COUNTEREXAMPLES
CONCERNING A WEIGHTED $L^2$ PROJECTION

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ABSTRACT. Counterexamples are given to show that some results concerning a weighted $L^2$ projection presented earlier by Bramble and the author are sharp, i.e., that certain error and stability estimates are impossible in some cases.

1. Introduction

Motivated by the numerical solution of second-order elliptic boundary value problems with discontinuous coefficients, certain weighted $L^2$ projections were studied in [1]. Owing to some technical difficulties, the error and stability estimates obtained in [1] are contingent upon some additional assumptions. In this paper, we study the problem further. The results we obtain are negative and demonstrate that the main results in [1] cannot be improved.

Let $\Omega \subset \mathbb{R}^d$ ($1 \leq d \leq 3$) be a bounded domain. For simplicity, we assume that $\Omega$ is a polyhedral domain, i.e., an interval for $d = 1$, a polygon for $d = 2$ and a polyhedron for $d = 3$. Assume the domain $\Omega$ admits the following decomposition:

$$\overline{\Omega} = \bigcup_{i=1}^{J} \overline{\Omega}_i,$$

where $\overline{\Omega}_i$ are mutually disjoint polyhedrons.

Given a set of positive constants $\{\omega_i\}_{i=1}^J$, we introduce two weighted inner products,

$$\begin{align*}
(u, v)_{L^2(\Omega)} &= \sum_{i=1}^{J} \omega_i \int_{\Omega_i} uv \, dx, \\
(u, v)_{H^1_0(\Omega)} &= \sum_{i=1}^{J} \omega_i \int_{\Omega_i} \nabla u \cdot \nabla v \, dx,
\end{align*}$$

Received October 25, 1990; revised December 5, 1990.
Key words and phrases. Finite element space, weighted $L^2$ projections.
This work is supported by the National Science Foundation under grant DMS 8805311-04.

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0025-5718/91 $1.00 + .25$ per page
with the induced norms denoted by \( \| \cdot \|_{L^2_w(\Omega)} \) and \( | \cdot |_{H^1_w(\Omega)} \), respectively. Moreover, we define a full weighted \( H^1 \) norm by
\[
\| \cdot \|_{H^1_w(\Omega)}^2 = \| \cdot \|_{L^2_w(\Omega)}^2 + | \cdot |_{H^1_w(\Omega)}^2.
\]
If \( \omega_i = 1 \) for each \( i \), we have the usual Sobolev space and the symbol \( \omega \) will then be dropped.

Next we introduce a finite element space. For \( 0 < h < 1 \), let \( \mathcal{T}_h \) be a triangulation of \( \Omega \) with simplices \( K \) of diameter less than or equal to \( h \). An additional assumption is that this triangulation be lined up with each subdomain \( \Omega_i \). Namely, \( \Omega_i \) is the union of a set of elements of \( \mathcal{T}_h \). We assume that the family \( \{ \mathcal{T}_h \} \) is quasi-uniform, i.e., there exist positive constants \( c_0 \) and \( c_1 \) such that
\[
\frac{\max_{K \in \mathcal{T}_h} h_K}{\min_{K \in \mathcal{T}_h} h_K} \leq c_0, \quad \frac{\max_{K \in \mathcal{T}_h} h_K}{\min_{K \in \mathcal{T}_h} h_K} \leq c_1, \quad \forall h.
\]
Here, \( h_K \) is the diameter of \( K \) and \( \rho_K \) the diameter of the largest ball contained in \( K \). Corresponding to each triangulation \( \mathcal{T}_h \), we define a finite element subspace \( S_h \subset H^1_0(\Omega) \) that consists of continuous piecewise (with respect to the elements in \( \mathcal{T}_h \)) linear polynomials vanishing on \( \partial \Omega \). For \( G \subset \Omega \), \( S_h(G) \) denotes the space of functions in \( S_h \) restricted to \( G \).

The weighted \( L^2 \) projection \( Q^w_h : L^2(\Omega) \to S_h \) is defined by
\[
(Q^w_h u, v)_{L^2(\Omega)} = (u, v)_{L^2(\Omega)}, \quad \forall u \in L^2(\Omega), v \in S_h.
\]
If \( \omega_i = 1 \) for all \( i \), we get the usual \( L^2 \) projection, denoted by \( Q_h \). The following estimate is known (cf. [1, 3] and the reference cited therein):
\[
\| u - Q_h u \|_{L^2(\Omega)} + h | Q_h u |_{H^1(\Omega)} \leq C h | u |_{H^1(\Omega)}, \quad \forall u \in H^1_0(\Omega).
\]
We are interested in similar estimates for the weighted \( L^2 \) projections with the regular norms replaced by the weighted norms, and with the constant \( C \) independent of the weights \( \omega_i \)'s. This problem has been carefully studied in [1].

