

## POWER SERIES EXPANSIONS OF RIEMANN'S $\zeta$ FUNCTION

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**ABSTRACT.** We show how high-precision values of the coefficients of power series expansions of functions related to Riemann's  $\zeta$  function may be calculated. We also show how the Stieltjes constants can be evaluated using this scheme and how the Riemann hypothesis can be expressed in terms of the behavior of two of the sequences of coefficients. High-precision values for the coefficients of these power series are found using *Mathematica*<sup>TM</sup>.

### 1. INTRODUCTION

The functional equation for the  $\zeta$  function is normally expressed [3, p. 16] as

$$(1) \quad \zeta(s) = \zeta(1-s),$$

where

$$(2) \quad \zeta(s) = \frac{s}{2}(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

It is this function  $\zeta$  and related functions which we will examine via power series expansions.

Riemann showed [5] that

$$(3) \quad 2\zeta(s) = 1 + \int_1^\infty \left( \sum_{n=1}^\infty e^{-n^2\pi t} \right) \frac{s(s-1)}{t} (t^{(1-s)/2} + t^{s/2}) dt$$

and suggested that  $\zeta$  can be expanded as a power series in  $(s-1/2)^2$  "which converges very rapidly." We will instead look at the power series expansion about the point  $s=1$ .

In addition to  $\zeta(s)$  we also look at the expansions of  $\zeta'(s)/\zeta(s)$ ,  $\zeta'(1/s)/\zeta(1/s)$ , and  $\log \zeta(1/s)$ . Note that the zeros of  $\zeta$ , i.e., the nontrivial zeros of  $\zeta$ , are singularities of the functions  $\log \zeta(s)$  and  $\zeta'(s)/\zeta(s)$  and that the mapping  $s \mapsto 1/s$  maps the critical line to the circle of radius 1 centered at  $s=1$ . Thus, the Riemann hypothesis can be expressed in terms of the growth of the coefficients of the power series expansion of  $\zeta'(1/s)/\zeta(1/s)$  or of

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Received February 1, 1991; revised March 26, 1991.

1991 *Mathematics Subject Classification.* Primary 11M26.

*Key words and phrases.* Riemann hypothesis, zeta function.

$\log \zeta(1/s)$ : the Riemann hypothesis is equivalent to the radius of convergence being 1.

## 2. POWER SERIES EXPANSION OF $\zeta(s)$ ABOUT $s = 1$

To find the power series coefficients of  $\zeta(s)$ , we examine the derivatives of  $\zeta(s)$  at  $s = 1$ . In particular, if we let

$$(4) \quad 2\zeta(s) = \sum_{j=0}^{\infty} \alpha_j (s-1)^j,$$

then by induction on (3), we get

$$(5) \quad \alpha_0 = 1 \quad \text{and} \quad \alpha_j = \beta_{j-2} + \beta_{j-1} \quad \text{for } j \geq 1,$$

where

$$(6) \quad \beta_{-1} = 0, \quad \beta_0 = 1 + \frac{\gamma}{2} - \log(2\sqrt{\pi}),$$

and

$$(7) \quad \beta_j = \frac{1}{j!} \int_1^{\infty} \left( \sum_{n=1}^{\infty} e^{-n^2 \pi t} \right) \frac{(\log \sqrt{t})^j}{t} (\sqrt{t} + (-1)^j) dt \quad \text{for } j \geq 1.$$

## 3. POWER SERIES EXPANSION OF $\zeta'(s)/\zeta(s)$

We also wish to get the values of  $\sigma_{k+1}$ , where

$$(8) \quad \frac{\zeta'(s)}{\zeta(s)} = \sum_{k=0}^{\infty} \sigma_{k+1} (1-s)^k.$$

(The reason for the factor of  $(-1)^k$  and the subscript of  $k+1$  will become apparent below.) To find the values of  $\sigma_{k+1}$ , we can multiply the power series expansion of  $2\zeta'(s)$  by that of  $1/(2\zeta(s))$ . We note that the power series expansion of  $1/(2\zeta(s))$  is given by

$$(9) \quad 1/(2\zeta(s)) = \sum_{n=0}^{\infty} a_n (s-1)^n,$$

where

$$(10) \quad a_0 = 1 \quad \text{and} \quad a_n = - \sum_{j=1}^n \alpha_j a_{n-j},$$

as can be seen by examining the product of the expansions of  $2\zeta(s)$  and  $1/(2\zeta(s))$ .

