

ON GAUSS-KRONROD QUADRATURE FORMULAE OF CHEBYSHEV TYPE

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ABSTRACT. We prove that there is no positive measure $d\sigma$ on (a, b) such that the corresponding Gauss-Kronrod quadrature formula is also a Chebyshev quadrature formula. The same is true if we consider measures of the form $d\sigma(t) = \omega(t)dt$, where $\omega(t)$ is even, on a symmetric interval $(-a, a)$, and the Gauss-Kronrod formula is required to have equal weights only for n even. We also show that the only positive and even measure $d\sigma(t) = d\sigma(-t)$ on $(-1, 1)$ for which the Gauss-Kronrod formula has all weights equal if $n = 1$, or has the form $\int_{-1}^1 f(t) d\sigma(t) = w \sum_{\nu=1}^n f(\tau_\nu) + w_1 f(1) + w \sum_{\mu=2}^n f(\tau_\mu^*) + w_1 f(-1) + R_n^K(f)$ for all $n \geq 2$, is the Chebyshev measure of the first kind $d\sigma_C(t) = (1 - t^2)^{-1/2} dt$.

1. INTRODUCTION

Let $d\sigma$ be a positive measure on the interval (a, b) , whose moments all exist,

$$(1.1) \quad \mu_i = \int_a^b t^i d\sigma(t) < \infty, \quad i = 0, 1, 2, \dots$$

The Gauss-Kronrod quadrature formula for $d\sigma$ has the form

$$(1.2) \quad \int_a^b f(t) d\sigma(t) = \sum_{\nu=1}^n \sigma_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1} \sigma_\mu^* f(\tau_\mu^*) + R_n^K(f),$$

where $\tau_\nu = \tau_\nu^{(n)}$ are the zeros of the n th-degree (monic) orthogonal polynomial $\pi_n(\cdot) = \pi_n(\cdot; d\sigma)$, and the $\tau_\mu^* = \tau_\mu^{*(n)}$, $\sigma_\nu = \sigma_\nu^{(n)}$, and $\sigma_\mu^* = \sigma_\mu^{*(n)}$ are determined such that (1.2) has maximum degree of exactness (at least) $3n + 1$, i.e., $R_n^K(f) = 0$ for all $f \in \mathbb{P}_{3n+1}$. Then the τ_μ^* must be the zeros of a (monic) polynomial $\pi_{n+1}^*(\cdot) = \pi_{n+1}^*(\cdot; d\sigma)$, called the Stieltjes polynomial, which satisfies the orthogonality condition

$$(1.3) \quad \int_a^b \pi_n(t) \pi_{n+1}^*(t) t^i d\sigma(t) = 0, \quad i = 0, 1, \dots, n,$$

that is, π_{n+1}^* is orthogonal to all polynomials of lower degree with respect to the oscillatory measure $d\sigma^*(t) = \pi_n(t)d\sigma(t)$ on (a, b) . It can be shown that

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π_{n+1}^* is uniquely defined by (1.3) (see [1, §4]). However, since $d\sigma^*$ changes sign on (a, b) , the reality of the τ_μ^* cannot in general be assumed.

A quadrature rule

$$(1.4) \quad \int_a^b f(t) d\sigma(t) = \sum_{i=1}^n w_i f(t_i) + R_n^C(f)$$

with equal weights

$$(1.5) \quad w_1^{(n)} = w_2^{(n)} = \dots = w_n^{(n)}$$

is called a Chebyshev quadrature rule, if the nodes $t_k = t_k^{(n)}$ are real and if (1.4) has degree of exactness (at least) n . By setting $f(t) = 1$ in (1.4), we find, in view of (1.5),

$$(1.6) \quad w_i = \frac{\mu_0}{n}, \quad i = 1, 2, \dots, n.$$

It is well known that the only equally weighted (for all n) Gauss formula is the one relative to the Chebyshev measure of the first kind, $d\sigma_C(t) = (1 - t^2)^{-1/2} dt$ on $(-1, 1)$ (see, e.g., [2, §4]). If the Gauss formula has the equicoefficient property only for n even, then among the positive measures of the form $d\sigma(t) = \omega(t) dt$, where $\omega(t)$ is even, with symmetric support, the only one admitting such a formula is, up to a linear transformation,

$$(1.7) \quad d\sigma_\xi(t) = \begin{cases} |t|(t^2 - \xi^2)^{-1/2}(1 - t^2)^{-1/2} dt, & t \in [-1, -\xi] \cup [\xi, 1], \\ 0 & \text{elsewhere} \end{cases}$$

(see [3, §6]).

