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A FABER SERIES APPROACH TO CARDINAL INTERPOLATION

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ABSTRACT. For a compactly supported function φ in $\mathbb{R}^d$ we study quasi-interpolants based on point evaluations at the integer lattice. We restrict ourselves to the case where the coefficient sequence λf, for given data f, is computed by applying a univariate polynomial q to the sequence $\varphi|_{\mathbb{Z}^d}$, and then convolving with the data $f|_{\mathbb{Z}^d}$. Such operators appear in the well-known Neumann series formulation of quasi-interpolation. A criterion for the polynomial q is given such that the corresponding operator defines a quasi-interpolant.

Since our main application is cardinal interpolation, which is well defined if the symbol of φ does not vanish, we choose q as the partial sum of a certain Faber series. This series can be computed recursively. By this approach, we avoid the restriction that the range of the symbol of φ must be contained in a disk of the complex plane excluding the origin, which is necessary for convergence of the Neumann series. Furthermore, for symmetric φ, we prove that the rate of convergence to the cardinal interpolant is superior to the one obtainable from the Neumann series.

1. INTRODUCTION

Let $\varphi \in C_0(\mathbb{R}^d)$ be a compactly supported complex-valued function and $\Phi := (\varphi(j))_{j\in\mathbb{Z}^d}$. We say that *cardinal interpolation* with φ is poised, if for any sequence $f := (f_j)_{j\in\mathbb{Z}^d} \in l_2$, there is a unique sequence $a := (a_j)_{j\in\mathbb{Z}^d} \in l_2$ with

$$a * \Phi = f.$$

This condition is equivalent to the property that the symbol $\tilde{\varphi}$ of φ defined by

$$\tilde{\varphi}(t) := \sum_{j\in\mathbb{Z}^d} \varphi(j) e^{ij\cdot t}, \qquad t \in \mathbb{R}^d, \tag{1.1}$$

does not vanish anywhere (cf. [2]). We denote the range of $\tilde{\varphi}$ by $R(\tilde{\varphi})$, and throughout the paper we assume that cardinal interpolation with φ is poised, or equivalently,

$$0 \notin R(\tilde{\varphi}). \tag{1.2}$$

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One condition that insures the validity of (1.2) is the requirement that $\tilde{\varphi}$ satisfies the *metric condition*

$$R(\tilde{\varphi}) \subset \{z \in \mathbb{C}\colon |z-1| < 1\}. \tag{1.3}$$

For instance, if φ is symmetric, real-valued, and satisfies $\tilde{\varphi}(0) > 0$, then φ satisfies (1.3) (cf. [3]). In general, if the metric condition is satisfied, then the Neumann series approach introduced in [4] gives an efficient scheme for approximating the fundamental sequence $\Lambda = (\lambda_j)_{j\in\mathbb{Z}^d}$, defined by

$$\Lambda * \Phi = (\delta_{0j})_{j\in\mathbb{Z}^d}, \tag{1.4}$$

where δ_{0j} denotes the Kronecker symbol. Of course, since we assume (1.2), (1.4) is equivalent to $\widetilde{\Lambda} = 1/\tilde{\varphi}$. However, in [6] we found that, in contrast with the univariate case, condition (1.3) is often not satisfied for multivariate cardinal spline interpolation. For example, if φ is a bivariate three-direction box spline whose center is close to the boundary lines of the *correctness region*

$$G := \{(x, y) \in \mathbb{R}^2\colon -\tfrac{1}{2} < x, y, x-y < \tfrac{1}{2}\}, \tag{1.5}$$

then (1.3) fails to hold. Therefore, a more general procedure for numerical computation of the fundamental sequence Λ is needed without imposing condition (1.3) on φ.

As in [3], the cardinal interpolation operator is studied as the inverse of the "Schoenberg operator"

$$S\colon l_2 \to l_2, \qquad a \to a * \Phi. \tag{1.6}$$

Using the fact that φ has compact support, so that Φ is a finite sequence, the relation (1.6) can be written in symbol notation as

$$(Sa)^{\sim} = \tilde{a}\tilde{\varphi}. \tag{1.7}$$

Assuming condition (1.2) on φ, the inverse $T := S^{-1}$ is therefore given by

$$(Tf)^{\sim} = \tilde{f}/\tilde{\varphi}, \qquad f \in l_2, \tag{1.8}$$

or equivalently, $Tf = \Lambda * f$ with the fundamental sequence Λ in (1.4). In order to construct Tf numerically, we find approximations $\Lambda^{(n)} \in l_1$ to Λ with uniform convergence of the symbols

$$\|\widetilde{\Lambda} - \widetilde{\Lambda}^{(n)}\|_\infty \to 0 \quad \text{for } n \to \infty. \tag{1.9}$$

Then, by Parseval's identity and some elementary calculations we have that

$$\|Tf - \Lambda^{(n)} * f\|_{l_2} = \|(\widetilde{\Lambda} - \widetilde{\Lambda}^{(n)})\tilde{f}\|_{L_2} \le \|\widetilde{\Lambda} - \widetilde{\Lambda}^{(n)}\|_\infty \|f\|_{l_2} \to 0. \tag{1.10}$$

The numerical computation of $\Lambda^{(n)}$, as given in [3], uses linear combinations of successive convolutions of the finite sequence Φ. For a complex univariate polynomial $p(z) = \sum_{k=0}^n c_k z^k$ we use the notation

$$p(\Phi) := \sum_{k=0}^{n} c_k \underbrace{\Phi * \Phi * \cdots * \Phi}_{k \text{ times}}. \tag{1.11}$$

The following observation gives a rather general tool for constructing appropriate numerical procedures for the approximation of the fundamental sequence Λ.

Proposition 1.1. *Let $\varphi \in C(\mathbb{R}^d)$ be compactly supported, $\tilde{\varphi}$ its symbol with $0 \notin R(\tilde{\varphi}) \subset \mathbb{C}$, and Λ the fundamental sequence* (1.4). *Then for any univariate polynomial p,*

$$\|\Lambda - p(\Phi)\|_{l_2} \le \|\widetilde{\Lambda} - (p(\Phi))^{\sim}\|_{\infty} = \|1/z - p(z)\|_{R(\tilde{\varphi})}. \tag{1.12}$$

Here and throughout, $\|\cdot\|_K$ denotes the supremum norm on the set K.

Proof. The equality in (1.12) holds by the relations $\widetilde{\Lambda} = 1/\tilde{\varphi}$ and $(p(\Phi))^{\sim} = p(\tilde{\varphi})$. The inequality is an immediate consequence of Parseval's identity. □

By (1.10) and Proposition 1.1, we see that any polynomial approximation scheme for the complex function $1/z$ on the domain $R(\tilde{\varphi})$ yields a corresponding procedure for approximate cardinal interpolation with φ. Since the degree of the polynomial p determines the size of the support of the finite sequence $p(\Phi)$, we are interested in approximations for $1/z$ which yield small errors for uniform approximation on $R(\tilde{\varphi})$ by polynomials of low degree. The set $R(\tilde{\varphi})$, which is a compact and connected set in $\mathbb{C}$, plays a central role. If it is a subset of the disk of radius 1 around 1 (cf. (1.3)), then a possible polynomial approximation of $1/z$ may be obtained by taking the partial sums of the geometric series

$$\frac{1}{z} = \sum_{n=0}^{\infty}(1-z)^n, \qquad |1-z| < 1. \tag{1.13}$$

This corresponds to the Neumann series approach for cardinal interpolation (cf. [3]). If φ is symmetric, i.e., $\varphi(x) = \varphi(-x)$ for $x \in \mathbb{R}^d$, and is real-valued, then $\tilde{\varphi}$ is also real-valued, and approximation of $1/z$ by a Chebyshev series converges much faster than the geometric series in (1.13) (cf. §5).