Before we review the results of [1], we introduce the following notation:
\[
x \lesssim y, \quad f \gtrsim g \quad \text{and} \quad u \asymp v
\]
meaning, respectively,
\[
x \leq C y, \quad f \geq c g \quad \text{and} \quad cv \leq u \leq C v,
\]
where \( C \) and \( c \) are positive constants independent of the variables appearing in the inequalities and the other parameters related to meshes, spaces and especially the weights \( \omega_i \)'s. We shall use the term "interface" to denote the union of the boundaries of all \( \Omega_i \) inside of \( \Omega \).

The first result shows that optimal estimates can be obtained in a special case.
Theorem 1.1 [1]. Assume that \( d = 1 \), or that the decomposition (1.1) has no internal cross points, i.e., there is no point on the interface that belongs to more than two \( \Omega_i \)’s. Then, for all \( u \in H_0^1(\Omega) \),

\[
\| (I - Q_h^\omega) u \|_{L^2_a(\Omega)} + h |Q_h^\omega u|_{H^1_a(\Omega)} \lesssim h |u|_{H^1_a(\Omega)}.
\]

If there are internal cross points, nearly optimal estimates can be obtained under additional conditions.

Theorem 1.2 [1]. If for all \( i \), the \( (d - 1) \)-dimensional Lebesgue measure of \( \partial \Omega_i \cap \partial \Omega \) is positive, then for all \( u \in H_0^1(\Omega) \)

\[
\| (I - Q_h^\omega) u \|_{L^2_a(\Omega)} + h |Q_h^\omega u|_{H^1_a(\Omega)} \lesssim \log h |u|_{H^1_a(\Omega)}.
\]

In order to obtain estimates without the restriction on the measure of \( \partial \Omega_i \cap \partial \Omega \), as in the above theorem, we consider a special class of functions instead of all of \( H^1. \) For a given triangulation \( \mathcal{T}_h \), we consider a finer quasi-uniform mesh \( \mathcal{T}_{h'} \) with \( h < h \) which is obtained by refining \( \mathcal{T}_h \) in such a way that

\[
S_h \subset S_{h'}.
\]

Here, \( S_h \subset H_0^1(\Omega) \) is the finite element space corresponding to \( \mathcal{T}_h \).

We have shown previously:

Theorem 1.3 [1]. For any \( u \in S_h \),

\[
\| (I - Q_h^\omega) u \|_{L^2_a(\Omega)} + h |Q_h^\omega u|_{H^1_a(\Omega)} \lesssim \begin{cases} h \left( \log \frac{h}{h} \right)^\frac{1}{3} |u|_{H^1_a(\Omega)}, & \text{if } d = 2; \\ h \left( \frac{h}{h} \right)^\frac{1}{2} |u|_{H^1_a(\Omega)}, & \text{if } d = 3. \end{cases}
\]

The purpose of this paper is to show that the assumption in Theorem 1.2 concerning the measure of \( \partial \Omega_i \cap \partial \Omega \) is necessary and that the estimate for \( d = 3 \) va. Theorem 1.3 is sharp.

2. Counterexamples

In Theorem 1.2, the estimate (1.6) is established only under the condition that all the subregions meet the boundary of the original region on a subset of a positive \( (d - 1) \)-dimensional measure. A natural question is then if this constraint is essential. The following two theorems show that this is the case.

Theorem 2.1. Assume that there is an \( \omega_i \) such that the \( (d - 1) \)-dimensional Lebesgue measure of \( \partial \Omega_i \cap \partial \Omega \) is zero. Then, there is no constant \( C \) independent of the \( \omega_i \)’s such that

\[
\| (I - Q_h^\omega) u \|_{L^2_a(\Omega)} \leq C |u|_{H^1_a(\Omega)}, \quad \forall u \in H_0^1(\Omega).
\]

Proof. For convenience, we shall use \( \text{meas}_k(G) \) to denote the \( k \)-dimensional Lebesgue measure of \( G \).
Case 1: $d = 3$ and $\text{meas}(\partial \Omega_0 \cap \partial \Omega) = 0$. Without loss of generality, we assume that $i_0 = 1$ and that $\Omega_1$ is the unit cube $(0,1)^3$. It suffices to consider two cases. In the first, there is another subdomain, $\Omega_2$, that touches $\Omega_1$ only at the origin $O$. In the second case, $O \in \partial \Omega$. Because of the similarity in the proofs, we only present the proof for the first case.