The only Gauss-Kronrod formula known, which is almost of Chebyshev type, is the one relative to the Chebyshev measure of the first kind,

$$(1.8) \quad \begin{aligned} \int_{-1}^1 f(t)(1 - t^2)^{-1/2} dt &= \frac{\pi}{3} \sum_{i=1}^3 f\left(\cos \frac{2i-1}{6}\pi\right) + R_1^{KC}(f), \\ \int_{-1}^1 f(t)(1 - t^2)^{-1/2} dt &= \frac{\pi}{2n} \left[\frac{1}{2} f(-1) + \sum_{i=1}^{2n-1} f\left(\cos \frac{i\pi}{2n}\right) + \frac{1}{2} f(1) \right] \\ &\quad + R_n^{KC}(f), \quad n \geq 2 \end{aligned}$$

(see, e.g., [6, equation (43)]). Incidentally, the second formula in (1.8) is the same as the Gauss-Lobatto formula for $d\sigma_C(t) = (1 - t^2)^{-1/2} dt$ with $2n + 1$ points, hence it has elevated degree of exactness $4n - 1$.

Equally-weighted quadrature rules are useful in practice because they minimize the effect of random errors in the values of $f(t_i)$ in (1.4). Therefore, it is interesting to examine if there are positive measures admitting Gauss-Kronrod formulae of Chebyshev type. In the next section we show that such measures do not exist. This is also the case if we consider measures of the form $d\sigma(t) = \omega(t) dt$, where $\omega(t)$ is even, with symmetric support, and the Gauss-Kronrod formula is required to have equal weights only for n even. Naturally, one then wonders if there are at least other Gauss-Kronrod formulae of the form (1.8). In §3, we prove that the only positive and even measure $d\sigma(t) = d\sigma(-t)$ on $(-1, 1)$, which gives rise to a Gauss-Kronrod formula of the type (1.8), is $d\sigma_C(t) = (1 - t^2)^{-1/2} dt$. In both sections we follow the technique used by

Geronimus [4, 5] to show that the only Gauss formula of Chebyshev type is the one relative to the Chebyshev measure of the first kind.

2. NONEXISTENCE OF GAUSS-KRONROD QUADRATURE FORMULAE OF CHEBYSHEV TYPE

It is known that for any positive measure $d\sigma$ on (a, b) the respective (monic) orthogonal polynomials $\{\pi_i(\cdot; d\sigma)\}$ satisfy a three-term recurrence relation of the form

$$(2.1) \quad \begin{aligned} \pi_{-1}(t) &= 0, & \pi_0(t) &= 1, \\ \pi_{i+1}(t) &= (t - \alpha_i)\pi_i(t) - \beta_i\pi_{i-1}(t), & i &= 0, 1, 2, \dots, \end{aligned}$$

where the recursion coefficients $\alpha_i = \alpha_i(d\sigma)$ and $\beta_i = \beta_i(d\sigma)$ are given by the formulae

$$(2.2) \quad \begin{aligned} \alpha_i &= \frac{\int_a^b t[\pi_i(t)]^2 d\sigma(t)}{\int_a^b [\pi_i(t)]^2 d\sigma(t)}, & i &= 0, 1, 2, \dots, \\ \beta_0 &= \int_a^b d\sigma(t), & \beta_i &= \frac{\int_a^b [\pi_i(t)]^2 d\sigma(t)}{\int_a^b [\pi_{i-1}(t)]^2 d\sigma(t)}, & i &= 1, 2, \dots, \end{aligned}$$

hence $\beta_i > 0$, $i = 0, 1, 2, \dots$. Using (2.1) and induction, we can show that

$$(2.3) \quad \begin{aligned} \pi_n(t) &= t^n - \left(\sum_{i=0}^{n-1} \alpha_i \right) t^{n-1} \\ &+ \left(\sum_{\substack{i,j=0 \\ i < j}}^{n-1} \alpha_i \alpha_j - \sum_{i=1}^{n-1} \beta_i \right) t^{n-2} + \dots, & n &\geq 1. \end{aligned}$$

A similar formula can be obtained for the corresponding Stieltjes polynomial.

Lemma 2.1. *The Stieltjes polynomial $\pi_{n+1}^*(\cdot; d\sigma)$ has the form*

$$(2.4) \quad \begin{aligned} \pi_{n+1}^*(t) &= t^{n+1} - \left(\sum_{i=0}^n \alpha_i \right) t^n \\ &+ \left(\sum_{\substack{i,j=0 \\ i < j}}^n \alpha_i \alpha_j - \sum_{i=1}^{n+1} \beta_i \right) t^{n-1} + \dots, & n &\geq 1. \end{aligned}$$

Proof. Expanding π_{n+1}^* in terms of π_i , we have

$$(2.5) \quad \pi_{n+1}^*(t) = \pi_{n+1}(t) + c_0\pi_n(t) + \dots + c_{n-1}\pi_1(t) + c_n\pi_0(t),$$

and then substituting into (1.3) with $i = 0, 1$ yields, by means of (2.2) and orthogonality,

$$(2.6) \quad c_0 = 0, \quad c_1 = -\beta_{n+1}.$$

These, together with (2.3) and (2.5), imply (2.4). \square

We can now prove our main result.