An outline of this paper is as follows. In the next section we deal with the most general assumption on $R(\tilde{\varphi})$ for this paper, namely: the polynomially convex hull of $R(\tilde{\varphi})$ (i.e., the complement of the unbounded component of $\mathbb{C}\backslash R(\tilde{\varphi})$) does not contain the origin. It is within this framework that efficient approximation schemes based on the theory of Faber polynomials can be derived. Also, in §2, a brief introduction to Faber polynomials is given. In §3, an algorithmic construction of Faber polynomials in regions G, which are either sectors of a disk or Moebius transform of the disk, is given. This algorithm is applied in §4 to derive a Faber series approximation to the cardinal interpolation operator. In §5, results of the previous sections are applied to the case where the symbol φ is symmetric. Here, the Faber series turns out to be a Chebyshev expansion that converges more quickly than the Neumann series. The paper concludes in §6 with some quasi-interpolation schemes based on the material of the previous sections.

2. Faber polynomials on simply connected regions

For the reader's convenience we give a short introduction to the theory of Faber polynomials. For details we refer the reader to [7, 8].

Let G be a simply connected bounded region in $\mathbb{C}$. Then, by the Riemann Mapping Theorem, there is a unique conformal mapping ψ from the exterior of the unit disk to the exterior of G of the form

$$\psi(u) = d\left(u + \sum_{n=0}^{\infty} d_n u^{-n}\right), \qquad |u| > 1, \tag{2.1}$$

with $\psi(\infty) = \infty$ and $\psi'(\infty) = d > 0$. The positive number d in (2.1) is called the *capacity* of G. For notational reasons we often use the conformal mapping $\theta(u) := \psi(1/u)$, $|u| < 1$, which maps the unit disk $\mathbb{D}$ onto the exterior of G. The Faber polynomials p_n of G are defined by the generating function formula

$$\frac{u\psi'(u)}{\psi(u) - z} = 1 + \sum_{n=1}^{\infty} p_n(z) u^{-n}, \qquad |u| > 1,\ z \in G. \tag{2.2}$$

Equivalently, p_n can be defined as the polynomial part of the Laurent series expansion of $[\psi^{-1}(z)]^n$, $z \in G$. Obviously, p_n is a polynomial of exact degree n with leading coefficient d^{-n}.

Curtiss [7] gives the following recursion formula for p_n using the coefficients of the Laurent expansion (2.1):

$$\begin{cases} p_1(z) = \frac{z}{d} - d_0, & \text{and for } n = 1, 2, \ldots \\ p_{n+1}(z) = p_1(z)p_n(z) - \sum_{k=1}^{n-1} d_k p_{n-k}(z) - (n+1)d_n. \end{cases} \tag{2.3}$$

To find a uniform bound for the polynomials p_n on G, we assume some additional properties for the boundary ∂G of G, namely that ∂G has a piecewise differentiable unit tangent field $\vartheta(s)$, where s denotes the arc-length parameter of ∂G, and that the *total rotation*

$$V_G := \int_{\partial G} |d\vartheta(s)|\, ds \tag{2.4}$$

of ∂G is finite. It then follows, as shown in [8, p. 50] for example, that

$$\sup_{n \geq 1} \|p_n\|_{\overline{G}} \leq \frac{V_G}{\pi}. \tag{2.5}$$

Faber polynomials are useful for finding series expansions of analytic functions on arbitrary simply-connected regions,

$$f \sim f_0 + \sum_{n=1}^{\infty} f_n p_n \quad \text{on } G.$$

The convergence properties of these *Faber series* are extensively studied in the literature (cf. [8] and references therein). Thus, we omit these details, since we are only interested in a very special analytic function on G, namely $f(z) = 1/z$. For this function, relation (2.2) already contains the necessary information.

In light of the details discussed above, we now assume that G is a simply-connected bounded region in $\mathbb{C}$, that ∂G has a piecewise differentiable unit tangent field with $V_G < \infty$, and that $0 \notin \overline{G}$. Then there is $s^* \in \mathbb{C}$, $|s^*| > 1$, with $\psi(s^*) = 0$, hence by (2.2), we have

$$\frac{1}{z} = -\frac{1}{s^* \psi'(s^*)}\left(1 + \sum_{n=1}^{\infty} (s^*)^{-n} p_n(z)\right), \qquad z \in G. \tag{2.6}$$

By (2.5) the series on the right-hand side of (2.6) converges as a geometric series uniformly on $\overline{G}$. In fact, if q_n is given by

$$q_n := -\frac{1}{s^* \psi'(s^*)} \left(1 + \sum_{k=1}^{n} (s^*)^{-k} p_k\right), \tag{2.7}$$

then it follows from (2.5) that

$$\left\| \frac{1}{z} - q_n(z) \right\|_{\overline{G}} \le \frac{V_G}{\pi |\psi'(s^*)| (|s^*| - 1)} |s^*|^{-n-1}. \tag{2.8}$$

Combining these estimates together with Proposition 1.1 yields the following theorem.

Theorem 2.1. *Let $\varphi \in C(\mathbb{R}^d)$ be a compactly supported function with $R(\tilde{\varphi}) \subset G$, where G has the following properties:*

(i) *G is a simply-connected bounded region with $0 \notin G$, and*
(ii) *∂G has a piecewise differentiable tangent field and $V_G < \infty$.*

Furthermore, let ψ be the conformal mapping in (2.1) associated with G, p_n the Faber polynomials associated with G, and $s^* = \psi^{-1}(0)$. Then q_n given in (2.7) is a polynomial of exact degree n which satisfies

$$\begin{aligned} \|\Lambda - q_n(\Phi)\|_{l_2} &\le \|\widetilde{\Lambda} - (q_n(\Phi))^{\sim}\|_\infty \\ &\le \frac{V_G}{\pi |\psi'(s^*)| (|s^*| - 1)} |s^*|^{-n-1}, \end{aligned} \tag{2.9}$$

where Λ is the fundamental sequence given in (1.4).

The main restriction on G, and hence on $R(\tilde{\varphi})$, in the above theorem is that G must be simply connected. In [12], an example was given of a univariate real-valued function g with the following property: the range of the symbol of any integer shift $\varphi := g(\cdot - j)$, $j \in \mathbb{Z}$, cannot be imbedded into any simply-connected region G with $0 \notin G$. If, however, φ is real-valued and $\tilde{\varphi}(0) > 0$, and if there is a region G as in Theorem 2.1 (e.g., the polynomially convex hull of $R(\tilde{\varphi})$), then by the symmetry property $\tilde{\varphi}(z) = \overline{\tilde{\varphi}(-z)}$ the set $R(\tilde{\varphi})$ is a subset of the region $\mathbb{C} \backslash \mathbb{R}_-$. Then, by the compactness of $R(\tilde{\varphi})$, we can imbed this set into a sector of a disk centered on the positive real axis. So the assumption (ii) on ∂G can simply be realized by an appropriate choice for G. Since the main work in the approximation procedure will consist in finding the explicit coefficients in (2.1), we restrict our attention to circular triangles which can be obtained as Moebius transforms of a sector of a disk. For these geometric figures, a derivation of the conformal mapping and the numbers d, d_n which appear in (2.1), is given in §3. This setting seems general enough for dealing with all the known cases of cardinal spline interpolation, including the case where φ is a bivariate shifted three-direction spline in $\mathbb{R}^2$.

