

EIGENVALUE APPROXIMATION BY A MIXED METHOD FOR RESONANT INHOMOGENEOUS CAVITIES WITH METALLIC BOUNDARIES

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ABSTRACT. For an inhomogeneous cavity bounded by a perfect conductor, we prove that the approximation of the eigenvalues for the Maxwell problem leads to a second-order rate of convergence when using mixed finite elements. If the cavity has a disconnected boundary, the problem has null eigenvalues. We verify the existence of null eigenvalues for the approximate problem. They do not mix with the others that still converge at the same rate.

INTRODUCTION

The approximation of eigenvalues involves problems that are of great interest. We mention, for example, the works of Osborn [10], Mercier, Osborn, Rappaz, and Raviart [8], and Brezzi and Raviart [5]. More specifically, mixed methods are used for this purpose (see Ishihara [7]). The aim of this paper is to apply results of Mercier, Osborn, Rappaz, and Raviart [8] on eigenvalues, and of Nedelec [9] on mixed finite elements, to prove a second-order rate of convergence for the harmonic Maxwell equation.

Another part of the paper concerns cavities with disconnected boundaries. The operator of the problem is then no longer elliptic. Work on this kind of geometry can be found in Bossavit [3], and in Bendali, Dominguez, and Gallic [2].

First, we deal with a connected boundary to get an elliptic problem, which is necessary in order to use known results. Then we extend the work to disconnected boundaries and prove that the zero eigenvalues of the curl operator have approximate equivalents that do not mix with the other converging eigenvalues.

To conclude, we show numerical experiments on a cube, for which eigenvalues are known analytically.

1. THE CONTINUOUS PROBLEM

Let Ω be a bounded cavity, with a connected boundary, filled with an inhomogeneous medium. We consider the following problem:

Received December 18, 1989; revised January 22, 1991.
1991 *Mathematics Subject Classification.* Primary 65N30.

$$\left\{ \begin{array}{l} \text{Find } (E, H) \in H(\text{curl}, \Omega) \text{ satisfying} \\ \text{curl } E + i\omega H = 0, \\ \text{curl } H - i\omega E = 0, \\ \text{div}(\mu H) = 0, H \cdot n|_{\partial\Omega} = 0, \\ \text{div}(\varepsilon E) = 0, E \times n|_{\partial\Omega} = 0, \end{array} \right.$$

where ε and μ are strictly positive bounded real functions. Let E' be a test function belonging to $H(\text{Curl}, \Omega)$, satisfying $E' \times n = 0$ on the boundary $\partial\Omega$. Then applying the Green formula

$$\int_{\Omega} \text{curl } H \cdot \bar{E} = \int_{\Omega} H \cdot \text{curl } \bar{E} - \int_{\partial\Omega} (\bar{E} \times n) \cdot H$$

to the first system, we obtain the variationally posed problem:

$$\int_{\Omega} \mu^{-1} \text{curl } E \cdot \text{curl } \bar{E}' - \omega^2 \int_{\Omega} \varepsilon E \cdot \bar{E}' = 0.$$

Given two regular complex functions ε and μ , with strictly positive bounded real parts, this is an eigenvalue problem in a variational form:

$$(P) \quad \left\{ \begin{array}{l} \text{Find } E \in H(\text{curl}, \Omega), E \times n|_{\partial\Omega} = 0, \text{div } \varepsilon E = 0, \lambda \in \mathbb{R} \\ \text{such that } \forall E' \in H(\text{curl}, \Omega) \text{ with } E' \times n = 0, \\ \int_{\Omega} \mu^{-1} \text{curl } E \cdot \text{curl } \bar{E}' = \lambda \int_{\Omega} \varepsilon E \cdot \bar{E}'. \end{array} \right.$$

Therefore, we introduce the following spaces:

$$\begin{aligned} W &= H_0^1(\Omega), \\ X &= \{E \in L^2(\Omega)^3, \text{curl } E \in L^2(\Omega)^3, E \times n|_{\partial\Omega} = 0\}, \\ H &= L^2(\Omega)^3 \end{aligned}$$

and the continuous bilinear forms

$$\begin{aligned} a: X \times X &\rightarrow \mathbb{C} \\ (E, E') &\rightarrow \int_{\Omega} \frac{1}{\mu} \text{rot } E \cdot \text{rot } \bar{E}' + \int_{\Omega} \varepsilon E \cdot \bar{E}', \\ b: X \times W &\rightarrow \mathbb{C} \\ (E, q) &\rightarrow \int_{\Omega} \varepsilon E \cdot \text{grad } \bar{q}, \\ r: H \times H &\rightarrow \mathbb{C} \\ (E, E') &\rightarrow 2 \cdot \int_{\Omega} \varepsilon E \cdot \bar{E}' \end{aligned}$$

that satisfy the properties:

H1: r is continuous and bounded on $H \times H$,

H2: $b(v, q) = 0$ for every v in X implies $q = 0$.