Assume that there is a constant $C$ independent of the $\omega_i$'s such that (2.1) holds. By letting $\omega_i = \omega$ for $i > 2$, we then would have

$$\|(I - \mathcal{Q}_h^\omega)u\|_{L^2(\Omega_1 \cup \Omega_2)} \leq C(|u|_{H^1(\Omega_1 \cup \Omega_2)} + \omega|u|_{H^1(\Omega_1 \setminus (\Omega_1 \cup \Omega_2))}).$$

In particular, the above inequality implies that $\|\mathcal{Q}_h^\omega u\|_{L^2(\Omega_1 \cup \Omega_2)}$ is bounded with respect to $\omega$, hence it has a subsequence that converges to a function $\mathcal{Q}_h^\omega u \in S_h(\Omega_1 \cup \Omega_2)$. Consequently, letting $\omega \to 0$ yields

$$\|(I - \mathcal{Q}_h^\omega)u\|_{L^2(\Omega_1 \cup \Omega_2)} \leq C|u|_{H^1(\Omega_1 \cup \Omega_2)}, \quad \forall u \in H^1_0(\Omega).$$

Take a function $\phi \in C^\infty(\mathbb{R}^1)$ such that $\phi = 0$ for $x \leq \frac{1}{2}$, $\phi = 1$ for $x \geq 1$ and $|\phi'(x)| \leq 4$ for any $x$. It is easy to see, for any $\varepsilon > 0$, that there exists a function $u_\varepsilon \in H^1_0(\Omega)$ such that

$$u_\varepsilon = \begin{cases} \phi(\frac{|x|}{\varepsilon}) & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2. \end{cases}$$

For example, in the rest of $\Omega$, $u_\varepsilon$ can be defined by solving $-\Delta u_\varepsilon = 0$ with some properly prescribed continuous boundary data. A direct calculation shows that

$$|u_\varepsilon|_{H^1(\Omega_1 \cup \Omega_2)} = |u_\varepsilon|_{H^1(\Omega_1)} \lesssim \varepsilon\varepsilon\varepsilon$$

and

$$\|u_\varepsilon - \bar{u}\|_{L^2(\Omega_1 \cup \Omega_2)} \lesssim \varepsilon\varepsilon\varepsilon,$$

where $\bar{u}$ equals 1 in $\Omega_1$ and 0 in $\Omega_2$. We first observe that $\|\mathcal{Q}_h^\omega u_\varepsilon\|_{L^2(\Omega_1 \cup \Omega_2)}$ is bounded with respect to $\varepsilon$. Hence, there exists a function $w_h \in S_h(\Omega_1 \cup \Omega_2)$ and a sequence $\{\varepsilon_m \to 0\}$ such that

$$\lim_{m \to \infty} \|\mathcal{Q}_h^\omega u_{\varepsilon_m} - w_h\|_{L^2(\Omega_1 \cup \Omega_2)} = 0.$$ 

Consequently, we conclude from (2.2), with $u_\varepsilon = u$, and (2.3) that

$$\|\bar{u} - w_h\|_{L^2(\Omega_1 \cup \Omega_2)} = 0.$$ 

This implies $\bar{u} = w_h$, which is a contradiction, since $w_h$ is continuous at $O$ but $\bar{u}$ is not.