Theorem 2.2. *There is no positive measure $d\sigma$ on (a, b) relative to which (1.2) is also a Chebyshev quadrature formula for each $n = 1, 2, \dots$*

Proof. Assume that there exists a positive measure $d\sigma$ on (a, b) for which (1.2) is a Chebyshev quadrature formula, that is, has the form

$$(2.7) \quad \int_a^b f(t) d\sigma(t) = \frac{\mu_0}{2n+1} \left[\sum_{\nu=1}^n f(\tau_\nu) + \sum_{\mu=1}^{n+1} f(\tau_\mu^*) \right] + R_n^K(f).$$

Since for each $n = 1, 2, \dots$, (2.7) is exact for $f(t) = t$ and $f(t) = t^2$, we obtain

$$(2.8) \quad \sum_{\nu=1}^n \tau_\nu + \sum_{\mu=1}^{n+1} \tau_\mu^* = (2n+1)m_1, \quad \sum_{\nu=1}^n \tau_\nu^2 + \sum_{\mu=1}^{n+1} \tau_\mu^{*2} = (2n+1)m_2^2,$$

where $m_1 = \mu_1/\mu_0$ and $m_2^2 = \mu_2/\mu_0$. Let

$$(2.9) \quad p_{2n+1}(t) = \pi_n(t)\pi_{n+1}^*(t)$$

with $\pi_n(t) = \prod_{\nu=1}^n (t - \tau_\nu)$ and $\pi_{n+1}^*(t) = \prod_{\mu=1}^{n+1} (t - \tau_\mu^*)$. First, because of (2.8) we must have

$$(2.10) \quad p_{2n+1}(t) = t^{2n+1} - (2n+1)m_1 t^{2n} + \frac{2n+1}{2} [(2n+1)m_1^2 - m_2^2] t^{2n-1} + \dots$$

Also, substituting $\pi_n(t)$ and $\pi_{n+1}^*(t)$ in (2.9) from (2.3) and (2.4), one finds, after a simple computation,

$$(2.11) \quad p_{2n+1}(t) = t^{2n+1} - \left(2 \sum_{i=0}^{n-1} \alpha_i + \alpha_n \right) t^{2n} + \left[\left(\sum_{i=0}^{n-1} \alpha_i \right)^2 + 2 \sum_{\substack{i,j=0 \\ i < j}}^n \alpha_i \alpha_j - \left(2 \sum_{i=1}^{n-1} \beta_i + \beta_n + \beta_{n+1} \right) \right] t^{2n-1} + \dots$$

Equating the coefficients of t^{2n} and t^{2n-1} gives

$$(2.12) \quad \begin{aligned} 2 \sum_{i=0}^{n-1} \alpha_i + \alpha_n &= (2n+1)m_1, \quad n \geq 1, \\ \left(\sum_{i=0}^{n-1} \alpha_i \right)^2 + 2 \sum_{\substack{i,j=0 \\ i < j}}^n \alpha_i \alpha_j - \left(2 \sum_{i=1}^{n-1} \beta_i + \beta_n + \beta_{n+1} \right) &= \frac{2n+1}{2} [(2n+1)m_1^2 - m_2^2], \quad n \geq 1. \end{aligned}$$

Now from (2.2) we find

$$(2.13) \quad \alpha_0 = m_1, \quad \beta_1 = m_2^2 - m_1^2,$$

which, inserted into (2.12), yields, for $n = 1$,

$$(2.14) \quad \alpha_1 = m_1, \quad \beta_2 = \frac{1}{2}(m_2^2 - m_1^2),$$

and for $n = 2$,

$$(2.15) \quad \alpha_2 = m_1, \quad \beta_3 = 0.$$

This contradicts the fact that $\beta_3 > 0$. \square

The negative result of Theorem 2.2 leads us to explore the possibility of having Gauss-Kronrod formulae with equal weights only for n even. We restrict our search among the positive measures of the type $d\sigma(t) = \omega(t)dt$, where $\omega(t)$ is even, with symmetric support $(-a, a)$. Thus, we want (1.2) to have the form

$$(2.16) \quad \int_{-a}^a f(t)\omega(t) dt = \frac{\mu_0}{2n+1} \left[\sum_{\nu=1}^n f(\tau_\nu) + \sum_{\mu=1}^{n+1} f(\tau_\mu^*) \right] + R_n^K(f), \quad n = 2k.$$