(Simply consider $v = \text{grad } q$ that belongs to X : since ε and μ have strictly positive bounded real parts, we apply the Poincaré inequality to conclude that $\text{grad } q = 0$ implies $q = 0$ in Ω .)

Let us now introduce the following problem: with g given in H , find $(E, p) \in X \times W$ such that, $\forall (E', p') \in X \times W$:

$$(II) \quad \begin{cases} a(E, E') + \bar{b}(E', p) = r(g, E'), \\ b(E, p') = 0. \end{cases}$$

We have

Lemma 1.1. *For each g given in H , there is a unique solution (E, p) of problem (II) that depends continuously on g . As g belongs to H , we have the higher regularity:*

$$\|E\|_{H^2} + \|p\|_{H_0^1} \leq c\|g\|_{L^2}.$$

Furthermore, if $\operatorname{div} E \in L^2$, then p belongs to H^2 , and we have

$$\|E\|_{H^2} + \|p\|_{H^2} \leq c(\|g\|_{L^2} + \|\operatorname{div} g\|_{L^2}).$$

Proof. The bilinear form a is V -elliptic: it is equivalent to the $H(\operatorname{curl})$ norm, since ε and μ have positive real parts. We have the following property:

H3: b satisfies the Brezzi-Babuška condition (cf. Brezzi [4])

$$\inf_{\|q\| \neq 0} \sup_{\|E\| \neq 0} \frac{|b(E, q)|}{\|E\| \cdot \|q\|} > 0.$$

Indeed, consider $u = \operatorname{grad}(q)$; since the seminorm of the gradient is equivalent to the norm, the inequality is verified. Brezzi's result [4] now gives us the existence and uniqueness of (u, p) . The first regularity inequality is a classical result that can be found in Agmon, Douglis, and Nirenberg [1]. Finally, if $\operatorname{div} g \in L^2$, then $\Delta p \in L^2$, and so $p \in H^2$. \square

2. THE APPROXIMATE PROBLEM

We define a triangulation of the domain Ω , with tetrahedra, and to avoid the problem of volume and surface approximation, we will suppose that Ω is a polyhedral open set. We define the spaces

$$X_h = \{E \in H(\operatorname{rot}, \Omega), \forall K \in \mathcal{T}_h, E|_K(\vec{r}) = \vec{\alpha} + \vec{\beta} \times \vec{r}, E \times n|_{\partial\Omega} = 0\},$$

$$W_h = \{q \in H_0^1(\Omega), q|_k \in P^1\},$$

$$V_h = \{E_h \in X_h, b(E_h, q_h) = 0, \forall q_h \in W_h\}.$$

The system that is numerically solved is

$$(E) \quad \begin{cases} \text{Find } E \in X_h \text{ and } \lambda \in C \text{ such that } \forall E' \in X_h \\ \int_{\Omega} \mu^{-1} \operatorname{curl} E \cdot \operatorname{curl} \bar{E}' = \lambda \int_{\Omega} \varepsilon E \cdot \bar{E}'. \end{cases}$$

We have introduced V_h in order to be sure that the solutions of the eigenproblem associated with (II.a) below are also solutions of (E). Therefore, we associate the following approximate problem:

With g given in L^2 , find (E_h, p_h) in $X_h \times W_h$ such that for all (E'_h, p'_h) in X_h, W_h we have

$$(II.a) \quad \begin{cases} a(E_h, E'_h) + \bar{b}(E'_h, p_h) = r(g, E'_h), \\ b(E_h, p'_h) = 0. \end{cases}$$