3. Construction of the Faber polynomials for circular triangles

As mentioned above, we restrict the algorithmic construction of the Faber polynomials to regions G which are the sector of a disk, or the Moebius transform of it. Since dilations, rotations, and translations of G have a simple

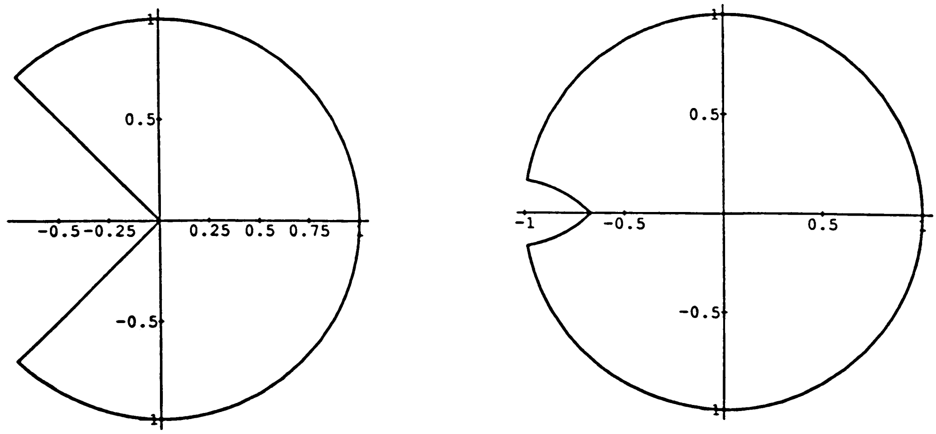

FIGURE 1
The regions $G_{1/2}$ *and* $G_{1/2,2/3}$

effect on the conformal mapping and the coefficients d and d_n in (2.1), we first derive an explicit expression for the two regions

$$G_t := \left\{ z\colon |z| < 1\,,\ |\arg z| < \pi\left(1 - \frac{t}{2}\right) \right\}, \qquad 0 < t < 2, \tag{3.1}$$

$$G_{t,a} := \left\{ \frac{z-a}{1-az}\colon z \in G_t \right\}, \qquad -1 < a < 1. \tag{3.2}$$

Typical shapes of these regions are shown in Figure 1. The exterior angle at the "critical points" 0 of G_t and $-a$ of $G_{t,a}$ has size πt.

Our aim is to find a recursive scheme for the coefficients d_n, $n \geq 1$, in order to apply the equation given in (2.3) for fast computations of the Faber polynomials p_n on $G = G_t$ or $G_{t,a}$. For notational reasons we prefer to specify the conformal mapping θ from the inner unit disk $\mathbb{D}$ to the exterior of G,

$$\theta(u) = d\left(\frac{1}{u} + \sum_{n=0}^{\infty} d_n u^n\right), \qquad |u| < 1. \tag{3.3}$$

Lemma 1. *Let* $0 < t < 2$, $-1 < a < 1$. *Then the following statements hold.*

(i) *The conformal mapping* θ_t *from* $\mathbb{D}$ *onto the exterior of* G_t *with* $\theta_t(0) = \infty$ *and* $\operatorname{res}(\theta_t, 0) > 0$ *has the form*

$$\begin{aligned} \theta_t(u) = \frac{d_t}{4u} &\left(\frac{-t(1-u) + 2\sqrt{(1+u)^2 - t^2 u}}{(2-t)(1+u)} \right)^t \\ &\cdot \left(1 - u + \sqrt{(1+u)^2 - t^2 u} \right)^2, \qquad |u| < 1, \end{aligned} \tag{3.4}$$

where $d_t = 4(2-t)^{-1+t/2}(2+t)^{-1-t/2}$ *is the capacity of* G_t. *Furthermore,* $\theta_t'(u) < 0$ *for all real* $u \in \mathbb{D}$.

(ii) *If* $b := \theta_t^{-1}(1/a)$, *and if* m_1 *and* m_2 *are Moebius transformations given by*

$$m_1(z) := \frac{z-a}{1-az}, \qquad m_2(u) := \frac{u+b}{1+bu}, \tag{3.5}$$

then the conformal mapping $\theta_{t,a}$ *from* $\mathbb{D}$ *onto the exterior of* $G_{t,a}$ *with* $\theta_{t,a}(0) = \infty$ *and* $\operatorname{res}(\theta_{t,a}; 0) > 0$ *is given by*

$$\theta_{t,a}(u) = m_1 \circ \theta_t \circ m_2(u), \qquad |u| < 1. \tag{3.6}$$

Proof. As shown in [10, p. 221], we find that the conformal mapping of the upper half plane to the exterior of the rotated sector $H_t := e^{i\pi(1-t/2)}G_t$ is given by

$$\xi(w) = \left(\frac{1-\sqrt{w}}{1+\sqrt{w}}\right)^t \frac{1+t\sqrt{w}+w}{1-t\sqrt{w}+w}, \qquad \operatorname{Im} w > 0. \tag{3.7}$$

The "critical points" of H_t are obtained as $\xi(0) = 1$, $\xi(1) = 0$, and $\xi(\infty) = e^{-i\pi t}$. Note that $\xi(c) = \infty$ for

$$c = -1 + \frac{t^2}{2} + it\sqrt{1 - \frac{t^2}{4}}, \qquad |c| = 1.$$

The Moebius transform

$$w(u) = \frac{c-u}{1-cu}, \qquad |u| < 1, \tag{3.8}$$

with c as above, satisfies $w(-1) = 1$, $w(c) = 0$, $w(\bar{c}) = \infty$, and hence w maps $\mathbb{D}$ conformally onto the upper half plane. Inserting (3.8) into (3.7) and rotating by a factor $e^{-i\pi(1-t/2)}$ gives the conformal mapping from $\mathbb{D}$ onto the exterior of G_t, namely

$$\theta(u) = e^{-i\pi(1-t/2)} \left(\frac{1-\sqrt{w(u)}}{1+\sqrt{w(u)}}\right)^t \frac{1+t\sqrt{w(u)}+w(u)}{1-t\sqrt{w(u)}+w(u)}, \qquad |u| < 1, \tag{3.9}$$

with $\theta(0) = \infty$. In order to simplify (3.9), we use the relations $(1+c)^2 = t^2 c$, $|c| = 1$. A careful analysis of the argument of the complex root shows that

$$t\frac{1-cu}{1+c}\sqrt{w(u)} = \sqrt{\frac{(c-u)(1-cu)}{c}} = \sqrt{(1+u)^2 - t^2 u}. \tag{3.10}$$

So the last term in (3.9) can be written as

$$\begin{aligned} \frac{1+t\sqrt{w(u)}+w(u)}{1-t\sqrt{w(u)}+w(u)} &= \frac{1-u+\sqrt{(1+u)^2-t^2u}}{1-u-\sqrt{(1+u)^2-t^2u}} \\ &= -\frac{1}{(4-t^2)u}\left(1-u+\sqrt{(1+u)^2-t^2u}\right)^2. \end{aligned} \tag{3.11}$$

Since $w(u)$ lies in the upper half plane, the term $(1-\sqrt{w(u)})/(1+\sqrt{w(u)})$ has negative imaginary part. If we fix the branch cut of the complex logarithm along the negative real axis, we therefore have

$$e^{i\pi t/2}\left(\frac{1-\sqrt{w(u)}}{1+\sqrt{w(u)}}\right)^t = \left(\frac{i(1-\sqrt{w(u)})}{1+\sqrt{w(u)}}\right)^t.$$

Simplifying the last term, using (3.10) and the relation $(1+c)/(t(1-c)) = i(4-t^2)^{-1/2}$, yields

$$\begin{aligned} \frac{i(1-\sqrt{w(u)})}{1+\sqrt{w(u)}} &= \frac{i(1-2\sqrt{w(u)}+w(u))}{1-w(u)} \\ &= \frac{i(1+c)\big(t(1-u)-2\sqrt{(1+u)^2-t^2u}\big)}{t(1-c)(1+u)} \\ &= (4-t^2)^{-1/2}\cdot\frac{2\sqrt{(1+u)^2-t^2u}-t(1-u)}{1+u}. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) finally gives $\theta(u) = \theta_t(u)$. Obviously, d_t given in the lemma is the residue of θ_t at 0.

