e^{A_h^* t}\Pi_h - B^*\Pi_h e^{A^* t}\|_{\mathcal{L}(H;U)} \le C\, \frac{e^{(\omega_0+\varepsilon)t}}{t^{\gamma}}, \quad t > 0. \tag{S.3.9}$$

On the other hand, by assumption (A.6) = (1.19) and analyticity of $e^{A^* t}$, we obtain, recalling the notation in the standing assumption (ii) below (1.2):

$$\|B^*\Pi_h e^{A^*t}\|_{\mathcal{L}(H;U)} \le C\|(\hat{A}^*)^\gamma e^{A^*t}\|_{\mathcal{L}(H)}$$

$$= C\, e^{\omega t}\|(\hat{A}^*)^\gamma e^{-\hat{A}t}\|_{\mathcal{L}(H)} \le C\,\frac{e^{(\omega-\hat{\omega})t}}{t^\gamma} = C\,\frac{e^{(\omega_0+\varepsilon)t}}{t^\gamma}. \qquad (S.3.10)$$

Combining (S.3.9) with (S.3.10), we obtain (3.4) as desired.

(v) First, from assumption (1.3) and analyticity of e^{A^*t} we have

$$\|B^* e^{A^*t}\|_{\mathcal{L}(H;U)} \le C\|(\hat{A}^*)^\gamma e^{A^*t}\|_{\mathcal{L}(H)} \le C\,\frac{e^{(\omega_0+\varepsilon)t}}{t^\gamma}, \quad t > 0, \qquad (S.3.11)$$

recalling the computations leading to (S.3.10). Then (S.3.11) and (3.4) of part (iv) imply

$$\|B_h^* e^{A_h^*t}\Pi_h - B^* e^{A^*t}\|_{\mathcal{L}(H;U)} \le C\,\frac{e^{(\omega_0+\varepsilon)t}}{t^\gamma}, \quad t > 0. \qquad (S.3.12)$$

Finally, we raise (3.1) of part (i) to the power Θ, we raise (S.3.12) to power $(1-\Theta)$, and multiply the resulting expressions together. This way we obtain (3.5) .

(vi) Eq. (3.6) follows from (3.5) and (3.2) via the interpolation (moment) inequality.

Lemma 3.1 is completely proved. ■

<u>Proof of Theorem 3.2</u>. (i) We compute from (2.3), (2.12) after recalling the estimate (3.5) of Lemma 3.1(v), with $\beta = \Theta+(1-\Theta)\gamma$, where $\gamma \le \beta < 1$, for any $\Theta < 1$ and $\gamma < 1$:

$$\|L_h^* v - L^* v\|^2_{L_2(0,T;U)} = \left\|\int_t^T [B_h^* e^{A_h^*(\tau-t)}\Pi_h - B^* e^{A^*(\tau-t)}]v(\tau)d\tau\right\|^2_{L_2(0,T;U)}$$

$$\le C_T^2\, h^{2s(1-\gamma)\Theta}\int_0^T\left[\int_t^T \frac{1}{(\tau-t)^\beta}\,\|v(\tau)\|_H d\tau\right]^2 dt$$

$$\le C_T^2\, h^{2s(1-\gamma)\Theta}\int_0^T\left(\int_t^T \frac{1}{(\tau-t)^\beta}\, d\tau\right)\left(\int_t^T \frac{\|v(\tau)\|_H^2}{(\tau-t)^\beta}\, d\tau\right)dt$$

$$\le C_T^2\, T^{1-\beta}\, h^{2s(1-\gamma)\Theta}\int_0^T \|v(\tau)\|_H^2\left(\int_\tau^T \frac{1}{(\tau-t)^\beta}\, dt\right)d\tau$$

$$\le C_{T,\beta}\, h^{2s(1-\gamma)\Theta}\|v\|^2_{L_2(0,T;H)} \qquad (S.3.13)$$

after using the Schwarz inequality and changing the order of integration. Then (S.3.13) proves (3.7).

(ii) Similarly, from (2.1) and (2.11), again by use of estimate (3.5) with $\gamma \le \beta = \Theta+(1-\Theta)\gamma < 1$:

$$\|L_h u - Lu\|_{C([0,T];H)} = \left\|\int_0^t (e^{A_h(t-\tau)}B_h - e^{A(t-\tau)}B)u(\tau)d\tau\right\|_{C([0,T];H)}$$

$$\le C\, h^{s(1-\gamma)\Theta}\sup_{0\le t\le T}\left[\int_0^T \frac{1}{(t-\tau)^\beta}\,\|u(\tau)\|_U d\tau\right]$$

$$\le C_{T,\beta}\, h^{s(1-\gamma)\Theta}\|u\|_{L_\infty(0,T;U)}, \qquad (S.3.14)$$

and (S.3.14) proves (3.8). ■

<u>Proof of Theorem 3.3</u>. (i) From (2.5) and (2.13), we compute with $v \in L_2(0,\infty;H)$

$$\|\hat{L}_h^* v - \hat{L}^* v\|^2_{L_2(0,\infty;U)} = \int_0^\infty\left\|\int_t^\infty [B_h^* e^{-\hat{A}_h^*(\tau-t)}\Pi_h - B^* e^{-\hat{A}^*(\tau-t)}]v(\tau)d\tau\right\|^2_U dt$$

$$\le C^2 h^{2s(1-\gamma)\Theta}\int_0^\infty\left[\int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta}\,\|v(\tau)\|_H d\tau\right]^2 dt, \qquad (S.3.15)$$

after recalling (3.5) with $\gamma \le \beta = \Theta+(1-\Theta)\gamma < 1$, for any $\Theta < 1$ and $\gamma < 1$ (as in the proof of Theorem 3.2), as well as $-\hat{A}^* = A^*-\omega I$, $\omega_0+\varepsilon-\omega = -\hat{\omega} < 0$ from (ii) below (1.2), and $-\hat{A}_h^* = A_h^*-\omega I$ from (2.15). Next, as in the proof of Theorem 3.2(i),

$$\int_0^\infty\int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta}\|v(\tau)\|_H d\tau\, dt \le \int_0^\infty(\int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta}d\tau)(\int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta}\|v(\tau)\|^2 d\tau)dt \tag{S.3.16}$$

$$\le C_\beta\int_0^\infty(\int_0^\tau \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta}d\tau)\|v(\tau)\|^2 d\tau \tag{S.3.17}$$

$$\le C_\beta\|v\|^2_{L_2(0,\infty;H)}, \tag{S.3.18}$$

since the first internal integral in (S.3.16) is bounded by a constant, and after changing the order of integration from (S.3.16) to (S.3.17). Then (S.3.15) and (S.3.18) prove (3.9).

(ii) The proof is similar to that of part (i). ∎

Section 4

4.1.1. Uniform analyticity

Proof of Lemma 4.1. **Step 1.** We have stipulated above (4.3) that, for the sake of definiteness, we are taking the case $\Sigma(F) \cap \Sigma_{app}(A)$. Thus, for all $\lambda \in \Sigma^c_{app}(A) = \Sigma^c_{app}(A;a;\Theta_a)$, the complement of $\Sigma_{app}(A)$, we have that $R(\lambda,A_F)$ and $R(\lambda,A_h)$ are well-defined, bounded operators on $L_2(\Omega)$ and V_h respectively, $\forall\, h \le h_a$.

Step 2. Given $\delta > 0$, there exist $r_\delta > 0$ and $0 < h_\delta \le h_a$, such that

$$\|R(\lambda,A_h)B_hF_h\|_{\mathcal{L}(V_h)} < \delta, \quad \text{for all } \lambda \text{ and } h \text{ as in (4.4).} \tag{S.4.1}$$

Indeed, (S.4.1) follows from

$$\|B_h^*R(\bar{\lambda},A_h^*)\|_{\mathcal{L}(H;U)} \le \frac{C}{|\bar{\lambda}-a|^{1-\gamma}}, \quad \lambda \in \Sigma^c_{app}(A), \tag{S.4.2}$$

which is the Laplace transform version in the λ-domain of estimate (3.4), Lemma 3.1(iv), in the t-domain (to be proved by contour integration, as usual), by taking $C\|F_h\|/|\lambda-a|^{1-\gamma} < \delta$ with $\|F_h\|$ uniformly bounded.

Step 3. We have

$$R(\lambda,A_F) = [I-R(\lambda,A)BF]^{-1}R(\lambda,A), \quad \forall\, \lambda \in \Sigma^c(A_F); \tag{S.4.3}$$

$$R(\lambda,A_{h,F_h}) = [I-R(\lambda,A_h)B_hF_h]^{-1}R(\lambda,A_h), \quad \forall\, \lambda \in \Sigma^c(A); \tag{S.4.4}$$

and, in both cases, $|\lambda|$ sufficiently large, and h sufficiently small in (S.4.4). In fact, (S.4.3) and (S.4.4) are the standard perturbation identities for the perturbed operators in (4.1) and (4.3) respectively, where we note, in the case of (S.4.4), that

$$\|[I-R(\lambda,A_h)B_hF_h]^{-1}\|_{\mathcal{L}(V_h)} \le \frac{1}{1-\|R(\lambda,A_h)B_hF_h\|_{\mathcal{L}(V_h)}}$$

$$\text{(by (S.4.1))} \qquad \le \frac{1}{1-\delta}, \ \lambda \text{ and } h \text{ as in (4.4).} \tag{S.4.5}$$

In the case of (S.4.3), we recall (1.3) and obtain the continuous version of (S.4.1),

$$\|R(\lambda,A)BF\| = \|R(\lambda,A)\hat{A}^\gamma\hat{A}^{-\gamma}BF\| \le \frac{C}{|\lambda-\omega_0|^{1-\gamma}} \to 0 \text{ as } |\lambda| \to \infty, \quad \lambda \in \Sigma^c_{app}(A). \tag{S.4.6}$$

Then the analog of (S.4.5) needed for (S.4.3) follows in the same way.

Step 4. The desired estimate (4.4) of part (i) follows from (S.4.4) and (S.4.5). Then (4.4) implies the desired estimate (4.5) of part (ii) via $\|R(\lambda,A_h)\|_{\mathcal{L}(V_h)} \le C/|\lambda-a|$, $\lambda \in \Sigma^c_{app}(A)$, see (A.1) = (1.14b).

Finally, the desired estimate (4.6) of part (iii) follows from (4.5) in the usual way: We write $R(\lambda,A_{h,F_h})A_{h,F_h} = \lambda R(\lambda,A_{h,F_h})-I$, and use on this (4.5), thereby obtaining (4.6) for $\Theta = 1$. Then the cases $\Theta = 0$ and $\Theta = 1$ imply the cases $0 < \Theta < 1$ via the interpolation (moment) inequality. Lemma 4.1 is proved. ∎

4.1.2. Uniform exponential stability of A_{h,F_h} and A_{F_h}

Proof of Theorem 4.2. **Orientation**. From Lemma 4.1 we know, *a fortiori*, that $e^{A_{h,F_h}t}$ are uniformly (in h) analytic on H and that the spectrum $\sigma(A_{h,F_h})$ is uniformly (in h) contained in a common sector, which preliminarily can be taken to be $\Sigma^c_{app}(A)$. The next step is to show that as a consequence of (4.8), in fact, $\sigma(A_{h,F_h})$ satisfies (4.11), i.e., via (4.10), $\sigma(A_{h,F_h})$ is contained on a three-sided sector on the left-hand side of the complex plane. Finally, uniform analyticity combined with the 'correct' location of the spectrum will imply the remaining parts (ii) = (4.12) and part (iii) = (4.13) of Theorem 4.2 via operator calculus. Details follows. We begin with

Lemma S.4.1.