For small values of  $k$  this method of computing the coefficients  $\sigma_{k+1}$  works well. However, the cost of computing each coefficient increases linearly with the index. Because, as we shall see below, there is another method whose cost per term does not increase at all, we only use this method for small  $k$ . There is also an issue of numerical ill-conditioning in the calculation of the  $\sigma_{k+1}$  from the  $\beta_j$ . However, this ill-conditioning is not nearly so bad as that of Lehmer's method, which loses nearly 0.85 digits per coefficient [4, equation (12)].

We use a different approach for large  $k$ . From the well-known product over all nontrivial zeros of the  $\zeta$  function [3, p. 20],

$$(11) \quad \zeta(s) = \frac{1}{2} \prod_{\rho} \left(1 - \frac{s}{\rho}\right),$$

we get

$$(12) \quad \log \zeta(s) = -\log 2 + \sum_{\rho} \log \left(\frac{\rho-1}{\rho}\right) + \sum_{\rho} \log \left(1 - \frac{s-1}{\rho-1}\right)$$

$$(13) \quad = -\log 2 + \sum_{\rho} \log \left(1 - \frac{1}{\rho}\right) - \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{\rho} \left(\frac{s-1}{\rho-1}\right)^k\right).$$

Note that at  $s = 1$  we get the identity  $\sum_{\rho} \log(1 - 1/\rho) = 0$  since  $\zeta(1) = 1/2$ .

Differentiating (13), we get

$$(14) \quad \frac{\zeta'(s)}{\zeta(s)} = \sum_{k=1}^{\infty} \left[ \sum_{\rho} \frac{1}{(1-\rho)^k} \right] (1-s)^{k-1},$$

and we see that

$$(15) \quad \sigma_k = \sum_{\rho} \frac{1}{(1-\rho)^k} = \sum_{\rho} \frac{1}{\rho^k}.$$

Note that the functional equation for  $\zeta$  applied to (13) at  $s = 1$  yields

$$(16) \quad \sum_{k=1}^{\infty} \frac{1}{k} \sigma_k = 0,$$

and applied to (14) at  $s = 1$  yields

$$(17) \quad \sigma_1 = - \sum_{k=1}^{\infty} \sigma_k.$$

In general, the functional equation applied to the  $j$ th derivative of (14) yields

$$(18) \quad \sigma_{j+1} = (-1)^{j+1} \sum_{k=1}^{\infty} \binom{k-1}{j} \sigma_k.$$

We can use these identities as consistency checks on the values of  $\sigma_k$ .

For  $k$  large, the sum of the  $-k$  powers of the nontrivial zeros of the zeta function is rapidly convergent. For  $k$  small we must get the values of  $\sigma_k$  from the values of  $\alpha_j$ .

#### 4. POWER SERIES EXPANSIONS OF RELATED FUNCTIONS

To get the power series expansions for

$$(19) \quad \frac{\zeta'(1/s)}{\zeta(1/s)} = \sum_{k=0}^{\infty} \tau_k (1-s)^k$$

and

$$(20) \quad \log(2\zeta(1/s)) = \sum_{k=0}^{\infty} \lambda_k (1-s)^k,$$

we need the following: if

$$(21) \quad f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n,$$

then

$$(22) \quad f\left(\frac{1}{z}\right) = a_0 + \sum_{m=1}^{\infty} \left[ \sum_{n=1}^m \binom{m-1}{n-1} a_n \right] (1-z)^m,$$

which can be seen from the fact that

$$(23) \quad \left(\frac{1}{z} - 1\right)^n = \frac{(1-z)^n}{[1-(1-z)]^n} = (1-z)^n \left[ \sum_{k=0}^{\infty} (1-z)^k \right]^n$$

and collecting like powers of  $(1-z)$ . Using this, we get the following expressions for the coefficients:

$$(24) \quad \tau_0 = \sigma_1,$$

$$(25) \quad \tau_k = \sum_{j=1}^k \binom{k-1}{j-1} (-1)^j \sigma_{j+1} \quad \text{for } k \geq 1,$$

$$(26) \quad \lambda_0 = 0,$$

$$(27) \quad \lambda_k = \sum_{j=1}^k \frac{(-1)^{j-1}}{j} \binom{k-1}{j-1} \sigma_j \quad \text{for } k \geq 1.$$

As with  $\sigma_k$ , further identities can be derived, which can be used for checking consistency. In particular, substituting (18) into the expressions for  $\tau_k$  and  $\lambda_k$ , we get

$$(28) \quad \tau_k = - \sum_{j=2}^{\infty} \binom{j+k-2}{k} \sigma_j,$$

$$(29) \quad \lambda_k = \frac{-1}{k} \sum_{j=1}^{\infty} \binom{j+k-1}{k-1} \sigma_j.$$