We next wish to show that $\theta_t'(u) < 0$ for real u. First observe that $\theta_t'(u)$ is real for real u. This is an immediate consequence of the relations

$$(1+u)^2 - t^2u \geq (1-|u|)^2 > 0$$

and

$$2\sqrt{(1+u)^2-t^2u} - t(1-u) > 0$$

for $|u| < 1$, $0 < t < 2$, which can be readily verified. As a conformal mapping θ_t has a nonvanishing derivative, so $\theta_t'(u)$ has constant sign for real u. A comparison of the function values $\theta_t(-1) = 0$, $\theta_t(0) = \infty$, $\theta_t(1) = 1$ gives that $\theta_t'(u) < 0$ for real u, proving (i).

For the proof of (ii) note that m_1 and m_2 are automorphisms of the unit disk (and of its exterior). Hence, with $G_{t,a}$ in (3.2), m_1 maps the interior and exterior of G_t onto the interior and exterior of $G_{t,a}$, respectively. Thus $\theta_{t,a}$ is a conformal mapping from $\mathbb{D}$ onto the exterior of $G_{t,a}$, and $\theta_{t,a}(0) = m_1(\theta_t(b)) = m_1(1/a) = \infty$. Furthermore, we obtain its residue at 0 by elementary computations, namely

$$\begin{aligned} \operatorname{res}(\theta_{t,a}; 0) &= \operatorname{res}\left(\frac{\theta_t\circ m_2 - a}{1-a\theta_t\circ m_2}; 0\right) \\ &= \frac{\theta_t\circ m_2(0) - a}{-a(\theta_t\circ m_2)'(0)} = \frac{a^{-1}-a}{-a\theta_t'(b)m_2'(0)}. \end{aligned}$$

With $m_2'(0) = 1-b^2 > 0$ and $\theta_t'(b) < 0$, we conclude in view of (i) that $\operatorname{res}(\theta_{t,a}; 0) > 0$. □

By means of a decomposition of the mapping $\theta_{t,a}$ into more elementary parts we are able to compute the Laurent coefficients d_n in (3.3) recursively. This observation is made more precise in the following algorithm.

Algorithm 3.1. Assume that the conditions $0 < t < 2$, $-1 < a < 1$, and $b = \theta_t^{-1}(1/a)$ hold.

The initializations for the algorithm are as follows:

$$\begin{aligned} &\alpha_0 = \alpha_2 = 1 + b^2 - \frac{t^2}{2}b, \qquad \alpha_1 = 4b - \frac{t^2}{2}(1+b^2); \\ &\beta_0 = \beta_2 = b, \quad \beta_1 = 1+b^2, \quad \kappa = 2(4-t^2)^{-1-t/2}; \\ &f = (1+b)^2 - t^2b, \quad g = (2-t^2)(1+b^2) + 4b, \quad h = \sqrt{f}. \end{aligned}$$

The starting values for the algorithm are defined as:

$$f_0 = \frac{1}{h}, \quad f_1 = -\frac{f_0}{2f}; \qquad g_0 = 1, \quad g_1 = g; \qquad h_0 = 1, \quad h_1 = f;$$

$$r_0 = f_0, \quad r_1 = g f_1; \qquad s_0 = h, \quad s_1 = \frac{g f_0}{2};$$

$$v_0 = \left(\frac{-t(1-b)+2h}{1+b}\right)^t, \quad v_1 = t^2(1-b) f_0 v_0;$$

$$w_0 = \alpha_0 + (1-b)h, \quad w_1 = \alpha_1 + (1-b)(s_1 - s_0);$$

$$x_0 = v_0 w_0, \quad x_1 = v_1 w_0 + v_0 w_1;$$

$$y_0 = \frac{\kappa^2 x_0^2 - b\beta_0}{\kappa}, \quad y_1 = \frac{\kappa^2 x_1 x_0 - b\beta_1}{\kappa y_0}; \qquad z_1 = \beta_1 x_0 - b x_1;$$

$$d = \frac{y_0}{z_1}, \quad d_{-1} = 1.$$

The algorithm is recursively defined for $n \geq 2$ in the following way:

$$f_n = \frac{(1-2n) f_{n-1}}{2f}; \qquad g_n = \frac{g g_{n-1}}{n}; \qquad h_n = \frac{f h_{n-1}}{n};$$

$$r_n = \sum_{k=0}^{[n/2]} h_k g_{n-2k} f_{n-k}; \qquad s_n = \frac{g r_{n-1}}{2n} + \frac{f r_{n-2}}{n};$$

$$v_n = \frac{t^2(1-b)}{n} \sum_{k=0}^{n-1} r_k v_{n-1-k} - \frac{(n-1) v_{n-1}}{n};$$

$$w_n = \delta_{2,n}\alpha_2 + (1-b)(s_n - s_{n-1}) \quad (\delta_{2,n} = \text{Kronecker-symbol});$$

$$x_n = \sum_{k=0}^{n} v_k w_{n-k}; \qquad y_n = \frac{\kappa^2 x_n x_0 - \delta_{2,n} b \beta_2}{\kappa y_0};$$

$$z_n = \frac{\delta_{2,n}\beta_2 x_0 - b x_n}{z_1}; \qquad d_{n-2} = y_{n-1} - \sum_{k=2}^{n} z_k d_{n-1-k}.$$

Theorem 3.2. *For $0 < t < 2$ and $-1 < a < 1$ the above algorithm generates the Laurent coefficients d and d_n in (3.3) of the conformal mapping $\theta_{t,a}$ from $\mathbb{D}$ onto the exterior of the transformed sector $G_{t,a}$.*

Proof. Consider the following functions which are defined on $\mathbb{D}$:

$$f(u) = (1+b)^2(1+u)^2 - t^2(u+b)(1+bu),$$

$$r(u) = \frac{1}{\sqrt{f(u)}}, \qquad s(u) = \sqrt{f(u)},$$

$$v(u) = \left(\frac{-t(1-m_2(u)) + 2\sqrt{(1+m_2(u))^2 - t^2 m_2(u)}}{1+m_2(u)}\right)^t, \tag{3.13}$$

$$w(u) = \frac{(1+bu)^2}{2}\left(1 - m_2(u) + \sqrt{(1+m_2(u))^2 - t^2 m_2(u)}\right)^2,$$

$$x(u) = v(u) w(u).$$

After some elementary calculations we obtain

$$\theta_{t,a}(u) = \frac{\kappa x(0)x(u) - b(u+b)(1+bu)/\kappa}{(u+b)(1+bu)x(0) - bx(u)} \tag{3.14}$$

with κ as in the initializations in Algorithm 3.1. Furthermore, for real u we have

$$v(u) = \left(\frac{-t(1-b)(1-u) + 2\sqrt{f(u)}}{(1+b)(1+u)}\right)^t \tag{3.15}$$

and

$$w(u) = \tfrac{1}{2}\left((1-b)(1-u) + \sqrt{f(u)}\right)^2.$$

Now, with the aid of Lemma 3.3 below, the following relations are obtained for the coefficients in Algorithm 3.1:

$$\begin{aligned}
&\beta_n = \frac{[(u+b)(1+bu)]^{(n)}(0)}{n!}, \qquad n = 0, 1, 2;\\
&f = f(0) = f''(0)/2, \qquad g = f'(0);\\
&f_n = n!\binom{-1/2}{n}(f(0))^{-n-1/2};\\
&g_n = \frac{(f'(0))^n}{n!}; \qquad h_n = \frac{(f''(0))^n}{2^n n!}.
\end{aligned}$$