(i) Under the same assumptions as in Lemma 4.1, let, for the sake of definiteness, λ_0 be fixed with $\mathrm{Re}\,\lambda_0 > r_\delta$, r_δ as in (4.4). Then the following convergence holds true for all $\varepsilon_0 < s(1-\gamma)$:

$$\|R(\lambda_0,A_F)-R(\lambda_0,A_{h,F_h})\Pi_h\|_{\mathcal{L}(H)} \le h^{\varepsilon_0}+\|F_h-F\|_{\mathcal{L}(U;H)} \to 0 \quad \text{as } h \downarrow 0, \tag{S.4.7}$$

for all $\varepsilon_0 < s(1-\gamma)$;

(ii) in (S.4.7), one may replace the chosen λ_0 with any other $\lambda \in \rho(A_F)$;

(iii) for any compact set $\mathcal{K} \subset \rho(A_F)$, we have

$$\sup_{\lambda\in\mathcal{K};h} \|R(\lambda,A_{h,F_h})\|_{\mathcal{L}(H)} \le \mathrm{const}_{\mathcal{K}}. \tag{S.4.8}$$

Proof of Lemma S.4.1. (i) Recalling (S.4.3) and (S.4.4), we compute in the $\mathcal{L}(H)$-norm:

$$\|R(\lambda_0,A_F)-R(\lambda_0,A_{h,F_h})\Pi_h\| = \|[I-R(\lambda_0,A)BF]^{-1}R(\lambda_0,A)-[I-R(\lambda_0,A_h)B_hF_h]^{-1}R(\lambda_0,A_h)\Pi_h\|$$

$$= \|(1)+(2)\|, \tag{S.4.9}$$

where, after adding and subtracting,

$$(1) = [I-R(\lambda_0,A)BF]^{-1}(R(\lambda_0,A)-R(\lambda_0,A_h)\Pi_h); \tag{S.4.10}$$

$$(2) = \{[I-R(\lambda_0,A)BF]^{-1}-[I-R(\lambda_0,A_h)B_hF_h]^{-1}\}R(\lambda_0,A_h)\Pi_h. \tag{S.4.11}$$

Thus, by assumption (A.2) = (1.15),

$$\|(1)\| \le C\,h^s\|[I-R(\lambda_0,A)BF]^{-1}\| \to 0 \quad \text{as } h \downarrow 0. \tag{S.4.12}$$

As to (2), we use the identity

$$[I-T_1]^{-1}-[I-T_2]^{-1} = [I-T_1]^{-1}[T_1-T_2][I-T_2]^{-1} \tag{S.4.13}$$

in (S.4.11) with $T_1 = R(\lambda_0,A)BF$, $T_2 = R(\lambda_0,A_h)B_hF_h$. By (S.4.4), $R(\lambda_0,A_{h,F_h}) = [I-T_2]^{-1}R(\lambda_0,A_h)$. Using these and $K = \|[I-T_1]^{-1}\|$, we can write from (S.4.11),

$$\|(2)\| \le c_{\cdot}K\|R(\lambda_0,A_{h,F_h})\|\;\|R(\lambda_0,A)BF-R(\lambda_0,A_h)B_hF_h\|$$

$$\le \frac{c}{1-\delta}\,\frac{1}{|\lambda-a|}\,\|R(\lambda_0,A)BF-R(\lambda_0,A_h)B_hF_h\|, \tag{S.4.14}$$

where in the last step we have used (4.5) of Lemma 4.1(ii), with $0 < \delta < 1$, preassigned, and $0 < h \le h_\delta$. We next compute the term in (S.4.14),

$$\|R(\lambda_0,A)BF-R(\lambda_0,A_h)B_hF_h\| \le \|[F^*-F_h^*]B^*R(\lambda_0,A^*)\|+\|F_h^*[B^*R(\lambda_0,A^*)-B_h^*R(\lambda_0,A_h^*)\Pi_h]\|. \quad (S.4.15)$$

As to the first term in (S.4.15), we invoke assumption (4.9) to see that it tends to 0 as $h \downarrow 0$. As to the second term in (S.4.15), we use the assumption $\|F_h\| \le$ const, as well as the Laplace (λ-) version of the estimate (3.5) of Lemma 3.1(v) with $\Theta < 1$, to be proved by contour integration, thereby obtaining

$$\|F_h^*[B^*R(\lambda_0,A^*)-B_h^*R(\lambda_0,A_h^*)\Pi_h]\| \le \text{const } h^{s(1-\gamma)\Theta} \to 0 \text{ as } h \downarrow 0. \quad (S.4.16)$$

Thus, the term (2) in (S.4.14) also tends to zero as $h \downarrow 0$. Then, by (S.4.9), the desired convergence (S.4.7) in part (i) is proved.

(ii) The statement for any other $\lambda \in \rho(A_F)$ follows now from standard results [13, Thm. 3.15, p. 206; also Remark 3.13, p. 211].

(iii) Part (iii), Eq. (S.4.8) is a consequence of the joint continuity of the resolvent $R(\lambda,A_F)$ in both arguments [13, Thm. 3.15, p. 212]. ■

Continuing with the proof of Theorem 4.2, we return to Lemma 4.1(i), Eq. (4.4): Given $1 > \delta > 0$, there exist r_δ, $h_\delta > 0$ such that for all $0 < h < h_\delta$,

$$[\Sigma^c_{app}(A) \cap \{|\lambda| \ge r_\delta\}] \subset \rho(A_{h,F_h}), \quad (S.4.17)$$

$\rho(\cdot)$ denoting the resolvent set.

We next complement the statement in (S.4.17) by virtue of the following:

<u>Lemma S.4.2</u>. For any $\varepsilon' > 0$ there exists $h_{\varepsilon'} > 0$ such that

$$\sup \operatorname{Re} \sigma(A_{h,F_h}) \le -\omega_F+\varepsilon', \quad 0 < h \le h_{\varepsilon'}, \quad (S.4.18)$$

where ε' may be taken 0 if $-\omega_F \in \rho(A_F)$.

<u>Proof of Lemma S.4.2</u>. The proof of this result is the same as the proof of [18, Lemma 4.4] and is omitted here. ■

Thus, conclusion (4.10) of Theorem 4.2 has been proved. In order to complete the proof of Theorem 4.2, we combine the results of Lemma 4.1, Lemmas S.4.1 and S.4.2, and we integrate along a path in $\Sigma^c_{app}(A_F)$ in (4.10) which follows its boundary. The computations are the same as those given in [18, pp. 200-201] and will not be repeated here. ■

4.3. <u>Uniform stability of the feedback semigroup</u> $\exp(A_{h,P_h}t)$

<u>Proof of Theorem 4.6</u>. The first step is a consequence of Theorem 4.2.

<u>Step 1</u>. <u>Lemma S.4.3</u>. We have

$$\|P_h\|_{\mathcal{L}(V_h)} \le \text{const, uniformly in } h. \quad ■ \quad (S.4.19)$$

<u>Proof of Lemma S.4.3</u>. We note first that the assumption of uniform convergence (4.9) for Theorem 4.2 holds true for the choice $F_h = F\Pi_h$, by virtue of the compactness assumption (1.27a), or else (1.27b), see Remark 4.2. Thus, Theorem 4.2(ii), Eq. (4.12), implies that $\exp\{(A_{h,F_h})t\} = \exp\{(A_h+B_hF_h)t\}$ is uniformly stable. For the approximating optimal control problem (1.23), (1.24) on V_h with optimal pair

$$\Phi_h(t)x = y_h^0(t;x) = e^{A_{h,P_h}t}x; \quad u_h^0(t;x) = -B_h^*P_h e^{A_{h,P_h}t}x \quad (S.4.20)$$

and initial point $x \in V_h$, the feedback control $F_h e^{A_{h,F_h} t} x$ and corresponding solution $e^{A_{h,F_h} t} x$ form a competing pair. Thus, by (2.20) we get

$$J(u_h^0, y_h^0) = (P_h x, x)_H = \int_0^\infty \|u_h^0(t;x)\|_U^2 + \|R_h y_h^0(t;x)\|_Z^2 dt$$

$$\le \int_0^\infty \left\|F_h e^{A_{h,F_h} t} x\right\|_U^2 + \left\|R_h e^{A_{h,F_h} t} x\right\|_Z^2 dt$$

$$\le C \int_0^\infty e^{2(-\omega_F + \varepsilon)t} dt \|x\|^2 \le C\|x\|_H^2, \tag{S.4.21}$$

where in the last step we have invoked (4.12). Since P_h is non-negative, self-adjoint, (S.4.21) yields $\|P_h\|_{\mathcal{L}(V_h)} = \sup(P_h x, x) \le C$, over all $x \in V_h$ with $\|x\| \le 1$. ∎

<u>Step 2</u>. <u>Lemma S.4.4</u>. We have

$$\int_0^\infty \left\|e^{A_{h,P_h} t} x\right\|_H^2 dt \le c\|x\|_H^2, \quad x \in H. \tag{S.4.22}$$

<u>Proof of Lemma S.4.4</u>. If $R > 0$, then (S.4.22) is a direct consequence of (S.4.21) via (S.4.20). Otherwise, we shall use, as usual, the more general detectability assumption which leads to the uniform estimate (4.19). Writing by (4.16) and (4.26),

$$A_{h,P_h} = A_{h,K} - \Pi_h K R \Pi_h - B_h B_h^* P_h, \tag{S.4.23}$$

and recalling that $\dot{y}_h^0 = A_{h,P_h} y_h^0$ for the approximating problem we have by (S.4.23),

$$y_h^0(t,x) = e^{A_{h,P_h} t} x = e^{A_{h,K} t} x - \int_0^t e^{A_{h,K}(t-\tau)} \Pi_h K R \Pi_h y_h^0(\tau,x) d\tau$$

$$- \int_0^t e^{A_{h,K}(t-\tau)} B_h B_h^* P_h y_h^0(\tau,x) d\tau \tag{S.4.24}$$

$$= e^{A_{h,K} t} x - \{L_{h,K} \Pi_h K R \Pi_h y_h^0(\cdot,x)\}(t)$$

$$- \{L_{h,K} B_h B_h^* P_h y_h^0(\cdot,x)\}(t), \tag{S.4.25}$$

after recalling the operators $L_{h,K}$ in (4.22), whose regularity properties (4.23), (4.24) will now be invoked. In fact, by (4.24) we have the first step of

$$\|L_{h,K} \Pi_h K R \Pi_h y_h^0(\cdot,x)\|_{L_2(0,\infty;H)} \le C\|R \Pi_h y_h^0(\cdot,x)\|_{L_2(0,\infty;Z)}$$

$$\le C\|y_h^0(\cdot,x)\|_{L_2(0,\infty;H)}$$

$$\text{(by (S.4.20) and by (4.27))} \qquad = C\left\|e^{A_{h,P_h} \cdot} x\right\|_{L_2(0,\infty;H)} \le C\|x\|_H, \tag{S.4.26}$$

as desired. Similarly, by (4.23) we have the first step of

$$\|L_{h,K} B_h B_h^* P_h y_h^0(\cdot,x)\|_{L_2(0,\infty;H)} \le C\|B_h^* P_h y_h^0(\cdot,x)\|_{L_2(0,\infty;U)}$$

$$\text{(by (S.4.20) and by (S.4.21))} \qquad \le C\|u_h^0(\cdot,x)\|_{L_2(0,\infty;U)} \le C\|x\|_H, \tag{S.4.27}$$

as desired. Finally, using (S.4.26), (S.4.27), in (S.4.22), as well as (4.19) for the first term in (S.4.25), we obtain (S.4.22) as desired. ∎

<u>Step 3</u>. <u>Proposition S.4.5</u>. There are numbers $c > 0$ and $a > 0$, independent of h, such that

$$\left\| e^{A_{h,P_h}t} \right\|_{\mathcal{L}(V_h)} \le c\, e^{at}, \quad t \ge 0. \tag{S.4.28}$$

Proof of Proposition S.4.5. We shall prove (S.4.28), in fact with $a = \omega$, by using a 'bootstrap' argument, as in [18], based on the following equations for the optimal pair:

$$e^{-\omega t} e^{A_{h,P_h}t} x_h = \hat{y}_h^0(t,x_h) = e^{-\hat{A}_h t} x_h + \{\hat{L}_h \hat{u}_h^0\}(t) \tag{S.4.29}$$

$$\hat{u}_h^0(t,x_h) = -\{\hat{L}_h^*[R^*R + 2\omega P_h]\hat{y}_h^0(\cdot,x_h)\}(t) \tag{S.4.30}$$

with $x_h \in V_h$, see (2.21), (2.22) in Section 2. The 'bootstrap' argument uses the following result, which is of interest only in the more demanding situation where $\frac{1}{2} < \gamma < 1$.

Lemma S.4.6. For the operators $\hat{L}_h$ and $\hat{L}_h^*$ defined by (2.13), (2.14), we have

(i) $\hat{L}_h$: continuous $L_2(0,\infty;U) \to L_r(0,\infty;H)$ uniformly in $h \downarrow 0$, (S.4.31)

$$\text{i.e., } \sup_{h>0} \|\hat{L}_h\|_{2,r} \le \text{const},$$

where r is an arbitrary number satisfying $r < 2/(2\gamma-1)$, where $2/(2\gamma-1) > 2$ for $\frac{1}{2} < \gamma < 1$; for $0 \le \gamma \le \frac{1}{2}$, one can take $r = \infty$.

(ii) $\hat{L}_h^*$: continuous $L_r(0,\infty;H) \to L_{r'}(0,\infty;U)$, uniformly in $h \downarrow 0$, (S.4.32)

$$\text{i.e., } \sup_{h>0} \|\hat{L}_h^*\|_{r,r'} \le \text{const},$$

where r is as in (i), and r' is any number satisfying $r' < 2/(4\gamma-3)$, where $\frac{3}{4} < \gamma < 1$, $2/(4\gamma-3) > r$; for $0 < \gamma < \frac{3}{4}$, we can take $r' = \infty$.