We now have

Lemma 2.1. *If q is an element of W_h , then $\text{grad}(q)$ belongs to X_h .*

Proof. Nedelec [9] \square

The approximate problem satisfies properties H1, H2, and H3: indeed, the bilinear form a is V_h -elliptic because V_h is contained in $H(\text{curl})$. Furthermore, Lemma 2.1 implies that the Brezzi-Babuška condition and property H2 hold. \square

This leads to

Lemma 2.2. *For each g given in H , there is a unique solution (E_h, p_h) in $X_h \times W_h$ of problem (II.a) that depends continuously on g . The continuity constants do not depend on h .*

Remark. A part of the set of eigenvectors given by the variational formulation (E) is composed of curl-free functions that do not belong to V_h and that are exactly the gradients of W_h functions, and we know the dimension of this subspace (N_s , number of vertices inside Ω). They are associated with N_s zero eigenvalues. We will prove that each of the other eigenvalues converges toward the corresponding eigenvalue of the continuous problem (P), and we will establish the rate of convergence.

3. RATE OF CONVERGENCE

Taking into account the preceding paragraphs, we introduce the following definitions:

$$\begin{aligned} A: H &\rightarrow X \\ g &\rightarrow E, \\ B: H &\rightarrow W \\ g &\rightarrow p, \end{aligned}$$

where (E, p) is the solution of (II) for a given g , and

$$\begin{aligned} A_h: H &\rightarrow X_h \\ g &\rightarrow E_h, \\ B_h: H &\rightarrow W_h \\ g &\rightarrow p_h, \end{aligned}$$

where (E_h, p_h) is the solution of (II.a) for a given g . We also introduce the dual operators A^* , B^* , A_h^* , B_h^* , by means of the solutions of the two problems:

$$(II^*) \quad \begin{aligned} a(v, A^*g) + b(v, B^*g) &= r(v, g) \quad \forall v \in X, \\ b(A^*, g, q) &= 0 \quad \forall q \in W, \end{aligned}$$

$$(II_h^*) \quad \begin{aligned} a(v_h, A_h^*g) + b(v_h, B_h^*g) &= r(v_h, g) \quad \forall v_h \in X_h, \\ b(A_h^*, g, q) &= 0 \quad \forall q_h \in W_h. \end{aligned}$$

These “starred” operators satisfy the same regularity results as A, B .

We also assume that as h decreases:

(H): $\sup_k(\text{diam}(K)) \leq c \inf_K(\text{diam}(K))$, in order to use the following theorem.

Theorem 3.1. *Under hypothesis (H), and if the solutions are sufficiently regular, there exists $c \geq 0$ such that for all h*

$$\|Ag - A_h g\|_X + \|Bg - B_h g\|_W \leq c \cdot (\|Ag\|_{H^2} + \|Bg\|_{H^2}).$$

Proof. Nedelec [9] and Brezzi [4]. \square

We define a norm for the operators,

$$\|A\|_{XY} = \sup_{g \in X, g \neq 0} \frac{\|Ag\|_Y}{\|g\|_X}.$$

We recall some results about the eigenproblem: A is a compact operator from L^2 in L^2 , so its spectrum consists of a countable set of complex numbers, and each nonzero number is an isolated eigenvalue. Let μ be a nonzero eigenvalue; then there exists n_α such that

$$\text{Ker}(A - \alpha I)^{n_\alpha - 1} \neq \text{Ker}(A - \alpha I)^{n_\alpha} = \text{Ker}(A - \alpha I)^{n_\alpha + 1}.$$

Set $E_\alpha = \text{Ker}(A - \alpha I)^{n_\alpha}$, with $\dim E_\alpha = m$ the algebraic multiplicity of α . We verify that $\bar{\alpha}$ is an eigenvalue of A^* with the same multiplicity, and we let E_α^* be the set of the associated generalized eigenvectors.

Another result that we need is a theorem proved by Mercier, Osborn, Rappaz, and Raviart [8]. A version adapted to our problem is