Moreover, the scalars r_n, s_n, v_n, w_n, x_n are the Taylor coefficients of the corresponding functions in (3.13). Hence, we obtain the representations

$$y(u) = y_0\left(1 + \sum_{n=1}^{\infty} y_n u^n\right), \qquad z(u) = z_1 u\left(1 + \sum_{n=2}^{\infty} z_n u^{n-1}\right)$$

for the numerator and denominator in (3.14). Comparing coefficients of the power series in $y = \theta_{t,a} z$ gives the recursion for the d_n. □

Lemma 3.3. *For the functions f, v, w in (3.13) and for any $\gamma \in \mathbb{R}$, the following three conditions are satisfied*:

$$\text{(i)} \qquad \frac{(f^\gamma)^{(n)}}{n!} = \sum_{k=0}^{[n/2]} \frac{(f'')^k}{2^k k!} \cdot \frac{(f')^{n-2k}}{(n-2k)!} \cdot \gamma_{n-k} f^{\gamma-n+k}, \qquad n \ge 0,$$

where $\gamma_k := k!\binom{\gamma}{k}$;

$$\text{(ii)} \qquad \frac{v^{(n)}(0)}{n!} = \frac{t^2(1-b)(f^{-1/2}v)^{(n-1)}(0) - (n-1)v^{(n-1)}(0)}{n!}, \qquad n \ge 1;$$

$$\text{(iii)} \qquad w(0) = \alpha_0 + (1-b)\sqrt{f(0)},$$

$$\frac{w^{(n)}(0)}{n!} = \alpha_n + (1-b)\left[\frac{(\sqrt{f})^{(n)}(0)}{n!} - \frac{(\sqrt{f})^{(n-1)}(0)}{(n-1)!}\right], \qquad n \ge 1,$$

with α_0, α_1, α_2 as in Algorithm 3.1 and $\alpha_n = 0$ for $n \ge 3$.

Proof. (i) can be verified by induction after observing that $f''' = 0$. We omit further details. For the proof of (ii) we use the relation

$$\frac{v'(u)}{v(u)} = \frac{t^2(1-b)}{(1+u)\sqrt{f(u)}}.$$

Multiplying by $(1+u)v(u)$ and differentiating $(n-1)$ times gives the formula

$$(1+u)v^{(n)}(u) = t^2(1-b)(f^{-1/2}v)^{(n-1)}(u) - (n-1)v^{(n-1)}(u),$$

which proves (ii). With α_0, α_1, α_2 as given in Algorithm 3.1 we find

$$w(u) = \alpha_0 + \alpha_1 u + \alpha_2 u^2 + (1-b)(1-u)\sqrt{f(u)}.$$

Now (iii) follows from an application of the Leibniz formula. □

The given algorithm for the computation of the Laurent coefficients of $\theta_{t,a}$ is very efficient. To compute d_n in the $(n+2)$nd step, we need approximately $(4n+7)$ real multiplications. In contrast to the number of multiplications used in computing the convolution product $p_1(\Phi)p_n(\Phi)$, which appears in the recursion (2.3) for the Faber polynomials, the complexity of Algorithm 3.1 is negligible. (Here we assume that the finite sequence ϕ has support on the vertices of a d-cube with $d \geq 2$, as is the case for three-direction box splines.)

So far we have excluded the case $a = 0$ from our consideration of $\theta_{t,a}$. It can easily be seen that Algorithm 3.1 produces the Laurent coefficients of θ_t if we start the algorithm with $a = b = 0$. The effect of dilations and translations of the region $G_{t,a}$ can be described as follows. For $r > 0$ and $\tau \in \mathbb{C}$, let $\theta_{t,a,r,\tau}$ denote the conformal mapping from $\mathbb{D}$ onto the exterior of $rG_{t,a} + \tau$. Then $\theta_{t,a,r,\tau}$ has the expansion

$$\theta_{t,a,r,\tau}(u) = rd\left(\frac{1}{u} + \left(d_0 + \frac{\tau}{rd}\right) + \sum_{n=1}^{\infty} d_n u^n\right), \tag{3.16}$$

where d and d_n are the coefficients of $\theta_{t,a}$.

TABLE 1
Numerical values of the Laurent coefficients for $\theta_{t,a}$

	$t = 1/2,\ a = 0$	$t = 3/2,\ a = 0$	$t = 3/2,\ a = 1/2$
d	9.38786(−1)	5.31127(−1)	8.53371(−1)
d_0	1.25 (−1)	1.125	2.95165(−1)
d_1	−1.13281(−1)	−1.75781(−1)	−1.92083(−1)
d_2	1.00098(−1)	−1.53809(−1)	8.98302(−2)
d_3	−8.61359(−2)	1.04782(−1)	−1.03362(−2)
d_4	7.21111(−2)	−8.53157(−3)	−3.52386(−2)
d_5	−5.86988(−2)	−4.55868(−3)	4.83060(−2)
d_6	4.64729(−2)	−1.35177(−2)	−3.95184(−2)
d_7	−3.58594(−2)	9.95135(−3)	2.25249(−2)
d_8	2.71095(−2)	1.10246(−3)	−8.05059(−3)
d_9	−2.02934(−2)	3.96085(−4)	8.21071(−4)
d_{10}	1.53168(−2)	−4.75490(−3)	−1.29118(−6)
⋮	⋮	⋮	⋮
d_{50}	1.32845(−3)	−3.09303(−5)	−5.67362(−5)

As an example, we consider the pairs $(t, a) = (\frac{1}{2}, 0), (\frac{3}{2}, 0)$, and $(\frac{3}{2}, \frac{1}{2})$, and compute the coefficients d_n and d of the conformal map $\theta_{t,a}$ (cf. (3.6)) by Algorithm 3.1. Both the values of d and the first eleven values of d_n are listed in Table 1.

4. The Faber series approach to cardinal interpolation

As has been the case throughout this paper, we continue to assume that $\varphi \in C(\mathbb{R}^d)$ is compactly supported and $0 \notin R(\tilde{\varphi})$. We are now ready to formulate an algorithm for the approximation by Faber series to the cardinal interpolation operator

$$T: l_2 \to l_2, \qquad f \to \Lambda * f,$$

where Λ is the fundamental sequence appearing in (1.4). Hereafter, we assume that the parameters $0 < t < 2$, $-1 < a < 1$, $r > 0$, and $\tau \in \mathbb{C}$ are chosen to satisfy

$$R(\tilde{\varphi}) \subset rG_{t,a} + \tau := G, \qquad 0 \notin \overline{G}. \tag{4.1}$$

The approximations to Λ, namely $\Lambda^{(n)} := q_n^{(F)}(\Phi)$ are based on partial sums $q_n^{(F)}$ of the Faber series of $1/z$ on G (cf. (2.7)). For the computation of $\Lambda^{(n)}$, the following steps should be performed.