(iii) With $p > 1(1-\gamma)$,

$$\hat{L}_h\text{: continuous } L_p(0,\infty;U) \to C([0,\infty];H), \text{ uniformly in } h \downarrow 0, \tag{S.4.33}$$

$$\text{i.e., } \sup_{h>0} \|\hat{L}_h\|_{p,\text{contin}} < \infty.$$

Proof of Lemma S.4.6. As in [18], the proof is based on Young's inequality [23, p. 29]. By using Lemma 3.1, inequality (3.4), as well as $\hat{A}_h = -A_h + \omega I$, $-\hat{\omega} = \omega_0 + \varepsilon - \omega$, we have preliminarily

$$\|e^{-\hat{A}_h t} B_h\|_{\mathcal{L}(U;V_h)} = \|B_h^* e^{-\hat{A}_h t} \Pi_h\|_{\mathcal{L}(V_h;U)} = \mathcal{O}\left(\frac{e^{-\hat{\omega}t}}{t^{\gamma}}\right). \tag{S.4.34}$$

Thus, it follows from (2.13), respectively (2.14), via (S.4.34) that

$$\|(\hat{L}_h u)(t)\|_H \le C \int_0^t \frac{e^{-\hat{\omega}(t-\tau)}}{(t-\tau)^{\gamma}} \|u(\tau)\|_U d\tau; \tag{S.4.35}$$

$$\|(\hat{L}_h^* v)(t)\|_U \le C \int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^{\gamma}} \|v(\tau)\| d\tau. \tag{S.4.36}$$

Then parts (i) and (ii) follow immediately from (S.4.35), (S.4.36) via Young's inequality. Part (iii) follows likewise with $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} - 1 = 0$, where we have $\gamma q < 1$, and $\frac{1}{q} = 1 - \frac{1}{p} \downarrow \gamma$ as $p \downarrow \frac{1}{1-\gamma}$. The proof of Lemma S.4.6 is complete. ∎

To complete the proof of Proposition S.4.5, Eq. (S.4.28), we start with $\hat{u}_h^0 \in L_2(0,\infty;U)$ and apply a bootstrap argument on (S.4.29), (S.4.30) using Lemma S.4.6. After a finite number of iterations we obtain ($\hat{u}_h^0 \in L_\infty(0,\infty;U)$ and) $y_h^0 \in C([0,\infty];H)$ from which (S.4.28) follows from (S.4.29) with the constant $a = \omega$. ∎

Step 4. Starting from the uniform bound in (S.4.28) of Proposition S.4.5, we can complete the proof of Theorem 4.6 and obtain estimate (4.27) by simply proceeding as in the continuous case; see [22, p. 121]. Theorem 4.6 is proved. ∎

4.4. Uniform regularity of P_h

Proof of Theorem 4.7. (i) We return to identity (2.19) for P_h and obtain with $x \in V_h$,

$$(\hat{A}_h^*)^{\Theta} P_h x = \int_0^\infty (\hat{A}_h^*)^{\Theta} e^{-\hat{A}_h^* t} \Pi_h [R^* R + 2\omega P_h] \hat{\Phi}_h(t) x \, dt. \tag{S.4.37}$$

Invoking estimate (S.4.16) of Lemma S.4.3 and analyticity, we obtain

$$\|(\hat{A}_h^*)^{\Theta} P_h\|_{\mathcal{L}(H)} \le C \int_0^\infty \frac{e^{-\hat{\omega} t}}{t^{\Theta}} \|\hat{\Phi}_h(t)\|_{\mathcal{L}(H)} dt, \quad 0 \le \Theta < 1. \tag{S.4.38}$$

Then (4.28) of part (i) follows immediately from (S.4.35) via (2.16) and the uniform bound (4.27) of Theorem 4.6 for $\Phi_h(t)$.

(ii) The proof of (4.29) is similar: From (2.19) with $x \in V_h$,

$$B_h^* P_h x = \int_0^\infty B_h^* e^{-\hat{A}_h^* t} \Pi_h [R^* R + 2\omega P_h] \hat{\Phi}_h(t) x \, dt, \tag{S.4.39}$$

and invoking estimates (3.4) of Lemma 3.1 and (S.4.16) of Lemma S.4.3, and $-\hat{A}_h = A_h - \omega I$, $-\hat{\omega} = \omega_0 + \varepsilon - \omega$, we obtain from (S.4.36),

$$\|B_h^* P_h\|_{\mathcal{L}(H;U)} \le C \int_0^\infty \frac{e^{-\hat{\omega} t}}{t^{\gamma}} \|\hat{\Phi}_h(t)\|_{\mathcal{L}(H)} dt, \tag{S.4.40}$$

and (4.29) follows likewise via (2.16) and (4.27).

(iii) Part (iii) follows from part (i) via self-adjoint calculus as in [10, Lemma 3.3]. ∎

Section 5

5.1. Uniform convergence $P_h \Pi_h \to P$ of Riccati operators

Proof of Theorem 5.1.

Proof. **Step 1**. The following four operators will play a key role. The first and the fourth, defined by (1.12) and (4.26), refer to the optimal dynamics, continuous and discrete. The second and third are introduced here for the first time. They will define competitive dynamics:

$$A_P = A - BB^* P; \quad A_{h,P} = A_h - B_h B^* P; \tag{S.5.1}$$

$$A_{P_h} = A - BB_h^* P_h; \quad A_{h,P_h} = A_h - B_h B_h^* P_h. \tag{S.5.2}$$

The semigroup generated by A_P is analytic and stable, Section 1.1. As to the other operators, we have

Proposition S.5.1. The semigroups generated by the operators $A_{h,P}$, A_{P_h}, A_{h,P_h} are both uniformly analytic (in the sense of Lemma 4.1) and uniformly stable.

Proof. In the case of A_{h,P_h}, uniform analyticity was established in Corollary 4.8, while uniform stability was established in Theorem 4.6, Eq. (4.27). The same properties then hold true for $A_{h,P}$ as special case of the latter, where $F_h \equiv B^*P$. Next, uniform stability of $e^{A_{P_h}t}$ follows from Remark 4.4, where we already know that $e^{A_{h,P_h}t}$ is uniformly stable, and thus we take $F = F_h = B_h^*P_h$, so that the required assumption (4.9) holds true. Finally, uniform analyticity of $\exp(A_{P_h}t)$ follows from Lemma 4.1 with $A_h \equiv A$, $B_h \equiv B$, and $F_h \equiv B_h^*P_h$ which is uniformly bounded by (4.29), as required. ∎

Step 2. Proposition S.5.2. Let ε_0 be the same number $\varepsilon_0 < s(1-\gamma)$ as in Lemma S.4.1, Eq. (S.4.7) of the Supplement. Then

$$\|A_P^{-1}-A_{h,P}^{-1}\|_{\mathcal{L}(H)} \le C\, h^{\varepsilon_0} \to 0 \quad \text{as } h \to 0; \tag{S.5.3}$$

$$\|A_{P_h}^{-1}-A_{h,P_h}^{-1}\|_{\mathcal{L}(H)} \le C\, h^{\varepsilon_0} \to 0 \quad \text{as } h \to 0. \tag{S.5.4}$$

Proof. By use of the first resolvent equation for resolvent operators, it suffices to show (S.5.3), (S.5.4) with $R(0,\cdot)$ replaced by $R(\lambda_0,\cdot)$, for Re λ_0 sufficiently large. In this latter case then, these desired bounds follow from Lemma 4.3 with $F = F_h = B^*P$ in the case of (S.5.3), and with $F = F_h = B_h^*P_h$ in the case of (S.5.4), where we note that the fact that now F depends on h does not make any difference in the argument of Lemma S.4.1 as long as $\|F_h\| \le$ const, which is true by (4.29). ∎

Step 3. By (1.10) and (2.23), we have

$$|([P_h\Pi_h-P]x,x)_H| = |J(u_h^0(\cdot,\Pi_h x),y_h^0(\cdot,\Pi_h x)-J(u^0(\cdot,x),y^0(\cdot,x))|. \tag{S.5.5}$$

Now, with x and h fixed, if $J(u_h^0,y_h^0)-J(u^0,y^0) > 0$ we introduce the competing pair

$$\tilde{u}_h(t,\Pi_h x) = -B^*P\, e^{A_{h,P}t}\Pi_h x; \quad \tilde{y}_h(t,\Pi_h x) = e^{A_{h,P}t}\Pi_h x, \tag{S.5.6}$$

for the approximating problem so that in this case

$$|J(u_h^0,y_h^0)-J(u^0,y^0)| \le J(\tilde{u}_h,\tilde{y}_h)-J(u^0,y^0). \tag{S.5.7}$$

Instead, if $J(u^0,y^0)-J(u_h^0,y_h^0) > 0$, we introduce the competing pair

$$\bar{u}_h(t,x) = -B_h^*P_h\, e^{A_{P_h}t}\Pi_h x, \quad \bar{y}_h(t,x) = e^{A_{P_h}t}\Pi_h x, \tag{S.5.8}$$

for the continuous problem so that in this case

$$|J(u^0,y^0)-J(u_h^0,y_h^0)| < J(\bar{u},\bar{y})-J(u_h^0,y_h^0). \tag{S.5.9}$$

Thus, in all cases, we have from (S.5.7), (S.5.9),

$$|J(u_h^0,y_h^0)-J(u^0,y^0)| \le |J(\tilde{u}_h,\tilde{y}_h)-J(u^0,y^0)| + |J(\bar{u}_h,\bar{y}_h)-J(u_h^0,y_h^0)|. \tag{S.5.10}$$

Recalling (S.5.6), (S.5.8), we rewrite the right-hand side (R.H.S.) of (S.5.10) after recalling the costs (1.2) and (1.24), as well as (S.4.17)

$$\begin{aligned}\text{R.H.S. of (S.5.10)} = \Big|\int_0^\infty &\{\|R\, e^{A_P t}x\|_Z^2-\|R\, e^{A_{h,P}t}\Pi_h x\|_Z^2 \\ &+ \|B^*P\, e^{A_P t}x\|_U^2-\|B^*P\, e^{A_{h,P}t}\Pi_h x\|_U^2\}dt\Big| + \Big|\int_0^\infty\{\|R\, e^{A_{P_h}t}\Pi_h x\|_Z^2-\|R\, e^{A_{h,P_h}t}\Pi_h x\|_Z^2 \\ &+ \|B_h^*P_h e^{A_{P_h}t}x\|_U^2-\|B_h^*P_h e^{A_{h,P_h}t}\Pi_h x\|_U^2\}dt\Big|\end{aligned} \tag{S.5.11}$$

$$\leq C\left\{\int_0^\infty \|e^{A_P t}x - e^{A_{h,P_h} t}\Pi_h x\|_H dt + \int_0^\infty \|e^{A_{P_h} t}\Pi_h x - e^{A_{h,P_h} t}\Pi_h x\|_H dt\right\}\|x\|_H , \quad (S.5.12)$$

where in the last step we have used $|\|a\|^2 - \|b\|^2| \leq \|a-b\|(\|a\|+\|b\|)$, as well as $\|B^*P\| + \|B_h^*P_h\| \leq$ const, and also the uniform exponential decay of the semigroups. Finally, splitting the integration over $[0,h^s]$ and $[h^s,\infty]$ and using uniform bounds on $[0,h^s]$, we obtain from (S.5.12),

$$\text{R.H.S. of (S.5.10)} \leq C\|x\|_H\left\{h^s\|x\|_H + \left\|e^{A_P t}x - e^{A_{h,P_h} t}\Pi_h x\right\|_{L_1(h^s,\infty)} + \left\|e^{A_{P_h} t}\Pi_h x - e^{A_{h,P_h} t}\Pi_h x\right\|_{L_1(h^s,\infty)}\right\}. \quad (S.5.13)$$