## 5. FURTHER OBSERVATIONS

From (28) we see that

$$(30) \quad \tau_{m-1} = - \sum_{j=m}^{\infty} \binom{j-1}{m-1} \sigma_{j-m+2}$$

$$(31) \quad = - \sum_{\rho} \left[ \sum_{j=m}^{\infty} \binom{j-1}{m-1} \rho^{-j} \right] \rho^{m-2}$$

$$(32) \quad = - \sum_{\rho} (\rho-1)^{-m} \rho^{m-2}$$

$$(33) \quad = - \sum_{\rho} \left( \frac{\rho}{\rho-1} \right)^m \rho^{-2}.$$

Likewise, from (29),  $\lambda_k$  can be expressed in terms of a sum over  $\rho$ , namely

$$(34) \quad \lambda_m = \frac{1}{m} \sum_{\rho} \left[ 1 - \left( \frac{\rho}{\rho-1} \right)^m \right].$$

In fact,  $\tau_m$  is just the second central difference of  $m\lambda_m$ :

$$(35) \quad \tau_m = (m+1)\lambda_{m+1} - 2m\lambda_m + (m-1)\lambda_{m-1}.$$

From (33) it is clear that the Riemann hypothesis implies that the values of  $|\tau_k|$  are bounded by

$$(36) \quad \sum_{\rho} |\rho|^{-2} = 0.04619141793224206762862\dots$$

Conversely, if the  $|\tau_k|$  are bounded, then by (19) the Riemann hypothesis must be true. It is also clear that the failure of the Riemann hypothesis (if such is the case) would be rather difficult to observe in the growth of the numerical values of the coefficients  $\tau_k$ , since  $k$  would have to be *extremely* large before a (large)  $\rho$  which is off the critical line would yield  $|\rho/(\rho-1)|^k$  large.

Likewise, from (34) it is clear that the Riemann hypothesis implies that  $\lambda_m > 0$  for all positive  $m$ . In fact, if we assume the Riemann hypothesis, and further that the zeros are very evenly distributed, we can show that

$$(37) \quad \lambda_m \approx \frac{\log m}{2} - \frac{\log(2\pi) + 1 - \gamma}{2}.$$

This comes from the fact [3, p. 132] that the number of zeros  $\rho$  in the critical strip with  $0 < \text{Im } \rho < T$  is

$$(38) \quad N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$$

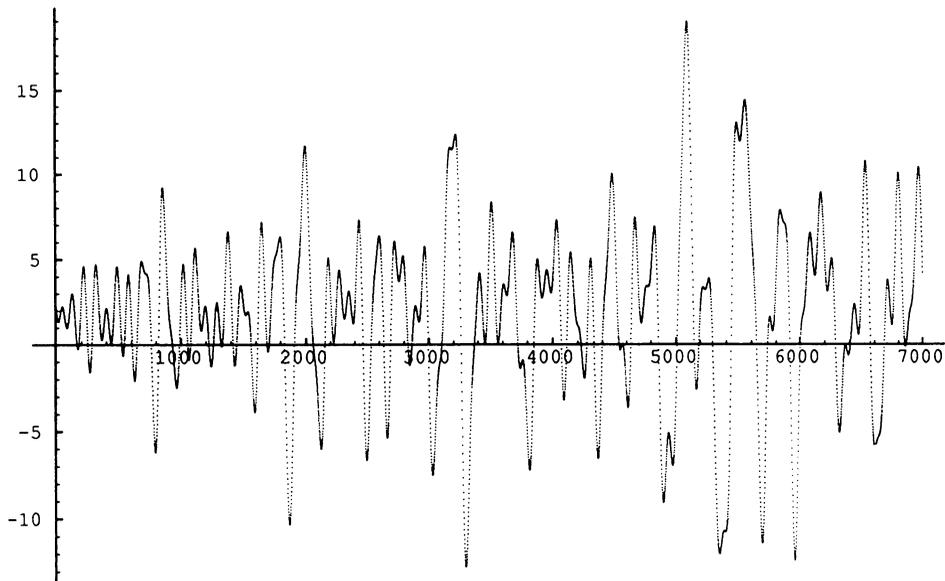


FIGURE 1  
 Plot of  $m \left( \lambda_m - \left( \frac{\log m}{2} - \frac{\log(2\pi) + 1 - \gamma}{2} \right) \right)$ .

and (34). Note that this asymptotic conjecture is much stronger than the Riemann hypothesis. Even the coefficient of the  $\log m$  (not to mention the constant term) could be altered by a slight preference of the zeros to cluster at, or avoid, the points  $1/2 + 2i \tan((2k + 1)\pi/(2m))$ . This approximation to  $\lambda_m$  agrees rather well with the observed behavior of these numbers (cf. Figure 1).