Theorem 3.2. *Under the hypotheses H1, H2, and H3 for both the continuous and the approximate problems, and assuming both $\lim \|A - A_h\|_{HH} = 0$ and the continuous dependence of E_h, E, p_h, p with respect to g , we have the following. Let α be a nonzero eigenvalue of A with multiplicity m ; then for h small enough, exactly m eigenvalues of A_h converge to α . Furthermore, let $\beta = 1/\alpha$ and $\beta_{i,h} = 1/\alpha_{i,h}$; then there exists C_β such that*

$$\left| \beta - \frac{1}{m} \sum_{1 \leq j \leq m} \beta_{j,h} \right| \leq C_\beta \cdot (\|(A - A_h)_{E_\alpha}\|_{HX} \cdot \|(A^* - A_h^*)_{E_\alpha^*}\|_{HX} \\ + \|(A^* - A_h^*)_{E_h^*}\|_{HX} \cdot \|(B - B_h)_{E_\alpha}\|_{HW} \\ + \|(A - A_h)_{E_\alpha}\|_{HX} \cdot \|(B^* - B_h^*)_{E_\alpha^*}\|_{HW}).$$

Proof. Mercier, Osborn, Rappaz, and Raviart [8]. \square

In order to prove our main result, we need the following lemma.

Lemma 3.3. *There holds $\lim_{h \rightarrow 0} \|A - A_h\|_{HH} = 0$.*

Proof. Let g be given in L^2 , $\|g\| = 1$. If $\text{div } g$ is in L^2 , we know that $\lim \|Ag - A_h g\|_{HH} = 0$, but not in the general case.

Let $\varepsilon \geq 0$; since Ag belongs to H^2 and satisfies $b(Ag, q) = 0$ for all q , we know that there exists h_2 such that for all $h \leq h_2$, there exists $w_h \in X_h$ such that

$$b(w_h, q_h) = 0 \quad \forall q_h \in W_h, \quad \|Ag - w_h\|_X \leq \frac{\varepsilon}{3}.$$

We have

$$\begin{aligned} a(A_h g - w_h, A_h g - w_h) &= -a(w_h, A_h g - w_h) + a(A_h g, A_h g - w_h) \\ &= -a(w_h, A_h g - w_h) + a(Ag, A_h g - w_h) \\ &\quad + b(A_h g - w_h, Bg) - b(A_h g - w_h, B_h g) \\ &= a(Ag - w_h, A_h g - w_h) + b(A_h g - w_h, Bg - B_h g), \end{aligned}$$

and also $b(w_h, q_h) = 0$ and $b(A_h g, q_h) = 0$ for all q_h in W_h , so we can replace $B_h g$ with any q_h in W_h . Since a is elliptic, continuous, and b is continuous, we have

$$\|A_h g - w_h\|_X^2 \leq c(\|A g - w_h\|_W \|A_h g - w_h\|_W + \|A_h g - w_h\|_0 \|B g - q_h\|_1).$$

Since $B g$ is in H^1 , there exists ϕ in H^2 such that

$$\|B g - \phi\|_1 \leq \frac{\varepsilon}{6},$$

using the dense inclusion of H^2 in H^1 , and we can fix $h_1 \geq 0$ such that

$$\forall h \leq h_1, \exists q_h \in W_h, \|\phi - q_h\|_1 \leq \frac{\varepsilon}{6},$$

using the interpolation property of the H^2 elements by W_h . So,

$$\|B g - q_h\|_1 \leq \frac{\varepsilon}{6},$$

and finally

$$\begin{aligned} \exists h_3, \forall h \leq h_3, \|A_h g - w_h\|_W &\leq \frac{2c\varepsilon}{3}, \\ \|A g - A_h g\|_W &\leq \frac{(2c+1)\varepsilon}{3}. \end{aligned}$$

Therefore,

$$\lim_{h \rightarrow 0} \|A g - A_h g\|_W = 0.$$

Since the unity sphere of L^2 is a compact set, we finally have

$$\lim_{h \rightarrow 0} \|A - A_h\|_{HX} = 0,$$

which is more than we need. \square

We can finally prove the following theorem.

Theorem 3.4. *Let λ be an eigenvalue for problem (P) with algebraic multiplicity m . Then for h small enough, m eigenvalues $\lambda_{i,h}$ of the approximate problem (E) converge towards λ and there exists c_λ such that*

$$\left| \lambda - \frac{1}{m} \sum_{1 \leq j \leq m} \lambda_{j,h} \right| \leq c_\lambda \cdot h^2.$$

The eigenvalues of (E) that do not converge towards an eigenvalue of (P) are zero and their number is N_s , the dimension of W_h .