Now, if Γ: $\rho\, e^{\pm i\Theta} - k$, $\frac{\pi}{2} < \Theta < \pi$, $0 \leq \rho < \infty$, $k > 0$, is the boundary of a triangular sector containing the spectrum of A_P, A_{h,P_h}, uniformly in h, as guaranteed by Proposition S.5.1, we compute

$$\|e^{A_P t} - e^{A_{h,P_h} t}\Pi_h\|_{\mathcal{L}(H)} = \|\int_\Gamma e^{\lambda t}R(\lambda,A_P) - R(\lambda,A_{h,P_h})d\lambda\| \quad (S.5.14)$$

$$= \|\int_\Gamma e^{\lambda t}R(\lambda,A_P)(A_P - A_{h,P_h})R(\lambda,A_{h,P_h})d\lambda\| \quad (S.5.15)$$

$$= \|\int_\Gamma e^{\lambda t}R(\lambda,A_P)A_P(A_P^{-1} - A_{h,P_h}^{-1})A_{h,P_h}R(\lambda,A_{h,P_h})d\lambda\|$$

$$\leq C\left(\int_\Gamma e^{(\mathrm{Re}\,\lambda)t}|d\lambda|\right)\|A_P^{-1} - A_{h,P_h}^{-1}\|_{\mathcal{L}(H)} , \quad (S.5.16)$$

where in going from (S.5.14) to (S.5.15) we have used an identity like the one in (S.4.13), while in going from (S.5.15) to (S.5.16), we have used analyticity of $e^{A_P t}$, uniform analyticity of $e^{A_{h,P_h} t}$, see Proposition S.5.1, in the sense of (4.6) Lemma 4.1 with $\Theta = 1$. Thus from (S.5.16) we obtain, with $k > 0$ (below (S.5.13)) and $t > 0$,

$$\left\|e^{A_P t} - e^{A_{h,P_h} t}\right\|_{\mathcal{L}(H)} \leq C\,\frac{e^{-kt}}{t}\left\|A_P^{-1} - A_{h,P_h}^{-1}\right\|_{\mathcal{L}(H)}, \quad (S.5.17)$$

and by a similar argument,

$$\left\|e^{A_{P_h} t}\Pi_h - e^{A_{h,P_h} t}\Pi_h\right\|_{\mathcal{L}(H)} \leq C\,\frac{e^{-kt}}{t}\left\|A_{P_h}^{-1} - A_{h,P_h}^{-1}\right\|_{\mathcal{L}(H)} . \quad (S.5.18)$$

Then, by (S.5.17), splitting the integration over $[h^s,1]$ and $[1,\infty]$ with $h^s < 1$ as $h \downarrow 0$, we obtain:

$$\int_{h^s}^\infty \left\|e^{A_P t}x - e^{A_{h,P_h} t}\Pi_h x\right\|_H dt \leq \|A_P^{-1} - A_{h,P_h}^{-1}\| \int_{h^s}^\infty \frac{e^{-kt}}{t}\,dt\,\|x\|_H$$

$$\text{(by (S.5.3))} \qquad \leq C\|A_P^{-1} - A_{h,P_h}^{-1}\|(s\,\ell n\,\tfrac{1}{h})\,\|x\|_H \leq C\,h^{\varepsilon_0 - \varepsilon}\|x\|_H, \quad (S.5.19)$$

for all $\varepsilon > 0$. The same estimate can be obtained starting from (S.5.18) and using (S.5.4)

$$\int_{h^s}^\infty \left\|e^{A_{P_h} t}\Pi_h x - e^{A_{h,P_h} t}\Pi_h x\right\|_H dt \leq C\,h^{\varepsilon_0 - \varepsilon}\|x\|_H. \quad (S.5.20)$$

Using estimates (S.5.19), (S.5.20), and (S.5.13), as well as recalling (S.5.5) and (S.5.10), we obtain

$$|([P_h\Pi_h - P]x,x)_H| \leq C\,h^{\varepsilon_0}\|x\|_H^2, \quad (S.5.21)$$

with any $\varepsilon_0 < s(1-\gamma) < s$, as desired. Then (S.5.21) implies (5.1) by taking the sup over all x, $\|x\| \le 1$, since P_h and P are self-adjoint. Theorem 5.1 is proved. ∎

<u>Proof of Theorem 5.2</u>. We interpolate between inequality (S.5.17) and (see (1.12) and (4.27))

$$\|e^{A_P t} - e^{A_{h,P_h} t}\Pi_h\|_{\mathcal{L}(H)} \le C\, e^{-kt}, \tag{S.5.22}$$

$k > 0$, to obtain for any $0 < \Theta < 1$,

$$\|e^{A_P t} - e^{A_{h,P_h} t}\Pi_h\|_{\mathcal{L}(H)} \le C\, \frac{e^{-kt}}{t^{\Theta}}\, \|A_P^{-1} - A_{h,P_h}^{-1}\|^{\Theta}_{\mathcal{L}(H)}. \tag{S.5.23}$$

Then (5.4) follows from (S.5.23) after recalling (S.5.3), as we can take $k = \overline{\omega}_P$, see (4.27). ∎

5.2. <u>Uniform convergence</u> $B_h^* P_h \Pi_h \to B^* P$ <u>of gain operators</u>

<u>Proof of Theorem 5.3</u>. From (2.19) (or (S.4.39)) and (2.10) we compute

$$B_h^* P_h \Pi_h - B^* P = \int_0^\infty B_h^* e^{-\hat{A}_h^* t} \Pi_h [R^* R + 2\omega P_h] \hat{\Phi}_h(t) \Pi_h dt$$

$$- \int_0^\infty B^* e^{-\hat{A}^* t} [R^* R + 2\omega P] \hat{\Phi}(t) dt = I_{1,h} + I_{2,h} + I_{3,h}, \tag{S.5.24}$$

where, after suitable adding and subtracting,

$$I_{1,h} = \int_0^\infty [B_h^* e^{-\hat{A}_h^* t} \Pi_h - B^* e^{-\hat{A}^* t}][R^* R + 2\omega P_h] \hat{\Phi}_h(t) \Pi_h dt; \tag{S.5.25}$$

$$I_{2,h} = \int_0^\infty B^* e^{-\hat{A}^* t} 2\omega [P_h \Pi_h - P] \hat{\Phi}_h(t) \Pi_h dt; \tag{S.5.26}$$

$$I_{3,h} = \int_0^\infty B^* e^{-\hat{A}^* t} [RR^* + 2\omega P][\hat{\Phi}_h(t)\Pi_h - \hat{\Phi}(t)] dt. \tag{S.5.27}$$

To handle $I_{1,h}$, we recall that from Lemma 3.1(v), Eq. (3.5), applied with $\Theta < 1$, we have (by the definitions of $\hat{A}^*$ and $\hat{A}_h^*$ below (1.2) and (2.15)):

$$\|B_h^* e^{-\hat{A}_h^* t} \Pi_h - B^* e^{-\hat{A}^* t}\|_{\mathcal{L}(H;U)} \le \frac{C\, h^{s(1-\gamma)\Theta}}{t^{\Theta + (1-\Theta)\gamma}}\, e^{-\hat{\omega} t}. \tag{S.5.28}$$

Thus, by (S.5.28) with $\Theta + (1-\Theta)\gamma < 1$, as well as the uniform bound (S.4.19) of Lemma S.4.3 for P_h, and (4.27) of Theorem 4.6 for $\Phi_h(t)$, we readily obtain from (S.5.25) that

$$\|I_{1,h}\|_{\mathcal{L}(H;U)} \le C\, h^{s(1-\gamma)\Theta} \downarrow 0 \quad \text{as } h \downarrow 0, \quad \Theta < 1. \tag{S.5.29}$$

To handle $I_{2,h}$, we recall Eq. (5.1) of Theorem 5.1, and the uniform bound (4.27), and note the bound

$$\|B^* e^{-\hat{A}^* t}\|_{\mathcal{L}(H;U)} = \|B^* (\hat{A}^*)^{-\gamma} (\hat{A}^*)^{\gamma} e^{-\hat{A}^* t}\|_{\mathcal{L}(H;U)} \le C\, \frac{e^{-\hat{\omega} t}}{t^{\gamma}} \tag{S.5.30}$$

(from (1.3) and analyticity) to conclude from (S.5.26) that

$$\|I_{2,h}\|_{\mathcal{L}(H;U)} \le C\, h^{\varepsilon_0} \to 0 \quad \text{as } h \downarrow 0, \quad \varepsilon_0 < s(1-\gamma). \tag{S.5.31}$$

To complete the proof of Theorem 5.3 by showing that $I_{3,h}$ also goes to zero, we need (part of) Lemma 5.4.

Proof of Lemma 5.4.

Proof (i) and (ii). Step 1. We return to (2.18), (2.19) rewritten here for convenience as

$$\hat{y}_h^0(\cdot,\Pi_h x) = [I+\hat{L}_h\hat{L}_h^*\Pi_h(R^*R+2\omega P_h)]^{-1}\{e^{-\hat{A}_h\cdot}\Pi_h x\}, \tag{S.5.32}$$

$$-\hat{u}_h^0(\cdot,\Pi_h x) = [I+\hat{L}_h^*\Pi_h(R^*R+2\omega P_h)\hat{L}_h]^{-1}\hat{L}_h^*\Pi_h[R^*R+2\omega P_h]\{e^{-\hat{A}_h\cdot}\Pi_h x\}, \tag{S.5.33}$$

and take the limit as $h \downarrow 0$. Using the uniform convergence,

$$\|e^{-\hat{A}_h t}\Pi_h - e^{-\hat{A}t}\|_{\mathcal{L}(H;L_2(0,\infty;H))} \le C\, h^{s\Theta} \to 0 \text{ as } h \downarrow 0,\ \Theta < 1,$$

which follows from assumption (1.20), the uniform convergence $\|P_h\Pi_h - P\| \le C\, h^{\varepsilon_0}$ in (5.1) of Theorem 5.1 and the uniform convergence $\|\hat{L}_h - \hat{L}\| \le C\, h^{s(1-\gamma)\Theta}$, $\Theta < 1$, of (3.9) of Theorem 3.3, we conclude via (2.8), (2.9) (since the rates of convergence are preserved by the inverses by an identity as (S.4.13)), that as $h \downarrow 0$,

$$\|\hat{u}_h^0(\cdot,\Pi_h x) - \hat{u}^0(\cdot,x)\|_{\mathcal{L}(H;L_2(0,\infty;U))} \le C\, h^{\varepsilon_0}\|x\|_H \to 0 \quad \text{as } h \downarrow 0; \tag{S.5.34}$$

$$\|\hat{\Phi}_h(\cdot)\Pi_h - \hat{\Phi}(\cdot)\|_{\mathcal{L}(H;L_2(0,\infty;H))} \le C\, h^{\varepsilon_0} \to 0 \quad \text{as } h \downarrow 0. \tag{S.5.35}$$

Step 2. *A fortiori*, from (S.5.34), (S.5.35) we have for any $0 < T < \infty$:

$$\|u_h^0(\cdot,\Pi_h x) - u_0(\cdot,x)\|_{\mathcal{L}(H;L_2(0,T;U))} \le \|\Phi_h(\cdot)\Pi_h - \Phi(\cdot)\|_{\mathcal{L}(H;L_2(0,T;H))}$$

$$\le C\, h^{\varepsilon_0} \to 0 \quad \text{as } h \downarrow 0. \tag{S.5.36}$$

On the other hand, we have

$$\int_T^\infty \|u_h^0(t,\Pi_h x) - u^0(t,x)\|_U^2 dt \le 2\int_T^\infty \|u_h^0(t,\Pi_h x)\|_U^2 + \|u^0(t,x)\|_U^2 dt$$

$$\text{(by ((S.4.20) and (1.7))} \quad = \int_T^\infty \|B_h^* P_h e^{A_{h,P_h} t}\Pi_h x\|_U^2 dt + \int_T^\infty \|B^* P e^{A_P t} x\|_U^2 dt$$

$$\le \int_T^\infty C(e^{-2\omega_P t} + e^{-2\bar{\omega}_P t})dt\,\|x\|_H$$

$$\le C\, e^{-\omega/h^{\varepsilon_0}} \to 0 \quad \text{as } T = 1/h^{\varepsilon_0} \to \infty, \tag{S.5.37}$$

where in the last step we have used the bound (4.29) of Theorem 4.7 and the exponential bound (4.27) of Theorem 4.6 in the first integral and (1.11), (1.12) for the second integral. A similar upper bound and a similar convergence as in (S.5.37) holds true *a fortiori* if u_h^0, u^0 are replaced by $\Phi_h(\cdot)\Pi_h x$ and $\Phi(\cdot)x$. Thus (S.5.37), combined with (S.5.36) for any T, yields the desired conclusions, (5.6) and (5.7). Thus, parts (i) and (ii) of Lemma 5.4 are proved.