Algorithm 4.1. Assume that t, a, r, τ are given as in (4.1).

Step 1. Compute $b := \theta_t^{-1}(1/a)$ with θ_t in (3.4), using Newton's method.

Step 2. Compute $u^* \in \mathbb{D}$, where

$$r\theta_{t,a}(u^*) + \tau = 0, \tag{4.2}$$

$\theta_{t,a}$ as in (3.6), using Newton's method. Note that the iteration is real for $\tau \in \mathbb{R}$.

Step 3. Compute d and d_0, using Algorithm 3.1. Thus, after initializing the recursions by

$$\begin{aligned}
&a_0 := (ru^*\theta'_{t,a}(u^*))^{-1}, \qquad a_1 := u^* a_0, \\
&p_1(\Phi) := P_1 := \frac{1}{rd}\Phi - \left(d_0 + \frac{\tau}{rd}\right) I, \quad \text{where } I := (\delta_{0,j})_{j \in \mathbb{Z}^d}, \\
&q_1^{(F)}(\Phi) := a_0 I + a_1 P_1,
\end{aligned}$$

the algorithm proceeds for $n \geq 2$ by means of the equations

$$\begin{aligned}
&a_n := u^* a_{n-1}, \\
&d_{n-1} \text{ as in Algorithm 3.1}, \\
&p_n(\Phi) := P_n := P_1 * P_{n-1} - \sum_{k=1}^{n-2} d_k P_{n-1-k} - n d_{n-1}, \\
&\Lambda^{(n)} := q_n^{(F)}(\Phi) := q_{n-1}^{(F)}(\Phi) + a_n P_n.
\end{aligned}$$

A bound on the total rotation (2.4) of $G = G_{t,a}$ is given by $V_G \leq 10\pi$, since G is a circular triangle with two exterior angles of size $3\pi/2$ and one

TABLE 2
Convergence parameters depending on a

t	$\lvert\tau - a\rvert^{1/t}$	a	u^*	a	u^*
1/2	0.01	0	−0.996289	0.854	−0.991298
	0.0025	0	−0.999071	0.903	−0.997492
	0.0001	0	−0.999963	0.981	−0.999541
1	0.1	0	−0.926	−0.212	−0.899
	0.05	0	−0.962	−0.108	−0.927
	0.01	0	−0.992	−0.021	−0.988
3/2	0.215	0	−0.767	−0.213	−0.719
	0.136	0	−0.846	−0.121	−0.827
	0.0464	0	−0.944	−0.029	−0.942

exterior angle of size πt. For $a = 0$, hence $G = G_t$, we even find the bound $V_G \leq 5\pi$, since the two straight boundary segments do not contribute to V_G. By employing Theorem 2.1 and the observation that $s^* = 1/u^*$, where s^* appears in (2.9), we can conclude the following.

Theorem 4.2. *Let $\varphi \in C(\mathbb{R}^d)$ be compactly supported, and let t, a, r, τ be chosen as in (4.1). Furthermore, let $u^* \in \mathbb{D}$ solve the equation $r\theta_{t,a}(u^*)+\tau = 0$. Then $\Lambda^{(n)} := q_n^{(F)}(\Phi)$ satisfies*

$$E_n^{(F)} := \|\widetilde{\Lambda} - \widetilde{\Lambda}^{(n)}\|_\infty \leq \frac{10|u^*|^n}{|\theta'_{t,a}(u^*)|(1-|u^*|)}. \tag{4.3}$$

For a real shift parameter τ, the size of the parameter $u^* = u^*(t, a, r, \tau)$ in (4.2) depends exponentially on the exterior angle πt of $G = rG_{t,a} + \tau$ at the boundary point $\tau - ar = r\theta_{t,a}(-1) + \tau$. As shown in [8], the following relation holds:

$$\left(\frac{1}{|u^*|} - 1\right) \sim \left|\frac{\tau}{r} - a\right|^{1/t}. \tag{4.4}$$

So, a small exterior angle πt of the sector G_t at 0 drawn in Figure 1 forces u^* to be close to -1. In typical examples involving three-directional shifted box-splines in $\mathbb{R}^2$, sectors with an exterior angle not smaller than $\pi/2$ seem to be most useful. A comparison of some numerical values for u^* when $r = 1$, t and $|\tau - a|$ are kept fixed, and a is varied (cf. Table 2) demonstrates that a careful choice of a can improve the convergence rate, although the exterior angle πt is not affected.

5. Cardinal interpolation with symmetric functions

In this section, $\varphi \in C(\mathbb{R}^d)$ represents a compactly supported *symmetric* function, i.e., $\varphi(x) = \varphi(-x)$, $x \in \mathbb{R}^d$. Thus, the symbol $\tilde{\varphi}$ is real-valued. Hence, if $0 \notin R(\tilde{\varphi})$ and $\tilde{\varphi}(0) > 0$, then $R(\tilde{\varphi})$ is a real interval, namely

$$R(\tilde{\varphi}) = [\tau, \tau + 2r], \qquad \tau, r > 0. \tag{5.1}$$

Although the Faber series approach does not apply directly, we note that the Chebyshev series can be obtained by a limiting process from the Faber series. For this reason, we compare in this section the Chebyshev series and Neumann series approach in generating approximations $\Lambda^{(n)}$ to the fundamental sequence Λ given in (1.4). The rate of convergence of the approximations $\Lambda^{(n)}$ is determined by the rate of approximation of the Chebyshev and Neumann series approximation to $1/z|_{[\tau, \tau+2r]}$.

As mentioned above, the series expansion in Chebyshev polynomials T_n of the first kind normalized so that $\|T_n\|_\infty = 1$ may be considered as a generalization of the Faber series (see Remark (ii) at the end of this section for details). Since the approximation properties of Chebyshev polynomials are well understood, our derivations and estimates are derived in terms of the Chebyshev polynomials T_n. However, in light of Remark (ii), our conclusions will relate to Faber series.

In order to use the Neumann series approach to cardinal interpolation, we can choose the midpoint of the interval $[\tau, \tau + 2r]$, namely $\tau + r$, as a scaling factor and obtain

$$\frac{1}{z} = \frac{1}{\tau + r} \sum_{n=0}^{\infty} \left(1 - \frac{z}{\tau + r}\right)^n, \qquad z \in [\tau, \tau + 2r]. \tag{5.2}$$

Letting $q_n^{(N)}$ represent the partial sum of (5.2), we have that

$$\left\| \frac{1}{z} - q_n^{(N)} \right\|_{[\tau, \tau+2r]} = \tau^{-1} \left(\frac{r}{\tau + r}\right)^{n+1}, \tag{5.3}$$

and this is the best possible error constant for a geometric series approximation of $1/z$ on $R(\tilde{\varphi})$. Since the constant in (5.3) determines the rate of convergence of the approximations $\Lambda^{(n)}$ to the fundamental sequence Λ in (1.4), we look for other schemes with a lower error bound.