Case 2: $d = 2$. Let $\Omega_1$ and $\Omega_2$ be similar as before, but $\Omega_1 = (0,1)^2$. In this case, the construction of an appropriate $u_\varepsilon$ is more difficult. Using the fact that
$C_0^\infty$ is dense in $H^\frac{1}{2}$, we can find a sequence of smooth functions $\phi_e$ on $\partial \Omega_1$ that vanish in a neighborhood of $(0, 0)$ and satisfy
\[(2.4) \lim_{\epsilon \to 0} \|\phi_e - 1\|_{H^\frac{1}{2}(\partial \Omega_1)} = 0.\]

As we did above, it is easy to find a $u_e \in H^1_0(\Omega)$ such that
\[(2.5) \begin{cases} -\Delta u_e = 0 & \text{in } \Omega_1, \\ u_e = \phi_e & \text{on } \partial \Omega_1 \end{cases} \]

and
\[u_e = 0 \text{ in } \Omega_2.\]

Notice that $u_e - 1$ is harmonic in $\Omega_1$, and therefore, as $\epsilon \to 0$,
\[(2.6) \|u_e\|_{H^\frac{1}{2}(\Omega_1 \cup \Omega_2)} = \|u_e - 1\|_{H^1(\Omega_1)} \lesssim \|\phi_e - 1\|_{H^\frac{1}{2}(\partial \Omega_1)} \to 0.\]

Here we have used (2.4).

The rest of the proof is the same as in the first case.

Case 3: $d = 3$ and $\text{meas}_2(\partial \Omega_i \cap \partial \Omega) = 0$. In this case, we may assume that $\Omega_i = \Omega_1 = (0, 1)^3$, and
\[\partial \Omega_1 \cap \partial \Omega = \{(0, 0, x_3): 0 < x_3 < 1\}.\]

We can construct a function $v_e \in H^1_0(\Omega)$ satisfying
\[v_e(x_1, x_2, x_3) = u_e(x_1, x_2), \quad 0 \leq x_i \leq 1, \quad i = 1, 2, 3,\]
where $u_e$ satisfies (2.5).

By (2.6), we have, as $\epsilon \to 0$,
\[|v_e|_{H^\frac{1}{2}(\Omega_1)} = |u_e|_{H^1((0, 1)^3)} \to 0.\]

The rest of the proof is similar as above.

The following result concerns the sharpness of the estimate in Theorem 1.3 for $d = 3$.

**Theorem 2.2.** Assume that $d = 3$ and that there is an index $i_0$ such that $\text{meas}_1(\partial \Omega_i \cap \partial \Omega) = 0$. Then, if $C_h$ is a constant satisfying
\[(2.7) \|(I - Q_h^\omega)u\|_{L^2_0(\Omega)} \leq C_h |u|_{H^\frac{1}{2}(\Omega)}, \quad \forall u \in S_h,\]
there holds
\[C_h \gtrsim h^{-\frac{1}{2}}.\]

**Proof.** As in the proof of Theorem 2.1, there exists, for any $u \in S_h$, a function $\overline{Q_h^\omega} u \in S_h(\Omega_1 \cup \Omega_2)$ such that
\[(2.8) \|u - \overline{Q_h^\omega} u\|_{L^2(\Omega_1 \cup \Omega_2)} \leq C_h |u|_{H^1(\Omega_1 \cup \Omega_2)}.\]
We now take \( u_h \in S_h \) such that \( u_h = 1 \) at all the nodes except \( O \) on \( \Omega_1 \), and \( u_h = 0 \) at all the nodes on \( \overline{\Omega}_2 \). A direct computation shows that
\[
|u_h|_{H^1(\Omega_1 \cup \Omega_2)} \lesssim \sqrt{h}.
\]
Using an argument similar to that in the proof of Theorem 2.1, we can find a \( w_h \in S_h \) such that
\[
\lim_{h \to 0} \|u_h - \overline{Q}_h^w u_h\|_{L^2(\Omega_1 \cup \Omega_2)} = \|\overline{u} - w_h\|_{L^2(\Omega_1 \cup \Omega_2)} = \alpha_h > 0.
\]
Consequently, for sufficiently small \( h \), we have
\[
C_h \gtrsim h^{-\frac{1}{2}} \|(I - \overline{Q}_h^w)u\|_{L^2(\Omega_1 \cup \Omega_2)} \gtrsim \frac{1}{2} \alpha h^{-\frac{3}{2}}.
\]
This completes the proof.

**Remark.** The questions concerning logarithmic factors appearing in the estimates of Theorems 1.2 and 1.3 are more subtle. The author does not know whether they are necessary.

**Acknowledgment**

It is a pleasure to express my thanks to Professor Franco Brezzi who helped to construct the function \( u_\varepsilon \) for \( d = 2 \) in the proof of Theorem 2.1. Thanks also go to Professors James Bramble, Ridgway Scott and Olof Widlund for helpful discussions.

**Bibliography**