(iii) From the representation (see (S.4.20) and point (6) of Theorem 1.0)

$$R(\lambda,A_{h,P_h})\Pi_h x - R(\lambda,A_P)x = \int_0^\infty e^{-\lambda t}[\Phi_h(t)\Pi_h x - \Phi(t)x]dt \tag{S.5.38}$$

for Re $\lambda > \min\{\omega_P, \bar{\omega}_P\}$ (defined in (1.12) and (4.27)), and from (5.7) of part (ii), we obtain with $\varepsilon_0 < s(1-\gamma)$:

$$\|R(\lambda,A_{h,P_h})\Pi_h - R(\lambda,A_P)\|_{\mathcal{L}(H)} \le C\, h^{\varepsilon_0} \to 0 \text{ as } h \downarrow 0. \tag{S.5.39}$$

Then (S.5.39), combined with the uniform bounds (1.12) and (4.27) for $\Phi(t)$ and $\Phi_h(t)$, allows us to invoke the Trotter-Kato Theorem [22, p. 87] and obtain

$$\|[\Phi_h(\cdot)\Pi_h-\Phi(\cdot)]x\|_{C([0,T];H)} \to 0 \quad \text{as } h \downarrow 0,\ x \in H. \tag{S.5.40}$$

Then (S.5.40), combined with the exponential decay of $\Phi(t)$ and $\Phi_h(t)$ (uniformly in h) from (1.12), (4.27), implies (5.8). ∎

Continuing with the proof of Theorem 5.3, we can now handle the term $I_{3,h}$ in (S.5.27): From (S.5.27) and (1.3) we obtain

$$\|I_{3,h}\| \le C\int_0^\infty \frac{e^{-\hat{\omega}t}}{t^\gamma}\,\|\hat{\Phi}_h(t)\Pi_h-\hat{\Phi}(t)\|_{\mathcal{L}(H)}dt. \tag{S.5.41}$$

But the norm inside the integral in (S.5.41) is dominated by a decaying exponential by (1.12) and (4.27). Thus, Lebesgue's dominated convergence theorem applies in (S.5.41) and yields

$$\|I_{3,h}\| \to 0 \quad \text{as } h \downarrow 0, \tag{S.5.42}$$

as desired. (Note that for $\gamma < ½$, one still obtains $|I_{3,h}| \le C\,h^{\varepsilon_0}$.) Then, (S.5.29), (S.5.31), and (S.5.42) used in (S.5.24) produce the claimed convergence in (5.5). Theorem 5.3 is proved. ∎

5.3. Uniform convergence $u_h^0 \to u^0$

Proof of Corollary 5.5. Step 1. We shall first show that (5.9) holds with u replaced by $\hat{u}$. Indeed, from (2.9b) and (2.22) = (S.4.30)

$$\begin{aligned}\hat{u}_h^0(\ ,\Pi_h x)-\hat{u}^0(\ ,x) &= \hat{L}_h^*(I+2\omega P_h)\hat{y}_h^0(\ ,\Pi_h x)-\hat{L}^*(I+2\omega P)\hat{y}^0(\ ,x)\\ &= [\hat{L}_h^*(I+2\omega P_h)-\hat{L}^*(I+2\omega P)]\hat{y}_h^0(\ ,\Pi_h x)\\ &\quad + \hat{L}^*(I+2\omega P)[\hat{y}_h^0(\ ,\Pi_h x)-\hat{y}^0(\ ,x)] \equiv (1)+(2).\end{aligned} \tag{S.5.43}$$

By (3.9b), Eq. (5.1), and uniform boundedness of y_h^0 in (4.27), we obtain

$$\|(1)\|_{C([0,\infty];U)} \le C\,h^{\varepsilon_0}\|x\|_H,\ \forall\ \varepsilon_0 < s(1-\gamma). \tag{S.5.44}$$

As for the term (2), we invoke the result (5.4) of Theorem 5.2 which, together with the estimate (via (2.4), (1.3))

$$|\hat{L}^*(t^{-\varepsilon}f)|_{C([0,\infty];U)} \le \sup_{t\in[0,\infty]}\int_t^\infty \frac{\|f(\tau)\|_U d\tau}{(\tau-t)^\gamma\tau^\varepsilon} \le C\|f\|_{L_\infty(0,\infty;H)}, \quad \text{for } \gamma+\varepsilon < 1,$$

gives for $\varepsilon_0 < s(1-\gamma)$,

$$\|(2)\|_{C([0,\infty];U)} \le C\,h^{\varepsilon_0}\|x\|_H. \tag{S.5.45}$$

Thus we obtain the desired estimate by using (S.5.43)-(S.5.45),

$$\|\hat{u}_h^0(\ ,\Pi_h x)-\hat{u}^0(\ ,x)\|_{\mathcal{L}(H;C([0,\infty];U)} \le C\,h^{\varepsilon_0}. \tag{S.5.46}$$

Step 2. From (S.5.46), it follows that for any fixed T > 0,

$$\|u_h^0(\ ,\Pi_h x)-u^0(\ ,x)\|_{\mathcal{L}(H;C([0,2T];U)} \le C_T\,h^{\varepsilon_0}.$$

Hence, in particular,

$$\sup_{0\le t\le 2T}\left\|B_h^*P_h\,e^{A_{h,P_h}T}\Pi_h - B^*P\,e^{A_Pt}\right\|_{\mathcal{L}(H;U)} \le C_T\,h^{\varepsilon_0}. \tag{S.5.47}$$

By the semigroup property, for any $t > 2T$ we compute via (S.4.20), (1.7),

$$\|u_h^0(t,\Pi_h x)-u^0(t,x)\|_{\mathcal{L}(H;U)} = \left\|B_h^* P_h\, e^{A_{h,P_h}T}\, e^{A_{h,P_h}(t-T)}\,\Pi_h - B^* P e^{A_P T} e^{A_P(t-T)}\right\|_{\mathcal{L}(H)}$$

$$\le \left\|B_h^* P_h\, e^{A_{h,P_h}T}\,\Pi_h - B^* P e^{A_P T}\right\|_{\mathcal{L}(H;U)} \left\|e^{A_{h,P_h}(t-T)}\,\Pi_h\right\|_{\mathcal{L}(H)}$$

$$+ \|B^* P\, e^{A_P T}\|_{\mathcal{L}(H;U)} \left\|e^{A_{h,P_h}(t-T)}\,\Pi_h - e^{A_P(t-T)}\right\|_{\mathcal{L}(H)}$$

$$\le C_T\, h^{\varepsilon_0} e^{-\bar{\omega}_P(t-T)} + C\, e^{-\omega_P T} \frac{h^{\varepsilon_0}}{t-T} e^{-\bar{\omega}_P(t-T)}, \qquad \text{(S.5.48)}$$

where in the last step we have used (S.5.47), (4.27) for the first term and (1.11), (1.12), and (5.4) with $\varepsilon = 1$ for the second term: this way we obtain the desired result in (5.9). Corollary 5.5 is proved. ∎

5.4. Convergence $(\hat{A}^*)^\Theta(P_h\Pi_h - P)x \to 0$

Proof of Proposition 5.6. (i) We return to (2.20) and get

$$(\hat{A}^*)^\Theta P_h \Pi_h x = \int_0^\infty (\hat{A}^*)^\Theta e^{-\hat{A}_h^* t}\,\Pi_h [R^* R + 2\omega P_h]\hat{\Phi}_h(t)x\, dt, \quad x \in H, \qquad \text{(S.5.49)}$$

where $(\hat{A}^*)^\Theta e^{-\hat{A}_h^* t} = (\hat{A}^*)^\Theta (\hat{A}_h^{*-1})^\Theta (\hat{A}_h^*)^\Theta e^{-\hat{A}_h^* t} = \mathcal{O}(e^{-\hat{\omega} t}/t^\Theta)$ by (5.10) and uniform analyticity (1.14), $\Theta < 1$. Moreover, $\hat{\Phi}_h(\cdot)x \to \hat{\Phi}(\cdot)x$ in $C([0,\infty];H)$ by (5.8) of Lemma 5.4(iii), and $P_h \to P$ by (5.1). Thus, letting $h \downarrow 0$ in (S.5.49) and recalling (2.10), we obtain the limit $(\hat{A}^*)^\Theta Px$ and (5.11) is proved.

(ii) By (4.33) of Corollary 4.8 for the discrete problem and the corresponding version for the continuous problem, we have

$$\|(\hat{A}^*)^\Theta (P_h\Pi_h - P)\hat{A}^\Theta\| \le \text{const}_\Theta, \quad 0 \le \Theta < \tfrac{1}{2}. \qquad \text{(S.5.50)}$$

Then (5.11) of part (i), combined with density of $\mathcal{D}(\hat{A}^\Theta)$ and with the uniform bound (S.5.50) yields (5.12) as desired. ∎

5.5. Completion of the proof of main Theorem 1.2

The conclusion (1.41) of Theorem 1.2 follows from Theorem 4.2 (see also Remark 4.2(ii)) and Theorem 5.3: the latter provides $\|B_h^* P_h \Pi_h - B^* P\| \to 0$ in $\mathcal{L}(H;U)$, see (5.5), and in the former we take $F = -B^* P$, $F_h = -B_h^* P_h$. Then, by virtue of the exponential decay (1.7) of $e^{A_P t}$, we obtain the counterpart of conclusion (4.12), which is precisely (1.41).

As for (1.42), we obtain from (5.2) = (S.5.1) and (5.3) = (S.5.2), using the same formula for the difference of inverses as the one in (S.4.13),

$$R(\lambda,A_P) - R(\lambda,A_{P_h}) = R(\lambda,A_P)B[B^* P - B_h^* P_h]R(\lambda,A_h), \qquad \text{(S.5.51)}$$

since the term in the bracket in (S.5.51) is $A_P - A_{P_h}$, where

$$R(\lambda,A_P)B = [I + R(\lambda,A)BB^* P]^{-1} R(\lambda,A)B. \qquad \text{(S.5.52)}$$

It follows that

$$\|R(\lambda,A_P)B\|_{\mathcal{L}(U;H)} \le \frac{C}{|\lambda|^{1-\gamma}}, \quad \lambda \in \Gamma_P, \qquad \text{(S.5.53)}$$

where Γ_p is the path $\rho\, e^{\pm i\Theta_P} - \bar{\omega}_P$, $\pi/2 < \Theta_P < \pi$, $0 \le \rho < \infty$. In fact, (S.5.53) is true for $|\lambda|$ sufficiently large, by (S.5.52), (1.11) and $\|R(\lambda,A)B\| \le C/|\lambda|^{1-\gamma}$, in view of analyticity and (1.3); and hence, for $\lambda \in \Gamma_P$ by using the first resolvent equation on

$R(\lambda, A_P)$ (which we already know to be well defined on Γ_p). Thus, (S.5.53) used in (S.5.51) yields by uniform analyticity (1.14b) with $\Theta = 0$:

$$\|R(\lambda, A_P) - R(\lambda, A_{P_h})\|_{\mathcal{L}(U;H)} \le \frac{C}{|\lambda|^{2-\gamma}} \|B^* P - B_h^* P_h\|_{\mathcal{L}(H)}. \tag{S.5.54}$$

From (S.5.54) we obtain, as usual,

$$\|e^{A_P t} - e^{A_{P_h} t}\|_{\mathcal{L}(H)} \le C \int_{\Gamma_P} e^{\lambda t} \frac{C}{|\lambda|} d\lambda \; \|B^* P - B_h^* P_h\|_{\mathcal{L}(H;U)}$$

$$\le C\, e^{-\bar{\omega}_P t} \|B^* P - B_h^* P_h\|_{\mathcal{L}(H;U)}, \tag{S.5.55}$$

and (1.42) follows. Theorem 1.2 is proved. ∎

Section 6: Approximation framework and verification of all required assumptions

Example 6.1: Heat equation with Dirichlet boundary control

Assumption (1.3). $(\hat{A})^{-\gamma} B \in \mathcal{L}(U, Y)$. Assumption (1.3) is satisfied in our present case with $\gamma = \frac{1}{4}+\varepsilon$, $\forall\, \varepsilon > 0$. In fact, we may take $\hat{A} = A_D$. From (6.7), we have

$$B = -AD_1 : \text{continuous } L_2(\Gamma) \to [\mathcal{D}(\hat{A}^{\frac{1}{4}+\varepsilon})]' = [\mathcal{D}(A_D^{\frac{1}{4}+\varepsilon})]', \tag{S.6.1}$$

and we then have with $\gamma = \frac{1}{4}+\varepsilon$ via (6.5) that our claim is verified.

$$\hat{A}^{-\gamma} B = -A_D^{-\gamma} A D_1 \in \mathcal{L}(L_2(\Gamma), L_2(\Omega)) = \mathcal{L}(U, H). \tag{S.6.2}$$

Stabilizability condition (1.5). The generator A has (for suitably large constant c^2 in (6.3)) only finitely many unstable eigenvalues of finite multiplicity, since its resolvent is compact and e^{At} is analytic. Thus, the stabilization theory as in [25], [21; App.], etc., applies: The problem is stabilizable on $L_2(\Omega)$ if and only if its projection onto the finite-dimensional unstable subspace is controllable. In particular, as shown in [27], one may prescribe the stabilizing feedback to be of the form

$$u(t) = \sum_{n=1}^{N} (y(t), w_n)_{L_2(\Omega)} g_n \tag{S.6.3}$$

for suitable vectors $w_k \in L_2(\Omega)$ and $g_k \in L_2(\Gamma)$ and suitable (minimal) N as described there, in order to stabilize uniformly the corresponding feedback system in the norm of $H^{\frac{1}{2}-\varepsilon}(\Omega)$ in fact. Thus, *a fortiori* the Finite Cost Condition on $L_2(\Omega)$ is satisfied.