The usual Chebyshev polynomials T_n of the first kind on the interval in (5.1), normalized to $\|T_n\|_\infty = 1$, are given by

$$\begin{aligned} &T_0(z) = 1, \qquad T_1(z) = \frac{z - r - \tau}{r}, \\ &T_{n+1} = 2T_1 T_n - T_{n-1}. \end{aligned} \tag{5.4}$$

Based on the generating function formula for such functions (cf. [1, p. 783]) we find, with $\alpha := (\tau + r)/r$ and $\beta := \alpha - \sqrt{\alpha^2 - 1} \in (0, 1)$, the relation

$$\frac{1}{z} = \frac{2\beta}{r(1 - \beta^2)} \left(1 + 2\sum_{n=1}^{\infty} (-\beta)^n T_n(z)\right), \qquad z \in [\tau, \tau + 2r]. \tag{5.5}$$

Letting $q_n^{(C)}$ represent the nth partial sum of the Chebyshev series (5.5), we obtain the error estimate

$$E_n^{(C)}\left(\frac{1}{z}\right) = \left\| \frac{1}{z} - q_n^{(C)} \right\|_{[\tau, \tau+2r]} = \frac{4\beta^{n+2}}{r(1 - \beta^2)(1 + \beta)}. \tag{5.6}$$

Figure 2 shows a comparison of the convergence rates $\gamma = r/(\tau + r)$ for the geometric series and

$$\beta(\gamma) = \frac{1 - \sqrt{1 - \gamma^2}}{\gamma}, \qquad \gamma \in (0, 1),$$

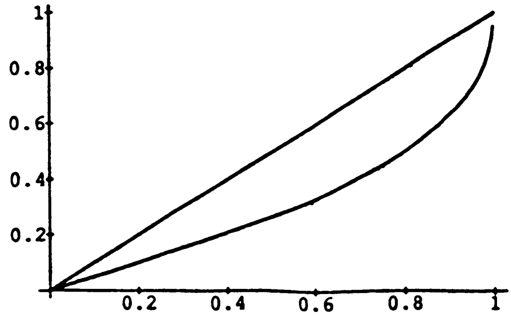

FIGURE 2
Convergence rates compared

for the Chebyshev series. We see that the Chebyshev series (5.5) gives, in all cases, a better convergence rate, while the computational effort for determining $q_n^{(C)}$, using recursion (5.4), is slightly larger than that required for $q_n^{(N)}$.

Remarks. (i) It is well known that the nth partial sum $q_n^{(C)}$ of the Chebyshev expansion (5.5) of $1/z$ is a "near best approximant" in the space Π_n of all polynomials of degree $\leq n$. In fact, comparing (5.6) to the error

$$E_n\left(\frac{1}{z}\right) = \inf_{p\in\Pi_n}\left\|\frac{1}{z} - p\right\|_{[\tau, \tau+2r]} = \frac{\beta^n}{r(\alpha^2-1)}$$

for best approximation, as given in [11, p. 32], we obtain

$$E_n^{(C)}\left(\frac{1}{z}\right) = \frac{2(\alpha+1)}{\alpha+1+\sqrt{\alpha^2-1}}E_n\left(\frac{1}{z}\right).$$

Hence, the approximations $q_n^{(C)}$ give rise to the best possible geometric convergence rate.

(ii) There is an interesting connection between the Chebyshev expansion (5.5) and the Faber series (2.6) of $1/z$. The conformal mapping from the unit disk $\mathbb{D}$ onto the cut plane $\mathbb{C}\backslash[\tau, \tau+2r]$ is given by

$$\theta(u) = \frac{r}{2}\left(\frac{1}{u} + 2\alpha + u\right)$$

(cf. [10, p. 79]). If we apply the recursion (2.3) for the Faber polynomials formally, we obtain the equations $p_n = 2T_n$, $n \geq 1$. Furthermore, for β as in (5.5), it follows that $\beta + 1/\beta = 2\alpha$, and hence $\theta(-\beta) = 0$. Thus, the expansion (5.5) coincides with the formal Faber series of $1/z$ on the interval (5.1).

6. QUASI-INTERPOLATION SCHEMES

As before, let $\varphi \in C(\mathbb{R}^d)$ be compactly supported with $0 \notin R(\tilde{\varphi})$. Without loss of generality we assume $\tilde{\varphi}(0) = 1$. For a complex sequence $a = (a_j)_{j\in\mathbb{Z}^d}$, we use the notation

$$a \mathbin{\tilde{*}} \varphi = \sum_{j\in\mathbb{Z}^d} a_j\varphi(\cdot - j)$$

for the semidiscrete convolution. Then the *commutator* (cf. [5]) of φ is given by the mapping

$$[f|\varphi] := (f|_{\mathbb{Z}^d}) \tilde{*} \varphi - \Phi \tilde{*} f, \qquad f \in C(\mathbb{R}^d). \tag{6.1}$$

The maximal integer k such that

$$[p|\varphi] = 0 \quad \text{for all } p \in \Pi_{k-1} \tag{6.2}$$

is called the *order* of the commutator of φ. It is well known that Π_{k-1} is the maximal space of polynomials of total degree which is contained in

$$\mathscr{S}(\varphi) := \{a \tilde{*} \varphi | a = (a_j) \text{ is polynomially bounded}\},$$

and that k is the maximal approximation order for "local approximation" (cf. [9]) with scaled versions of $\mathscr{S}(\varphi)$.

We now consider the cardinal interpolation operator on function spaces,

$$\mathscr{T} f := (\Lambda * f|_{\mathbb{Z}^d}) \tilde{*} \varphi, \qquad f \in C(\mathbb{R}^d), \tag{6.3}$$

for polynomially bounded f. For the finite approximations $\Lambda^{(n)}$ to the fundamental sequence Λ developed in §§4 and 5, let $\mathscr{T}_n$ be defined by

$$\mathscr{T}_n f := (\Lambda^{(n)} * f|_{\mathbb{Z}^d}) \tilde{*} \varphi, \qquad f \in C(\mathbb{R}^d). \tag{6.4}$$

A desirable property of these local approximation schemes is that they define *quasi-interpolants*, i.e., they reproduce all polynomials in Π_{k-1}. It is well known (cf. [3]) that with $\Lambda^{(n)} = q_n^{(N)}(\Phi)$, the corresponding operators $\mathscr{T}_n^{(N)}$ in (6.4) have this property for

$$n \geq \begin{cases} k-1 & \text{for general } \varphi, \\ (k-1)/2 & \text{for symmetric } \varphi. \end{cases} \tag{6.5}$$

In this section we give a systematic study of the quasi-interpolation property for operators of the form (6.4), where $\Lambda^{(n)} = q(\Phi)$ for some univariate polynomial q. Our aim will be to modify the Faber series approach (§4) to a quasi-interpolating scheme.

If k is the order of the commutator of φ, then the quasi-interpolation property of $\mathscr{T}_n$ in (6.4) can be described by the equation

$$(I - q(\Phi) * \Phi) * p|_{\mathbb{Z}^d} = 0 \quad \text{for all } p \in \Pi_{k-1}. \tag{6.6}$$

The next result gives a sufficient condition for the polynomial q to satisfy (6.6).

Theorem 6.1. *Let $\varphi \in C(\mathbb{R}^d)$ be compactly supported, k the order of the commutator of φ, and $\tilde{\varphi}(0) = 1$.*

(i) *For a univariate polynomial q, condition (6.6) is equivalent to*

$$D^\alpha(1 - (q \circ \tilde{\varphi}) \cdot \tilde{\varphi})(0) = 0 \quad \textit{for all } |\alpha| \leq k-1. \tag{6.7}$$

(ii) *If $D^\alpha \tilde{\varphi}(0) = 0$ for $1 \leq |\alpha| \leq m-1$, then a sufficient condition for (6.6) to hold is that*

$$[1 - zq(z)]^{(\mu)}(1) = 0 \quad \textit{for all } 0 \leq \mu \leq \frac{k-1}{m}. \tag{6.8}$$

Furthermore, if φ is a univariate function and $\tilde{\varphi}^{(m)}(0) \neq 0$, then (6.8) is necessary and sufficient for (6.6) to be satisfied.