### 6. STIELTJES CONSTANTS

The Stieltjes constants are the numbers  $\gamma_n$  in the Laurent expansion

$$(39) \quad \zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_n}{n!} (s-1)^n.$$

In this section we show how the coefficients  $\sigma_j$  can be used efficiently to evaluate the Stieltjes constants.

Taking the logarithm of both sides of (2), we see that

$$(40) \quad \log(s-1)\zeta(s) = \log 2\xi(s) + \frac{1}{2} \log \pi - \frac{1-s}{2} \log \pi - \log \left[ s\Gamma\left(\frac{s}{2}\right) \right].$$

Employing equation 6.1.33 (p. 256) of [1] with  $1+z = s/2$ , we find that

$$(41) \quad \begin{aligned} \log \left[ s\Gamma\left(\frac{s}{2}\right) \right] &= \left( \log 2 + \frac{\gamma-1}{2} \right) + \frac{\gamma-1}{2} (1-s) \\ &+ \sum_{n=2}^{\infty} \frac{\zeta(n)-1}{n2^n} [1+(1-s)]^n. \end{aligned}$$

Substituting this into (40), we get that

$$(42) \quad \begin{aligned} \log(s-1)\zeta(s) &= \log 2\xi(s) + \left[ \frac{1}{2} \log \pi - \log 2 + \frac{1-\gamma}{2} - \sum_{n=2}^{\infty} \frac{\zeta(n)-1}{n2^n} \right] \\ &- \left[ \frac{1}{2} \log \pi + \frac{\gamma-1}{2} + \sum_{n=2}^{\infty} \frac{\zeta(n)-1}{2^n} \right] (1-s) \\ &- \sum_{j=2}^{\infty} \frac{1}{j} \left[ \sum_{n=j}^{\infty} \binom{n-1}{j-1} \sum_{k=2}^{\infty} \frac{1}{(2k)^n} \right] (1-s)^j. \end{aligned}$$

Now we know that

$$\log(s-1)\zeta(s) = \gamma(s-1) + \dots$$

and that

$$\log 2\xi(s) = - \sum_{j=1}^{\infty} \frac{1}{j} \sigma_j (1-s)^j.$$

Hence, to get the coefficients of the constant and linear terms to be correct, we must have that

$$(43) \quad \sum_{n=2}^{\infty} \frac{\zeta(n)-1}{n2^n} = \frac{1-\gamma}{2} + \log \frac{\sqrt{\pi}}{2}$$

and

$$(44) \quad \sum_{n=2}^{\infty} \frac{\zeta(n)-1}{2^n} = \log 2 - \frac{1}{2}.$$

Furthermore, since for large  $x$

$$(45) \quad \sum_{n=j}^{\infty} \binom{n-1}{j-1} x^{-n} = (x-1)^{-j},$$

we get that (42) simplifies to

$$(46) \quad \log(s-1)\zeta(s) = \gamma(s-1) - \sum_{j=2}^{\infty} \frac{1}{j} \left[ \sigma_j + \sum_{k=2}^{\infty} (2k-1)^{-j} \right] (1-s)^j.$$

By taking the exponential of this series we get that  $\gamma_0 = \gamma$  and

$$(47) \quad \frac{\gamma_k}{k!} = \frac{1}{k+1} \left( \sigma_{k+1} + \frac{\zeta(k+1, 3/2)}{2^{k+1}} - \gamma \frac{\gamma_{k-1}}{(k-1)!} - \sum_{j=1}^{k-1} \frac{\gamma_{j-1}}{(j-1)!} \left[ \sigma_{k+1-j} + \frac{\zeta(k+1-j, 3/2)}{2^{k+1-j}} \right] \right).$$

It should be noted that we could also map  $s$  to  $1/s$  in (46) and get the series expansion of  $\log(1/s - 1)\zeta(1/s)$  about  $s = 1$ . If we do this, we find that the behavior of these coefficients is very similar to the behavior of the  $\lambda_m$ . In particular, we empirically find that these coefficients are approximated by  $\lambda_m - (0.75 - 0.134/m)/m$ . Since the evaluation of these coefficients is much more ill-conditioned than that of the  $\lambda_m$ , and the results are so similar, we do not pursue this further.