Detectability Condition (1.6). This is automatically satisfied since in our case R = I, see (6.5).

Conclusion. Theorem 1.0 applies to problem (6.1), (6.2).

6.1.2. Discrete problem

We present next the approximation framework for the problem (6.1), (6.2) [18].

Choice of V_h. We shall select the approximating space $V_h \subset H_0^1(\Omega)$ to be a space of splines (linear, quadratic, curvilinear, etc.) which comply with the usual approximation properties ([2], [24]):

$$\|\Pi_h y - y\|_{H^\ell(\Omega)} \le C\, h^{s-\ell} \|y\|_{H^s(\Omega)}, \quad s \le 2; \quad s-\ell \ge 0; \quad 0 \le \ell \le 1; \tag{S.6.4}$$

(inverse approximation properties)

$$\|y_h\|_{H^\alpha(\Omega)} \le C\, h^{-\alpha} \|y_h\|_{L_2(\Omega)}, \quad 0 \le \alpha \le 1, \tag{S.6.5}$$

$$\left\|\frac{\partial}{\partial \nu}(y - \Pi_h y)\right\|_{L_2(\Gamma)} \le C\, h^{s-\frac{1}{2}} \|y\|_{H^s(\Omega)}, \quad \tfrac{1}{2} < s \le 2, \tag{S.6.6}$$

$$\left\|\frac{\partial y_h}{\partial \nu}\right\|_{L_2(\Gamma)} \le C\, h^{-3/2}\|y_h\|_{L_2(\Omega)}, \quad y_h \in V_h, \tag{S.6.7}$$

where Π_h is the orthogonal projection of $L_2(\Omega)$ onto V_h.

Choice of A_h. We define $A_h: V_h \to V_h$ as usual, where the inner products are in L_2:

$$(A_h x_h, y_h)_\Omega = (Ax_h, y_h)_\Omega = \int_\Omega \nabla x_h \cdot \nabla y_h \, d\Omega + c^2 (x_h, y_h)_\Omega, \quad x_h, y_h \in V_h. \tag{S.6.8}$$

Choice of B_h. With reference to (6.5), we define $B_h: U \to V_h$ by

$$B_h = -\Pi_h A D_1, \tag{S.6.9}$$

D_i as in (6.6), (6.7), and we notice that (L_2-inner products)

$$(B_h u, y_h)_\Omega = -(AD_1 u, y_h)_\Omega = -(u, D_1^* A y_h)_\Gamma = (u, \frac{\partial y_h}{\partial \nu})_\Gamma. \tag{S.6.10}$$

Hence

$$B_h^* y_h = \frac{\partial y_h}{\partial \nu}. \tag{S.6.11}$$

Approximating control problem. This is given by the O.D.E. problem:

$$\begin{cases} (\dot{y}_h, \phi_h)_\Omega + \int_\Omega \nabla y_h \cdot \nabla \phi_h \, d\Omega + c^2 \int_\Omega y_h \phi_h \, d\Omega = (u, \frac{\partial}{\partial \nu}\phi_h)_\Gamma, \quad \phi_h \in V_h; \\ (y_h(0), \phi_h)_\Omega = (y(0), \phi_h)_\Omega. \end{cases} \tag{S.6.12}$$

The optimal feedback control for the approximating finite-dimensional problem is

$$u_h^0(t; y_0) = \frac{\partial}{\partial \nu} P_h y_h^0(t; y_0),$$

where P_h satisfies the following discrete Algebraic Riccati Equation

$$-\int_\Omega \nabla P_h x_h \cdot \nabla y_h \, d\Omega - \int_\Omega \nabla x_h \cdot \nabla P_h y_h \, d\Omega + (x_h, y_h)_\Omega = (\frac{\partial}{\partial \nu} P_h x_h, \frac{\partial}{\partial \nu} P_h y_h)_\Gamma, \quad \forall\, x_h, y_h \in V_h. \tag{S.6.13}$$

Eq. (S.6.13) leads to a standard matrix Riccati equation, which can be effectively solved by finite-dimensional methods [12].

Verification of assumptions of Theorem 1.1. Assumptions (1.26) and (1.27). These are plainly satisfied since $R = I$ and $\hat{A}^{-\gamma} B \in \mathcal{L}(U, H)$ with $\gamma = 3/4 + \varepsilon$ in our case. Because of the compactness of A^{-1} (since Ω is bounded), this then implies in turn that $\hat{A}^{-1}B$ is compact $U \to H$, and thus $B^*(\hat{A}^*)^{-1}$ is compact $H \to U$, as desired.

Assumption (A.1) = (1.14) (uniform analyticity). That this is satisfied follows from results on Galerkin approximations of elliptic operators, see [3] for the self-adjoint case and [14] for the general non-self-adjoint case.

Assumption (A.2) = (1.15). The standard elliptic approximation estimate is

$$\|\Pi_h \hat{A}^{-1} - \hat{A}_h^{-1} \Pi_h\|_{\mathcal{L}(L_2(\Omega))} \le C\, h^2,$$

so that (A.2) holds with $s = 2$.

Assumption (A.3) = (1.16). By (S.6.11) and (S.6.7), we obtain with $U = L_2(\Gamma)$ and $H = L_2(\Omega)$,

$$\|B^* y_h\|_U = \|B_h^* y_h\|_U = \|\frac{\partial}{\partial \nu} y_h\|_{L_2(\Gamma)} \le C\, h^{-3/2}\|y_h\|_{L_2(\Omega)} \tag{S.6.14}$$

and (A.3) follows since $\gamma s = 2(3/4 + \varepsilon) > 3/2$.

Assumption (A.4) = (1.17). By (S.6.11) and (S.6.6) applied with $s = 2$,

$$\|B^*(\Pi_h x - x)\|_{L_2(\Gamma)} = \|\frac{\partial}{\partial \nu}(\Pi_h x - x)\|_{L_2(\Gamma)} \le C\, h^{1/2}\|x\|_{H^2(\Omega)}, \tag{S.6.15}$$

which implies (A.4) in view of the fact that $\mathcal{D}(A) \subset H^2(\Omega)$ and $s(1-\gamma) = 2(1-\frac{3}{4}-\varepsilon) = \frac{1}{2}-2\varepsilon < \frac{1}{2}$.

<u>Assumption (A.5) = (1.18)</u>. Since in our case $B_h^*\Pi_h = B^*\Pi_h$, (A.4) coincides with (A.5).

<u>Assumption (A.6) = (1.19)</u>. From (S.6.6) applied with $s = \frac{3}{2}+\varepsilon$ and from the trace theorem, we obtain

$$\|B^*\Pi_h x\|_{L_2(\Gamma)} = \|\frac{\partial}{\partial\nu}\Pi_h x\|_{L_2(\Gamma)} \le \|\frac{\partial}{\partial\nu}(\Pi_h - I)x\|_{L_2(\Gamma)}$$

$$+ \|\frac{\partial}{\partial\nu}x\|_{L_2(\Gamma)} \le C\,h^{\varepsilon}\|x\|_{H^{\frac{3}{2}+\varepsilon}(\Omega)} + C\|x\|_{H^{\frac{3}{2}+\varepsilon}(\Omega)}$$

(A.6) follows now from $\mathcal{D}(A^{*\frac{3}{4}+\varepsilon}) \subset H^{\frac{3}{2}+2\varepsilon}(\Omega)$.

<u>Conclusion</u>. Thus, we have verified all the assumptions of Theorems 1.1 and 1.2 in the case of the heat equation problem with Dirichlet boundary control as in (6.1). Then, application of Theorem 1.1 yields the following convergence results (see also [18]):

(i) $\|P_h\Pi_h - P\|_{\mathcal{L}(L_2(\Omega))} \le C\,h^{\varepsilon_0}, \quad \varepsilon_0 < \frac{1}{2}$;

(ii) $\|\frac{\partial}{\partial\nu}[P_h\Pi_h - P]\|_{\mathcal{L}(L_2(\Omega);L_2(\Gamma))} \to 0$ as $h \downarrow 0$;,

(iii) $\|y_h^0 - y^0\|_{\mathcal{L}(L_2(\Omega);L_2(0,\infty;L_2(\Omega)))} + \sup_{t\ge 0} e^{\bar{\omega}_P t}\, t\|y_h^0(t) - y^0(t)\|_{\mathcal{L}(L_2(\Omega))} \le C\,h^{\varepsilon_0}, \quad \varepsilon_0 < \frac{1}{2}$;

(iv) $\sup_{t\ge 0} e^{\bar{\omega}_P t}\|u_h^0(t) - u^0(t)\|_{\mathcal{L}(L_2(\Omega);L_2(\Gamma))} \le C\,h^{\varepsilon_0}, \quad \varepsilon_0 < \frac{1}{2}$;

(v) since $\|\hat{A}^{*\Theta}x_h\| = \|\hat{A}_h^{*\Theta}x_h\|$ for $0 \le \Theta \le \frac{1}{2}$, (1.39) gives

$$\|(P_h\Pi_h - P)x\|_{H^1(\Omega)} \to 0, \quad x \in L_2(\Omega).$$

Application of Theorem 1.2 yields the following result: If we use the feedback law given by

$$u_h^*(t) = -\frac{\partial}{\partial\nu}P_h\, y^h(t),$$

which we insert into the original dynamics

$$\begin{cases} y_t^h = (\Delta + c^2)y^h, \\ y^h|_\Sigma = u_h^*, \end{cases} \tag{S.6.16}$$

then the corresponding system is exponentially stable in $L_2(\Omega)$ uniformly in the parameter h. Moreover,

$$\sup_{t\ge 0} e^{\bar{\omega}_P t}\|y^h(t) - y^0(t)\|_{\mathcal{L}(L_2(\Omega))} \to 0. \tag{S.6.17}$$

Other boundary conditions, like Neumann or Robin can be treated similarly (see [19]). In fact, the analysis here is even simpler as $\gamma = \frac{1}{4}+\varepsilon < \frac{1}{2}$, if one takes $H = L_2(\Omega)$; otherwise, $\gamma = \frac{3}{4}+\varepsilon$ if one takes $H = H^1(\Omega)$.

<u>Example 6.2: Structurally damped plates with point control</u>

<u>Assumption (1.3)</u>: $(-A)^{-\gamma}B \in \mathcal{L}(U,H)$. It is easy to verify that assumption (1.3) is satisfied with $\gamma = 1$. Indeed, from (6.9), we require that

$$(-A)^{-1}Bu = \begin{bmatrix} \rho\mathcal{A}^{-\frac{1}{2}} & \mathcal{A}^{-1} \\ -I & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \delta(x-x^0)u \end{bmatrix} = \begin{bmatrix} \mathcal{A}^{-1}\delta(x-x^0)u \\ 0 \end{bmatrix} \in H, \tag{S.6.18}$$

i.e., from (6.12), we require that $\mathcal{A}^{-\frac{1}{2}}\delta(x-x^0) \in L_2(\Omega)$, or that (*): $\delta(x-x^0) \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, the dual of $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ with respect to $L_2(\Omega)$. Since it is true that $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset H^2(\Omega)$ for the fourth-order operator $\mathcal{A}$ in (6.11) (in fact, regardless of the particular boundary conditions), and thus $[H^2(\Omega)]' \subset [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, then condition (*) is satisfied provided $\delta(x-x^0) \in [H^2(\Omega)]'$, i.e., provided $H^2(\Omega) \subset C(\bar{\Omega})$, which is indeed the case by Sobolev embedding provided $2 > \frac{n}{2}$, or $n < 4$, as required.

However, the above result is not sufficient for our purposes as--according to our assumption--we need to show that we can take $\gamma < 1$ in (1.3). As a matter of fact, we now show that assumption (1.3) holds true for any $\gamma > \frac{n}{4}$, which then for $n \leq 3$ yields $\gamma < 1$ as desired. To this end, we note that

$$(-A)^{-\gamma}B \in \mathcal{L}(U,H) \text{ if and only if } B \in \mathcal{L}(U,[\mathcal{D}((-A^*)^{\gamma})]') \tag{S.6.19}$$

with duality with respect to H. But $\mathcal{D}((-A^*)^{\gamma}) = \mathcal{D}((-A)^{\gamma})$: this follows since A is the direct sum of two normal operators on H, with possibly an additional finite-dimensional component (if 1 is an eigenvalue of $\mathcal{A}$) [4], [5, Lemma A.1, case v(a) with $\alpha = \frac{1}{2}$]. Moreover, [7, with $\alpha = \frac{1}{2}$], we have

$$\mathcal{D}((-A^*)^{\gamma}) = \mathcal{D}((-A)^{\gamma}) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}+\gamma/2})\times\mathcal{D}(\mathcal{A}^{\gamma/2}),\ 0 < \gamma < 1 \tag{S.6.20}$$

(the first component does not really matter in the argument below). Thus, from (S.6.20) and B as in (6.13), it follows that (S.6.19) holds true, provided $\delta(x-x^0) \in [\mathcal{D}(\mathcal{A}^{\gamma/2})]'$ (duality with respect to $L_2(\Omega)$), where $\mathcal{D}(\mathcal{A}^{\gamma/2}) \subset H^{2\gamma}(\Omega)$, and hence, provided $\delta(x-x^0) \in [H^{2\gamma}(\Omega)]' \subset [\mathcal{D}(\mathcal{A}^{\gamma/2})]'$. But this in turn is the case, provided $H^{2\gamma}(\Omega) \subset C(\bar{\Omega})$; i.e., by Sobolev embedding provided $2\gamma > \frac{n}{2}$, as desired. We conclude: <u>assumption</u> (1.3) $(-A)^{-\gamma}B \in \mathcal{L}(U,H)$ <u>holds true for problem</u> (6.9) <u>with</u> $\frac{n}{4} < \gamma < 1$, $n \leq 3$.