Proof. We first prove that elements E of E_α belong to H^2 . Such an E satisfies

$$A((A - \alpha I)^{n_\alpha - 1} E) = \alpha(A - \alpha I)^{n_\alpha - 1} E,$$

so $((A - \alpha I)^{n_\alpha - 1} E)$ belongs to H^2 , since A goes from L^2 to H^2 . We repeat the same process to conclude that $E \in H^2$ and $\operatorname{div} E \in L^2$. Then we have

$$\begin{aligned} \|AE - A_h E\|_X + \|BE - B_h E\|_W &\leq c \cdot h \cdot (\|AE\|_2 + \|BE\|_2) \\ &\leq c \cdot h \cdot (\|E\|_0 + \|\operatorname{div} E\|_0). \end{aligned}$$

Since $\dim E_\alpha$ is finite, the $H(\text{div})$ norm and L^2 norm are equivalent, so we have

$$\|(A - A_h)_{E_\alpha}\|_{HX} + \|(B - B_h)_{E_\alpha}\|_{HW} \leq c_\alpha \cdot h.$$

The same result holds for A^* , B^* .

The eigenvalues λ of (P) (resp. (E)) can be written as $(2 - \alpha)/\alpha$, where α is an eigenvalue of A (resp. (A_h)). Applying the theorem to (II)-(II.a), we translate the inequality to find the result for (P)-(E). The gradients of W_h 's functions are eigenvectors of the problem (E) associated with the zero eigenvalue, but they have no meaning for the continuous problem, since they are not divergence-free. \square

Remark. We notice that these parasitic eigenvalues do not mix with our approximate ones; this is the advantage of the mixed method, which does not introduce any uncontrollable eigenvalue into the set. The existence of parasitic values is common when using conforming finite elements; such eigenvalues are often called phenomena of parasite modes.

4. DISCONNECTED BOUNDARIES

Now we will deal with the presence of kernels: when the boundary is not connected, there exist functions in V that are curl-free. We want to study their approximate analogues more precisely.

We consider a polyhedral open set Ω , bounded by

$$\partial\Omega = \Gamma_0 + \sum_{i=1, \dots, p} \Gamma_i.$$

Here, Γ_0 is the boundary of the unbounded component of $R^3 - \Omega$. The Γ_i , $i = 1, \dots, p$, are the other connected components. This is the case when a cavity has metallic kernels inside. The space of curl-free functions such that $\text{div } \varepsilon E = 0$, denoted by H_n , is no longer reduced to the element $E = 0$. Specifically, we have the following

Lemma 4.1. *The dimension of H_n is exactly p .*

Proof. In Bendali, Dominguez, and Gallic [2], we find the proof of this result and a construction of a basis in the case of the Laplacian (i.e., $\varepsilon = 1$). We adapt easily their steps: to each kernel i , we associate q_i , the solution of the following problem:

$$\begin{cases} \text{div } \varepsilon \text{ grad } q_i = 0 & \text{in } \Omega, \\ q_i = 0 & \text{on } \Gamma_0, \\ q_i = \delta_{i,j} & \text{on } \Gamma_j. \end{cases}$$

Then $\text{grad } q_i$ belongs to H_n , and $(\text{grad } q_i, i = 1, \dots, p)$ forms a basis of this space. \square

We now introduce the new space

$$W = \{q \in H^1(\Omega), q|_{\Gamma_0} = 0, q|_{\Gamma_i} \equiv c_i\}.$$

The space X is the same as before, except for the definition of the boundary, which now includes the boundaries of the kernels. As Γ_i is a level surface of q , $\text{grad } q$ is orthogonal to Γ_i , and thus belongs to X . We consider the

same bilinear forms a, b, r as before for the saddle-point formulation of the problem: a is elliptic on V ; the Brezzi-Babuška condition is still satisfied, because the Poincaré inequality is still true for functions which are zero on part of the boundary. The property $(b(v, q) = 0, \forall v \in X \Rightarrow q = 0)$ also holds, since $\text{grad } q$ belongs to X and Poincaré's inequality implies $q = 0$.

The saddle-point continuous formulation can be treated as in §2, and we have to add a zero eigenvalue for each metallic kernel in order to find the whole set of solutions for the problem (P).

For the approximate formulation, we need to change a few details. We define W_h as we did W , with q constant on each Γ_i . And X_h is also defined as in §2. This definition implies $\text{grad } q$ belongs to X_h , for then $\text{grad } q \times n = 0$ on each Γ_i . Thus, we find the same properties satisfied by a, b, r as for the continuous and approximate formulations in the case of a connected boundary: the nonzero eigenvalues converge with second-order precision, since we can apply the same theorems, with a small modification in the proof of Nedelec [9] owing to the boundary condition no longer being $q|_{\Gamma_i} \equiv c_i$.