We note here that the values we found for the Stieltjes constants are not consistent with those presented in Table 2 of [2]. In particular, the number of correct digits presented there decreases linearly from about ten for  $\gamma_{36}$  to about three for  $\gamma_{50}$ . The value for  $\gamma_{55}$  is correct to ten digits, however. (It is clear that the error is not in our values, since we can use (39) to evaluate  $\zeta(30)$  with an absolute error of less than  $10^{-26}$  while the term involving  $\gamma_{50}$  has magnitude greater than  $10^{10}$ .)

### 7. PRACTICAL CONSIDERATIONS

The computations associated with this paper were performed over a period of several months using *Mathematica*<sup>TM</sup> on a Sun SPARC-station 1. (A few of the results are presented in the appendix. More complete results are available by e-mail from the author.) With careful coding these computations can probably be extended by at least an order of magnitude, more on a more powerful computer. Here we discuss some of the practical considerations in such computations.

A major difficulty is the evaluation of  $\beta_j$  using (7). In fact there are two things that can be done here to make the computation efficient. First the theta function  $\sum_{n=1}^{\infty} e^{-n^2\pi t}$  can be evaluated for any  $t$  with a single evaluation of the exponential function and relatively few multiplications. The sum converges rapidly and we include only the significant initial terms. The terms  $c_n$  in the sum are evaluated as follows:

1. Let  $c_1 = e^{-\pi t}$ ,  $a = c_1^2$ , and  $b_1 = ac_1$ .
2. Let  $c_{n+1} = c_n b_n$  and  $b_{n+1} = b_n a$ .

This algorithm is based on the fact that the differences of successive square integers are the successive odd integers.



k	$\sigma_k$	
1	+2.309570896612103381431024790649529162193212715205075952539207221297135647672457997080	(-2)
2	-4.61543172958046027571079903790773035302679623241449903488484535080426762496695547013	(-2)
3	-1.111582314521059227626682389145784739641892489865187702734526728912130006262402266830	(-4)
4	+7.362722126168951832677130703060151131283159627410029044634235763037817581718916822441	(-5)
5	+7.150933557626077358010939131324513224066493315275687689243077349973886795343345379881	(-7)
6	-2.814364169387662616067156404761237675859726423079049572032672000607516813449833830282	(-7)
7	-4.574191149704772111162692589888829067500353539908809420840533064282441704505957969250	(-9)
8	+1.26886811095076071901254757319813393032921111457864293367633398492232850203209798305	(-9)
9	+2.827437155055887089334396878061404036954927289641239391634424272672199406527476371645	(-11)
10	-5.997714847151874594892026289791394537726526162047544413570098071851241127703763514117	(-12)
20	+1.48239932962907928147936992117261052747149970744927574926928515355665179386555369385	(-23)
50	+1.16167915797498095115971305675998351333604072745152565985494150992931702385953078585	(-58)
100	-1.623533779132046013719927026422027056500639639687899676815890529499817817190817339207	(-115)
200	+1.08972059877509865119694218110721704667264958705043782173997019606510472169211478264	(-230)
400	-7.751997439598467511592699186293675133724454694447477262280361299099633364630822424485	(-463)

k	Stieltjes constants $\gamma_k$	
0	+5.772156649015328606065120900824024310421593359399235988057672348848677267776646709369	(-1)
1	-7.28158454836767248605863758749013191377363383433795259900655974140143357151148487809	(-2)
2	-9.690363192872318484530386035212529359065806101340749880701365451850755382280414171980	(-3)
3	+2.053834420303345866160046542753384285715804445410618245481483336913834492112970053571	(-3)
4	+2.32537006546730005746817017752606600904469413784850990758040907124841005315521900302	(-3)
5	+7.933238173010627017533348774444448307315394045848807757342562698231482118017152023797	(-4)
6	-2.38769345430199609872421841908004277783715156358078631476425307391067559929638714369	(-4)
7	-5.272895670577510460740975054788582819962534729698953310134042268856827324651411821440	(-4)
8	-3.521233538030395096020521650012087417291805337923503566573315073642817765060653010801	(-4)
9	-3.439477441808804817791462379822739062078953859444162975929190484315010334446152837096	(-5)
10	+2.053328149090647946837222892370653029598537741667643038402087143530090240710691751985	(-4)
20	+4.663435615115594494005948244335505251131434739256889976707266280985445821300329007011	(-4)
50	+1.268236026513227165967252536486575555384835759448901701980956993469732544041271419075	(2)
100	-4.253401571708026962314438519727835824702893105347346897162431985636244871067986720565	(17)
150	+8.0288537315068429932566464259898920148424673878973112390001195006142409263025740	(35)

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