Also, the operator A in (6.13) generates an s.c. contraction semigroup e^{At} on H, which moreover is analytic here for $t > 0$. (This is a special case of a much more general result [4-5]). This, along with the requirement $\gamma < 1$ proved above guarantees that problem (6.9) satisfies our preliminary assumption (ii) of the Introduction, below (1.2).

<u>Stabilizability Condition (1.5)</u>. With A as in (6.13), the semigroup e^{At} is uniformly (exponentially) stable in H [5], and thus the Finite Cost Condition (1.5) holds true with $u \equiv 0$.

<u>Remark 6.1</u>. Suppose that instead of Eq. (6.9a), one has

$$w_{tt}+(\Delta^2-k_1)w-(\Delta+k_2)w_t = \delta(x-x^0)u(t) \quad \text{in } Q, \tag{S.6.21}$$

along with (6.9b-c). Then, if $0 < k_1+k_2$ is sufficiently large, the generator A has finitely many unstable eigenvalues in $\{\mathrm{Re}\,\lambda > 0\}$. Since e^{At} is analytic on H, the usual theory [25] applies: The problem is stabilizable on H if [25] and only if [21] its projection onto the finite-dimensional unstable subspace is controllable.

For instance, if $\lambda_1,\ldots,\lambda_K$ are the unstable eigenvalues of A, assumed for simplicity to be simple, and $\Phi_1,\ldots,\Phi_K$ are the corresponding eigenfunctions in H, then the necessary and sufficient condition for stabilization is that $\Phi_k(x^0) \neq 0$, $k = 1,\ldots,K$.

If $\lambda_1,\ldots,\lambda_K$ are not simple, then their largest multiplicity M determines the smallest number of scalar controls needed for the stabilization of (S.6.21), where now the right-hand side is replaced by $\sum_{i=1}^{M} \delta(x-x^i)u_i(t)$, along with (6.9b-c). The necessary and sufficient condition for stabilization is now a well-known full-rank condition [25].

<u>Detectability Condition (D.C.)</u>. This is satisfied since in our case $R = I$, see (6.13).

<u>Conclusion</u>. Theorem 1.0 applies to problem (6.9)-(6.10), $n \leq 3$, and provides existence and uniqueness of the solution to the ARE (1.8), with Riccati operator $P \in \mathcal{L}(H,\mathcal{D}(A))$ (since A, as remarked above (S.6.20), is the direct sum of two normal operators on H, plus possibly a finite-dimensional component, in particular, A has a Riesz basis of

eigenvectors on H), where $\mathcal{D}(A) = \mathcal{D}(\mathcal{A})\times\mathcal{D}(\mathcal{A}^{1/2})$, see (6.11), (6.12) for the characterizations of these spaces. Thus, in particular, we have $B^*P \in \mathcal{L}(H;U)$, where $B^*\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_2(x^0)$.

6.2.2. Discrete problem

Choice of V_h. We shall select the approximating space $V_h \subset H^2(\Omega) \cap H_0^1(\Omega)$ to be a space of splines (e.g., quadratic or cubic splines or curvilinear), which comply with the usual approximation properties

$$\|Q_h z - z\|_{H^\ell(\Omega)} \le C\, h^{s-\ell}\|z\|_{H^s(\Omega)}, \quad z \in H^s(\Omega) \cap H_0^1(\Omega), \quad 0 \le \ell \le 2;\ \ell \le s < r; \tag{S.6.22}$$

$$\|z_h\|_{H^\alpha(\Omega)} \le C\, h^{-s}\|z_h\|_{H^{\alpha-s}(\Omega)}, \quad 0 \le \alpha \le 2, \tag{S.6.23}$$

where Q_h is the orthogonal projection of $L_2(\Omega)$ onto V_h and where r is the order of approximation.

Choice of $\mathcal{A}_h$. We let

$$\mathcal{A}_h = Q_h \mathcal{A} Q_h : V_h \to V_h;$$

i.e.,

$$(\mathcal{A}_h\phi_h, \psi_h)_\Omega = (\Delta\phi_h, \Delta\psi_h)_\Omega = (\mathcal{A}^{1/2}\phi_h, \mathcal{A}^{1/2}\psi_h)_\Omega, \quad \phi_h, \psi_h \in V_h, \tag{S.6.24}$$

$$\|\mathcal{A}_h^{1/2}\phi_h\|_{L_2(\Omega)} = \|\mathcal{A}^{1/2}\phi_h\|_{L_2(\Omega)}; \quad \|\mathcal{A}_h^{1/2}\phi_h\| \sim \|\phi_h\|_{H^2(\Omega)}, \quad \phi_h \in V_h, \tag{S.6.25}$$

where (S.6.25) is a consequence of (S.6.24). From the estimates for the biharmonic operator, see [2], we obtain

$$\begin{cases} \|(\mathcal{A}^{-1} - \mathcal{A}_h^{-1}Q_h)z\|_{H^2(\Omega)} \le C\, h^2\|z\|_{L_2(\Omega)}; \\ \|(\mathcal{A}^{-1/2} - \mathcal{A}_h^{-1/2}Q_h)z\|_{H^2(\Omega)} \le C\, h^2\|z\|_{H^2(\Omega)}, \quad z \in \mathcal{D}(\mathcal{A}^{1/2}). \end{cases} \tag{S.6.26}$$

Choice of A_h and B_h. To begin with, we let $H_h \equiv V_{h1}\times V_{h2}$, where V_{h1} consists of the elements of V_h equipped with norm $\|v_h\|_{V_h^1} = \|\mathcal{A}_h^{1/2}v_h\|_{L_2(\Omega)} = \|\mathcal{A}^{1/2}v_h\|_{L_2(\Omega)}$, and V_{h2} consists of the elements of V_h equipped with the $L_2(\Omega)$-norm. We shall write $x_h = [x_{h1}, x_{h2}] \in H_h$. Next, we define

$$A_h : H_h \to H_h : A_h = \begin{bmatrix} 0 & Q_h \\ -\mathcal{A}_h & -\mathcal{A}_h^{1/2} \end{bmatrix}; \tag{S.6.27}$$

$$B_h : L_2(\Gamma) \to H_h : B_h u = \begin{bmatrix} 0 \\ \mathcal{B}_h u \end{bmatrix}, \quad (\mathcal{B}_h u, v_h)_\Omega = v_h(x^0)u. \tag{S.6.28}$$

Finally, we let $\Pi_h : H \to H_h$ be defined as

$$\Pi_h = \begin{bmatrix} Q_h & 0 \\ 0 & Q_h \end{bmatrix}.$$

Computation of adjoints A_h^* and B_h^*. To compute the adjoints of A_h and B_h, we use the inner products generated by the topology on V_{h1} and V_{h2}. We find, as in the continuous case,

$$A_h^* = \begin{bmatrix} 0 & -Q_h \\ \mathcal{A}_h & -\mathcal{A}_h^{1/2} \end{bmatrix}; \quad B_h^* x_h = x_{h2}(x^0)$$

as it follows from $(A_h x_h, y_h)_{H_h} = (x_h, A_h^* y_h)_{H_h}$ and $(B_h u, x_h)_{H_h}$ and $(u, B_h^* x_h)_U$, respectively.

Approximating control problem. With the above notation, the approximating version of the dynamics is

$$\begin{cases} (\ddot{y}_h,\phi_h)+(\mathcal{A}_h y_h,\phi_h)+(\mathcal{A}_h^{1/2} y_h,\phi_h) = \phi_{h2}(x^0)u; \\ (\mathcal{A}_h y_h,\phi_h) = (\Delta y_n,\Delta\phi_h), \quad \text{all } \phi_h \in V_h; \\ (y_h(0),\phi_h) = (y_0,\phi_h);\ (\dot{y}_h(0),\phi_h) = (y_1,\phi_h), \end{cases} \tag{S.6.29}$$

where all inner products are in $L_2(\Omega)$.

The optimal feedback control for the finite-dimensional problem is given by $u_h^0(t) \equiv -[\overline{P}_{h2}y_h(t)][x^0]$, where

$$P_h y_h \equiv \begin{cases} P_{h1}y_{h1}+P_{h2}y_{h2} \equiv \overline{P}_{h1}y_h; \\ P_{h3}y_{h1}+P_{h4}y_{h2} \equiv \overline{P}_{h2}y_h; \end{cases} \tag{S.6.30}$$

and P_h satisfies the following algebraic equation with $L_2(\Omega)$-inner products

$$\begin{aligned} &(\mathcal{A}_h x_{h2},\overline{P}_{h1}y_h) + (\mathcal{A}_h x_{h1}-\mathcal{A}_h^{1/2}x_{h2},\overline{P}_{h2}y_h) - (\mathcal{A}_h\overline{P}_{h1}x_h,y_{h2}) \\ &+ (\overline{P}_{h2}x_h,\mathcal{A}_h y_{h1}-\mathcal{A}_h^{1/2}y_{h2}) + (\mathcal{A}_h x_{h1},y_{h1}) + (x_{h2},y_{h2}) \\ &= (\overline{P}_{h2}x_h)(x^0)(\overline{P}_{h2}y_h)(x^0). \end{aligned} \tag{S.6.31}$$

Again, (S.6.31) leads to a matrix Riccati equation, which can be effectively solved by finite-dimensional methods [12].

Verification of assumptions of Theorem 1.1. In order to apply Theorem 1.1, we need to verify the approximating assumptions (A.1)-(A.6), as well as assumptions (1.26), (1.27). Indeed, the last two are plainly satisfied: (1.26) since $R = I$, while (1.27) follows from (S.6.18) and the argument below it (in essence, $\mathcal{A}^{-(1/2-\varepsilon)}\delta \in L_2(\Omega)$), while $\mathcal{A}^{-\varepsilon}$ is compact on $L_2(\Omega)$).

Assumption (A.1). This follows by applying the arguments of [5] of the continuous case o the finite-dimensional operator given by (S.6.27).

Assumption (A.2). By (6.38), we have that (A.2) with $s = 2$ holds true:

$$\begin{aligned} \|(A_h^{-1}\Pi_h-A^{-1})x_h\|_H &= \|-(\mathcal{A}^{-1/2}-\mathcal{A}_h^{-1/2})x_{h1}+(\mathcal{A}^{-1}-\mathcal{A}_h^{-1})x_{h2}\|_{H^2(\Omega)} \\ &\le C\,h^2[\|x_{h1}\|_{H^2(\Omega)}+\|x_{h2}\|_{L_2(\Omega)}] = C\,h^2[\|x_h\|_H]. \end{aligned}$$

The same result holds for the adjoint A^*, in view of its definition.

Assumption (A.3). By Sobolev embedding and the inverse approximation property (S.6.23), we have for any $\varepsilon > 0$,

$$\begin{aligned} \|B_h^* x_h\|_U = \|B^* x_h\|_U = \|x_{h2}(x^0)\| &\le C\|x_{h2}\|_{H^{n/2+\varepsilon}(\Omega)} \\ &\le C\,h^{-n/2-\varepsilon}\|x_{h2}\|_{L_2(\Omega)} \le C\,h^{-n/2-\varepsilon}\|x_h\|_H, \end{aligned}$$

and (A.3) follows since $\gamma s = (n/4+\varepsilon)2 > n/2$.

Assumption (A.4). By (S.6.22) we compute

$$\begin{aligned} \|B^*(\Pi_h x-x)\|_U = |(Q_h x_2)(x^0)-x_2(x^0)|_{\mathbb{R}^1} &\le C\|Q_h x_2-x_2\|_{H^{n/2+\varepsilon}(\Omega)} \\ &\le C\,h^{2-n/2-\varepsilon}\|x_2\|_{H^2(\Omega)} \\ &\le C\,h^{2-n/2-\varepsilon}\|x\|_{\mathcal{D}(A^*)} \end{aligned}$$

Since $2(1-\gamma) = 2(1-\frac{n}{4}-\varepsilon) < 2-\frac{n}{2}-\varepsilon$, then (A.4) is satisfied.