The gradients of H_0^1 -functions of W_h generate N_s zero eigenvalues, and we still have to study the approximate analogues of the curl-free functions in V .

Theorem 4.2. *Let N_c be the number of metallic kernels inside the cavity. Then for h small enough there are exactly $N_s + N_c$ zero eigenvalues for the approximate problem (E). Since $\text{grad}(W_h)$ is not included in V , N_s eigenvectors are not in V . The N_s associated eigenvalues are meaningless for the continuous problem (P). The N_c other zero eigenvalues are the numerical equivalents of the N_c zero eigenvalues of the problem (P), and there is no need to study their convergence rates.*

Proof. Let $\prod q_i$ be the approximation of q_i in W_h . Then $\prod q_i$ is curl-free (it is a gradient); furthermore, it is in X_h , and it does not belong to $\text{grad } \prod(H_0^1)$. Indeed, suppose $\text{grad } \prod q_i = \text{grad } p$ with p in H_0^1 ; then $\text{grad}(\prod q_i - p) = 0$. Let M be a vertex on Γ_0 and N a vertex on Γ_i ; then

$$\begin{aligned} 0 &= \int_M^N \text{grad} \left(\prod q_i - p \right) \cdot \vec{ds} \\ &= \left(\prod q_i - p \right) (N) - \left(\prod q_i - p \right) (M) = \left(\prod q_i \right) (N) = 1, \end{aligned}$$

which makes no sense. This implies $\text{grad } \prod q_i$ has a projection on V which is not null and is curl-free (the difference between two gradients). For each Γ_i , we have constructed an eigenvector for the approximate problem associated with the zero eigenvalue. \square

In short, we have

Theorem 4.3. *Let N_c be the number of metallic kernels inside the cavity, and let N_s be the number of vertices of the triangulation inside. The eigenvalues of the approximate problem (E) can be divided into three sets*

(1) N_s zero eigenvalues not corresponding to the continuous problem (P), since they are not divergence-free. (They are introduced by the finite element space. But we know their number, so we can easily eliminate them.)

(2) N_c zero eigenvalues corresponding exactly to the N_c zero eigenvalues of (P).

(3) N nonzero eigenvalues ($N = \dim(X_h) - N_s - N_c$) that converge with second-order rate toward the nonzero eigenvalues of (P) .

5. NUMERICAL EXPERIMENTS

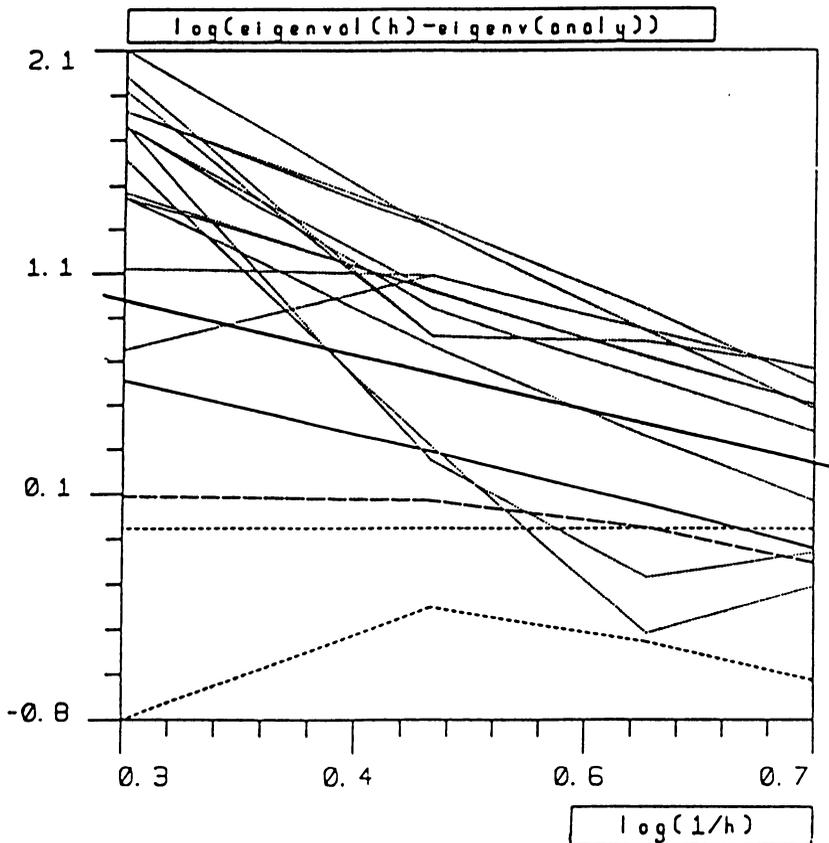
To illustrate our theory, we consider a cubic cavity, $L = 25$ centimeters long, filled with a homogeneous medium $\epsilon = \mu = 1$. In this case, the eigenvalues are easy to calculate. For each triplet (m, n, p) in $N \times N \times N$, of which at least two are positive integers, we have the eigenvalue