Note that in (6.8) only q, and not $\tilde{\varphi}$, is involved, and that (ii) generalizes (6.5).

Proof of Theorem 6.1. The sequence $(p|_{\mathbb{Z}^d})$ is the Fourier transform of the tempered distribution $\tilde{p} = \sum_{j\in\mathbb{Z}^d} p(j)\chi_j$, where $\chi_j\psi = (2\pi)^{-d}\int_{(0,2\pi)^d}\psi(x)e^{ij\cdot x}\,dx$ for any periodic test function ψ. After taking Fourier transforms, condition (6.6) is equivalent to

$$(1-\tilde{\varphi}q(\tilde{\varphi}))\cdot\tilde{p}\equiv 0 \quad \text{for all } p\in\Pi_{k-1}. \tag{6.9}$$

Letting $p_\alpha(x) = (ix)^\alpha$, $|\alpha|\le k-1$, we have that $\tilde{p}_\alpha = D^\alpha\tilde{\delta}$ with the periodic Dirac distribution $\tilde{\delta}$. Thus (6.9) is equivalent to the condition that for all periodic test functions ψ

$$\begin{aligned} 0 &= (1-\tilde{\varphi}\cdot q(\tilde{\varphi}))\tilde{p}_\alpha(\psi) \\ &= D^\alpha((1-\tilde{\varphi}\cdot q(\tilde{\varphi}))\psi)(0) \quad \text{for all } |\alpha|\le k-1, \end{aligned}$$

and this proves (i). We now show that (6.8) implies (6.7). For $\alpha = 0$ this is obvious. Moreover, if $R_j(x)$ is defined by $R_j(x) := [z\cdot q(z)]^{(j)}(\tilde{\varphi}(x))$, then clearly $\partial/\partial x_l R_j = R_{j+1}\partial/\partial x_l\tilde{\varphi}$, $j\ge 0$. Furthermore, for $0<\nu := |\alpha|\le k-1$ with $D^\alpha = \partial/\partial x_{i_1}\cdots\partial/\partial x_{i_\nu}$, let $\mathscr{M}_{\nu,j}$ be the collection of all partitions $(M_1,\dots,M_j)$ of $\{1,2,\dots,\nu\}$ with nonempty M_κ, and write $D_M f = \prod_{\rho\in M}\partial/\partial x_{i_\rho} f$. Then the following relation holds:

$$D^\alpha(\tilde{\varphi}\cdot(q\circ\tilde{\varphi}))(0) = \sum_{j=1}^{\nu} R_j(0)\left(\sum_{(M_1,\dots,M_j)\in\mathscr{M}_{\nu,j}}\prod_{\kappa=1}^{j}(D_{M_\kappa}\tilde{\varphi}(0))\right). \tag{6.10}$$

For $1\le j\le (k-1)/m$ the corresponding term in the above sum vanishes by assumption (6.8). The remaining cases are characterized by the inequality $(k-1)/m < j\le\nu\le k-1$. For these cases, any partition $(M_1,\dots,M_j)\in\mathscr{M}_{\nu,j}$ contains at least one M_κ with cardinality $\le m-1$; hence, $D_{M_\kappa}\tilde{\varphi}(0) = 0$. Thus, (6.7) is proved, and this gives (6.6).

The necessity of (6.8) for univariate φ with $\tilde{\varphi}^{(m)}(0)\ne 0$ can be seen by an inductive argument using (6.10). For $\mu = 0$, this is obvious, hence $R_0(0) = q(1) = 1$. For $(k-1)/m\ge\mu>0$, the relation (6.10) with $\alpha = m\mu$ simplifies to

$$0 = R_\mu(0)\cdot(\tilde{\varphi}^{(m)}(0))^\mu,$$

assuming that $R_j(0) = 0$ for $1\le j\le\mu-1$. Hence, $R_\mu(0) = [zq(z)]^{(\mu)}(1) = 0$, which proves (6.8) and completes the proof. □

The partial sums $q_n^{(F)}$ (or $q_n^{(C)}$) of the Faber (resp. Chebyshev) expansion of $1/z$ in (2.6) and (5.5) usually do not satisfy (6.8), even for $\mu = 0$. In order to combine the advantages of quasi-interpolation with the fast convergence of the Faber (or Chebyshev) series, we propose the following "blended" approach to cardinal interpolation.

The notation for the next theorem agrees with that of the previous theorem.

Theorem 6.2. *Let $\mu := [(k-1)/m]$ denote the integer part of $(k-1)/m$ and assume that $D^\alpha\tilde{\varphi}(0) = 0$ for $1\le|\alpha|\le m-1$. Then, if $q_n^{(F)}$ represents the partial sum of the Faber expansion of $1/z$, it follows that the polynomial*

$$q_{n+\mu+1}^{(B)} := \sum_{l=0}^{\mu}(1-z)^k + (1-z)^{\mu+1}q_n^{(F)}, \qquad n\ge 0, \tag{6.11}$$

has exact degree $n+\mu+1$, *and the quasi-interpolation property*

$$(6.12)\qquad (q^{(B)}_{n+\mu+1}(\Phi) * p|_{\mathbb{Z}^d}) \,\tilde{*}\, \varphi = p\,, \qquad p \in \pi_{k-1}\,,$$

holds for all $n \geq 0$. *Furthermore, the fundamental sequence* Λ *satisfies the inequality*

$$(6.13)\qquad \|\Lambda - q^{(B)}_{n+\mu+1}(\Phi)\|_{l_2} \leq \|\widetilde{\Lambda} - (q^{(B)}_{n+\mu+1}(\Phi))^{\sim}\|_\infty \leq \|1-z\|^{\mu+1}_{R(\tilde{\varphi})} E^{(F)}_n\,,$$

where $E^{(F)}_n$ *is defined in* (4.3).

Proof. The polynomial $q^{(B)}_{n+\mu+1}$ satisfies (6.8). The error bound in (6.13) is obtained by an application of Proposition 1.1 and Theorem 2.1, together with the observation that

$$\begin{aligned}\frac{1}{z} - q^{(B)}_{n+\mu+1}(z) &= \frac{1}{z}\left(1 - z\sum_{k=0}^{\mu}(1-z)^k - (1-z)^{\mu+1} z q^{(F)}_n(z)\right)\\ &= \frac{1}{z}(1-z)^{\mu+1}(1 - zq^{(F)}_n(z)). \quad \square\end{aligned}$$

Remark. The polynomial $q^{(B)}_{n+\mu+1}$ induces the local approximation operator

$$\mathscr{T}^{(B)}_n f := (q^{(B)}_{n+\mu+1}(\Phi) * f|_{\mathbb{Z}^d}) \,\tilde{*}\, \varphi\,, \qquad f \in C(\mathbb{R}^d).$$

This operator can also be described as the Boolean sum of $\mathscr{T}^{(N)}_\mu \oplus \mathscr{T}^{(F)}_n$, where these operators are derived from the polynomials $q^{(N)}_\mu$, $q^{(F)}_n$ as in (6.4). So the quasi-interpolation property (6.12) of $\mathscr{T}^{(B)}_n$ is inherited from the corresponding property of $\mathscr{T}^{(N)}_\mu$, which is obvious in view of condition (6.8).

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