Assumption (A.5). It coincides with (A.4).

Assumption (A.6).

$$\|B^*\Pi_h x\|_U = \|x_{h_2}(x^0)\|_{L_2(\Omega)} \le C\|x_{h_2}\|_{H^{n/2+\varepsilon}(\Omega)} \le C\|x_h\|_{\mathcal{D}(A^{*\gamma})}$$

(as in [7]), $\mathcal{D}(A^{*\gamma}) \subset H^{4\gamma}(\Omega)\times H^{2\gamma}(\Omega)$ and $2\gamma = 2(\frac{n}{4}+\varepsilon) = \frac{n}{2}+2\varepsilon > \frac{n}{2}+\varepsilon$.

Conclusion. Thus, we have verified all the assumptions of Theorem 1.1. Thus, Theorem 1.1 applies to our problem, and yields the following convergence results:

(i) $$\|P_h\Pi_h - P\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))} \le C\,h^{\varepsilon_0}, \quad \varepsilon_0 < \frac{4-n}{2};$$

(ii) $$\left\|(P_h\Pi_h - P)|_{x=x^0}\right\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);R^1)} \to 0 \quad \text{as } h \to 0,$$

or equivalently

$$\|B^*P_h\Pi_h - B^*P\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);R)} \to 0 \quad \text{as } h \downarrow 0,$$

where P_h is computed from (S.6.31).

(iii) $$\sup_{t\ge 0} e^{\bar{\omega}_P t}\|u_h^0(t)-u^0(t)\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);R)} \le C\,h^{\varepsilon_0};$$

(iv) $$\sup_{t\ge 0} t\,e^{\bar{\omega}_P t}\|y_h^0(t)-y^0(t)\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))} \le C\,h^{\varepsilon_0}, \quad \varepsilon_0 < \frac{4-n}{2}.$$

Application of Theorem 1.2 to our problem yields the following result: Let $u_h^*(t)$ be a feedback law given by

$$u_h^*(t) = -[\bar{P}_{h2}y(t)][x^0]$$

which we insert into the original dynamics (6.9) to obtain

$$w_{tt}+\Delta^2 w-\rho\Delta w_t = \delta(x-x^0)u_h^*;\ w|_\Gamma = \Delta w|_\Gamma = 0.$$

Then the corresponding feedback system is uniformly (in h) exponentially stable in the topology of $H^2(\Omega)\times L_2(\Omega)$ and uniformly approximates the original feedback dynamics. This means that the numerical algorithm provides a feedback control which yields uniform (in h) stability results for the original system.

We conclude this section by pointing out that the other examples of [19, Section 3.3] dealing with structurally damped plate problems can be dealt with by a similar approximating scheme.

Example 6.3: Kelvin-Voigt plate equation with point control

Assumption (1.3) $(-A)^{-\gamma}B \in \mathcal{L}(U,Y)$. Again, it is straightforward to verify that assumption (1.3) is satisfied with $\gamma = 1$: From (6.48), we require that

$$(-A)^{-1}Bu = \begin{pmatrix}\rho I & A^{-1}\\ -I & 0\end{pmatrix}\begin{pmatrix}0\\ \delta(x-x^0)u\end{pmatrix} = \begin{pmatrix}A^{-1}\delta(x-x^0)\\ 0\end{pmatrix} \in H, \tag{S.6.32}$$

i.e., from (6.47) we require that $A^{-1/2}\delta(x-x^0)$. The Sobolev imbedding then yields that (S.6.32) holds true if $n \le 3$.

However, in order to verify assumption (1.3), which requires that γ should be < 1, the most elementary way is to check that assumption (1.3) holds in fact true with $\gamma = 1/2$. In this case, we can in fact rely on the direct computation of $(-A)^{-1/2}$ (for simplicity of notation, we take henceforth $\rho = 1$)

$$(-A)^{-1/2} = \begin{pmatrix}(1) & A^{-1/4}(2I+A^{1/2})^{-1/2}\\ (2) & A^{-1/4}(2I+A^{1/2})^{-1/2}\end{pmatrix} \tag{S.6.33}$$

(where the entries $(1) = A^{-1/4}(2I+A^{1/2})^{-1/2}(I+A^{1/2})$ and $(2) = -A^{1/4}(2I+A^{1/2})^{-1/2}$ do not really count in the present analysis), and avoid the domain of fractional powers as in [7].

We need to compute

$$(-A)^{-\frac{1}{2}}Bu = \begin{bmatrix} \mathcal{A}^{-\frac{1}{4}}(2I+\mathcal{A}^{\frac{1}{2}})^{-\frac{1}{2}}\delta(x-x^0)u \\ \mathcal{A}^{-\frac{1}{4}}(2I+\mathcal{A}^{\frac{1}{2}})^{-\frac{1}{2}}\delta(x-x^0)u \end{bmatrix}. \tag{S.6.34}$$

From (S.6.34), we then readily see that $(-A)^{-\frac{1}{2}}Bu \in H = \mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times L_2(\Omega)$ provided (#): $\mathcal{A}^{-\frac{1}{2}}\delta(x-x^0) \in L_2(\Omega)$. But $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega)$ (and, in fact, only $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset H^2(\Omega)$ suffices for the present analysis) so that condition (#) is satisfied provided $\delta(x-x^0) \in [H^2(\Omega)]'$ (duality with respect to $L_2(\Omega)$); i.e., provided $H^2(\Omega) \subset C(\bar{\Omega})$, i.e., by Sobolev embedding provided $2 > \frac{n}{2}$, or $n < 4$, as desired. We have shown: <u>Assumption (1.3)</u> $(-A)^{-\gamma}B \in \mathcal{L}(U,Y)$ <u>holds true for problem (6.44) with</u> $n \leq 3$, <u>and</u> $\gamma = \frac{1}{2}$. The above argument shows some 'leverage. Indeed, $\gamma = \frac{1}{2}$ is not the least γ for which assumption (1.3) holds true. Indeed, one can show (see [19]) that <u>Assumption (1.3)</u> $(-A)^{-\gamma}B \in \mathcal{L}(U,H)$ <u>holds true for problem (6.44) provided</u> $\frac{n}{8} < \gamma \leq \frac{1}{2}$, $n \leq 3$.

Also, the operator A in (6.18) generates an s.c. contraction semigroup e^{At} on H, which moreover is analytic here for $t > 0$. (This is a special case of a much more general result [5].) This, along with the requirement $\gamma < 1$ proved above, guarantees that problem (6.14) satisfies the required assumption (ii) of the Introduction, below (1.2).

<u>Stabilizability Condition (1.5)</u>. With A as in (6.18), the semigroup e^{At} is uniformly (exponentially) stable in $H/\mathcal{N}(A)$, where $\mathcal{N}(A)$ is the finite-dimensional nullspace of A [5], and thus (1.5) is automatically satisfied on this space. For the eigenvalue $\lambda = 0$, we apply the same procedure as in the example of Section 6.2.

<u>Detectability Condition (1.6)</u>. This is satisfied since in our case $R = I$.

<u>Conclusion</u>. Theorem 1.0 applies to problem (6.14) for $n \leq 3$.

6.3.2. <u>Discrete problem</u>

<u>Approximation Framework</u>. The choice of the spaces V_h and H_h and of the operator $\mathcal{A}_h$ is the same as in the case of the example of Section 6.2 (damped plate equation).

<u>Choice of A_h and B_h</u>. We define

$$A_h: H_h \to H_h: \quad A_h \equiv \begin{bmatrix} 0 & Q_h \\ -\mathcal{A}_h & -\rho\mathcal{A}_h \end{bmatrix};$$

$$B_h: L_2(\Gamma) \to H_h: \quad B_h u = \begin{bmatrix} 0 \\ \mathcal{B}_h u \end{bmatrix},$$

where $(\mathcal{B}_h u, v_h)_{L_2(\Omega)} = v_h(x^0)u$.

Computations of adjoints A_h^* and B_h^*, as previously done yield

$$A_h^* = \begin{bmatrix} 0 & -Q_h \\ \mathcal{A}_h & -\rho\mathcal{A}_h \end{bmatrix}; \quad B_h^* x_h = x_{h2}(x^0).$$

<u>Approximating control problem</u>. The approximating dynamics is

$$(\ddot{y}_h,\phi_h)_\Omega + (\mathcal{A}_h y_h,\phi_h)_\Omega + \rho(\mathcal{A}_h y_h,\phi_h) = \phi_{h2}(x^0)u$$

with

$$(\mathcal{A}_h y_h,\phi_h) = (\Delta y_h,\Delta\phi_h)_\Omega;$$

$$(y_h(0),\phi_h)_\Omega = (y_0,\phi_h)_\Omega; \quad (\dot{y}_h(0),\phi_h)_\Omega = (y_1,\phi_h)_\Omega.$$

The optimal feedback control for the finite-dimensional problem is given by

$$u_h^0(t) = -[\bar{P}_{h2}Y_h(t)][x^0],$$

$$P_h y_h = \begin{bmatrix} \overline{P}_{h1} y_h \\ \overline{P}_{h2} y_h \end{bmatrix},$$

and P_h satisfies

$$-(\mathcal{A}_h x_{h2}, \overline{P}_{h1} y_h) + (\mathcal{A}_h x_{h1} - \rho \mathcal{A}_h x_{h2}, \overline{P}_{h2} y_h) - (\mathcal{A}_h \overline{P}_{h1} x_h, y_{h2})$$

$$+ (\overline{P}_{h2} x_h, \mathcal{A}_h y_{h1} - \rho \mathcal{A}_h y_{h2}) + (\mathcal{A}_h x_{h1}, y_{h1}) + (x_{h2}, y_{h2})$$

$$= (\overline{P}_{h2} x_h)(x^0) \overline{P}_{h2} y_h(x^0).$$

Verification of the approximating assumptions. Assumption (1.26a) is satisfied as $R = I$. Assumption (1.27a) follows from the fact that the operator

$$(-A)^{-1} B = \begin{bmatrix} \mathcal{A}^{-1} \delta(x - x^0) \\ 0 \end{bmatrix} : R \to H^2(\Omega) \times L_2(\Omega)$$

is compact, which, in turn, is a consequence of Sobolev imbeddings and compactness of $\mathcal{A}^{-1}$ (indeed, $\mathcal{A}^{-(1/2-\varepsilon)} \delta \in L_2(\Omega)$).

Assumption (A.1). This follows by applying the arguments of [5] of the continuous case to the finite dimensional operator A_h.

Assumption (A.2). Computing directly, we obtain

$$\|(A_h^{-1} \Pi_h - \Pi_h A^{-1}) x\|_H = \|(Q_h \mathcal{A}^{-1} - \mathcal{A}_h^{-1} \Pi_h) x_2\|_{H^2(\Omega)}$$

by (S.6.22) and (S.6.26) $\qquad \le C\, h^2 \|x_{2h}\|_{L_2(\Omega)} \le C\, h^2 \|x\|_H.$

Thus, we have $s = 2$ in this case.

Assumption (A.3). The argument is identical to that of the damped wave equation in the example of Section 6.2.

Assumption (A.4). We compute

$$\|B^*(\Pi_h x - x)\|_U = |(Q_h x_2)(x^0) - x_2(x^0)|_R$$

$$\le C \|Q_h x_2 - x_2\|_{H^{n/2+\varepsilon}(\Omega)} \le C\, h^{2-n/2-\varepsilon} \|x_2\|_{H^2(\Omega)}.$$

By the results of [7], $\mathcal{D}(A^*) \subset H^2(\Omega) \times H^2(\Omega)$, so (A.4) follows since

$$2 - \frac{n}{2} - \varepsilon > 2\left(1 - \frac{n}{8} - \varepsilon\right).$$

Assumption (A.5). Coincides with (A.4).

Assumption (A.6). By Sobolev's imbedding,

$$\|B^* \Pi_h x\|_U = \|x_{h2}(x^0)\|_{L_2(\Omega)} \le C \|x_{h2}\|_{H^{n/2+\varepsilon}(\Omega)}$$

By the results of [7], for $0 < \gamma < 1/2$, we have

$$\mathcal{D}(A^{*\gamma}) \subset H^2(\Omega) \times H^{4\gamma}(\Omega),$$

so (A.6) is a consequence of the inequality,

$$4\gamma = 4\left(\frac{n}{8} + \varepsilon\right) = \frac{n}{2} + 4\varepsilon > \frac{n}{2} + \varepsilon.$$

Thus, we have verified all the assumptions of Theorem 1.1 and the conclusion of Theorem 1.1 yields the desired convergence results which can be listed in an analogous way as in the case of the example of Section 6.2.