$$k^2 = \left(\frac{\pi}{L}\right)^2 (m^2 + n^2 + p^2).$$

If the three integers are positive, the multiplicity of this eigenvalue associated with the triplet is two. Otherwise, the multiplicity is one, but an eigenvalue can be generated by more than one triplet.

One problem caused by the existence of zero eigenvalues in the approximate problem is the calculation of the first eigenvalues. A power method cannot be used, since the kernel is not empty, and we have to calculate the whole set of eigenvalues, which takes time.

In order to point out the decrease of the eigenvalues, we use four different meshes: first, 3 vertices in each direction, then 4, 5, 6. For each simulation, we erase N_s ($= 1, 4, 27, 125$) zero eigenvalues in the approximate spectrum that come from the mixed elements. Then we trace the decrease of the first nineteen eigenvalues, $h = L/2, \dots, L/5$. In order to facilitate the interpretation of these curves, we trace a reference curve $y = a \cdot x - 2$, and we use logarithmic scales. We can see the global tendency of decreasing with second-order rate.



6. CONCLUSION

We have selected polyhedral open sets to prevent the long calculation arising from the approximation of the set, and to apply easily the results of Nedelec [9]. But more recent work by Dubois [6] shows that the norms $\|A - A_h\|$, $\|B - B_h\|$ also decrease with first-order rate when the boundary is regular enough and when curved finite elements are used. Thanks to this result, our study still holds when the open set is not polyhedral.

The main advantage of using mixed finite elements is to avoid the mixing of eigenvalues, a disadvantage of conforming finite elements. We find the analogues of the zero eigenvalues of the continuous problem by a subtraction: the number of zero eigenvalues minus the number of vertices inside the approximate set.

Finally, the convergence occurs with a second-order rate for the nonzero eigenvalues, and the convergence is trivial for the zero eigenvalues, when the boundary is not connected.

BIBLIOGRAPHY

1. S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary value conditions*. I, *Comm. Pure Appl. Math.* **12** (1959), 623–727.
2. A. Bendali, J. M. Dominguez, and S. Gallic, *A variational approach for the vector potential formulation of the Stokes and Navier-Stokes problems in three dimensional domains*, *J. Math. Anal. Appl.* **107** (1985), 537–560.
3. A. Bossavit, *Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism*, *IEEE Proc.-A* **135** (1988), 493–500.
4. F. Brezzi, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, *RAIRO* (1974), 129–151.
5. F. Brezzi and P. A. Raviart, *Mixed finite element methods for 4th order elliptic equations*, *Topics in Numerical Analysis III* (J. J. H. Miller, ed.), Academic Press, New York, 1977, 33–56.
6. F. Dubois, *Discrete vector potential of a representation of a divergence-free vector field in three-dimensional domains: numerical analysis of a model problem*, *SIAM J. Numer. Anal.* **27** (1990), 1103–1141.
7. K. Ishihara, *The buckling of plates by the mixed finite element method*, *Mem. Numer. Math.* **5** (1978), 73–82.
8. B. Mercier, J. Osborn, J. Rappaz, and P. A. Raviart, *Eigenvalue approximation by mixed and hybrid methods*, *Math. Comp.* **36** (1981), 427–453.
9. J. C. Nedelec, *Mixed finite elements in R^3* , *Numer. Math.* **35** (1980), 315–341.
10. J. Osborn, *Spectral approximation for compact operators*, *Math. Comp.* **29** (1975), 712–725.

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