Since $d\sigma$ is an even measure with symmetric support, using orthogonality and (1.3), one easily shows by uniqueness that π_n and π_{n+1}^* are always either even or odd depending on the parity of n , that is,

$$(2.17) \quad \begin{aligned} \pi_n(-t) &= (-1)^n \pi_n(t), & n = 0, 1, 2, \dots, \\ \pi_{n+1}^*(-t) &= (-1)^{n+1} \pi_{n+1}^*(t), & n = 0, 1, 2, \dots \end{aligned}$$

Consequently, the τ_ν and τ_μ^* in (2.16) are symmetric with respect to the origin. Setting $f(t) = g(t^2)$, $g \in \mathbb{P}_{[(3n+1)/2]}$, where $[\cdot]$ denotes the integer part of a real number, it follows by symmetry that

$$(2.18) \quad \int_0^a g(t^2)\omega(t) dt = \frac{\mu_0}{4k+1} \left[\sum_{\nu=1}^k g(\tau_\nu^2) + \sum_{\mu=1}^k g(\tau_\mu^{*2}) + \frac{1}{2}g(0) \right]$$

for all $g \in \mathbb{P}_{3k}$.

Letting $t = x^{1/2}$, so that $dt = \frac{1}{2}x^{-1/2}dx$, we get

$$(2.19) \quad \begin{aligned} \int_0^{a^2} g(x)\omega(x^{1/2})x^{-1/2} dx \\ = \frac{2\mu_0}{4k+1} \left[\sum_{\nu=1}^k g(\tau_\nu^2) + \sum_{\mu=1}^k g(\tau_\mu^{*2}) + \frac{1}{2}g(0) \right] \end{aligned}$$

for all $g \in \mathbb{P}_{3k}$.

If $\bar{\omega}(x) = \omega(x^{1/2})x^{-1/2}$ and $\bar{a} = a^2$, then $\bar{\mu}_0 = \mu_0$, $\bar{\tau}_\nu = \tau_\nu^2$, and $\bar{\tau}_\mu^* = \tau_\mu^{*2}$, where for the rest of this section all the quantities carrying a bar refer to the measure $d\bar{\sigma}(x) = \bar{\omega}(x)dx$ on $(0, \bar{a})$. Replacing x by t in (2.19), we obtain

$$(2.20) \quad \int_0^{\bar{a}} g(t)\bar{\omega}(t) dt = \frac{\bar{\mu}_0}{2k+1/2} \left[\sum_{\nu=1}^k g(\bar{\tau}_\nu) + \sum_{\mu=1}^k g(\bar{\tau}_\mu^*) + \frac{1}{2}g(0) \right]$$

for all $g \in \mathbb{P}_{3k}$.

Therefore, it comes down to examining if there exist Gauss-Kronrod formulae of the type (2.20), that is, one of the zeros of $\bar{\pi}_{k+1}^*$ is 0, and all the weights are equal except the one corresponding to the node at 0.

Our findings are given in the following:

Theorem 2.3. *There is no positive measure of the form $d\sigma(t) = \omega(t)dt$, where $\omega(t)$ is even, with symmetric support $(-a, a)$, for which (1.2) has equal weights for all n even.*

Proof. To prove the theorem, it suffices to show that there is no Gauss-Kronrod formula of the type (2.20). If we assume that such a formula exists, by much the same way as in the proof of Theorem 2.2, we find that \bar{p}_{2k+1} (cf. (2.9)) must have the form

$$(2.21) \quad \begin{aligned} \bar{p}_{2k+1}(t) = & t^{2k+1} - \left(2k + \frac{1}{2}\right) \bar{m}_1 t^{2k} \\ & + \frac{2k + 1/2}{2} \left[\left(2k + \frac{1}{2}\right) \bar{m}_1^2 - \bar{m}_2^2 \right] t^{2k-1} + \dots, \end{aligned}$$

where $\bar{m}_1 = \bar{\mu}_1/\bar{\mu}_0$ and $\bar{m}_2^2 = \bar{\mu}_2/\bar{\mu}_0$. Then from (2.21) and (2.11), with \bar{p}_{2k+1} in place of p_{2n+1} , we obtain the equations

$$(2.22) \quad \begin{aligned} 2 \sum_{i=0}^{k-1} \bar{\alpha}_i + \bar{\alpha}_k &= \left(2k + \frac{1}{2}\right) \bar{m}_1, \quad k \geq 1, \\ \left(\sum_{i=0}^{k-1} \bar{\alpha}_i \right)^2 + 2 \sum_{\substack{i,j=0 \\ i < j}}^k \bar{\alpha}_i \bar{\alpha}_j - \left(2 \sum_{i=1}^{k-1} \bar{\beta}_i + \bar{\beta}_k + \bar{\beta}_{k+1} \right) \\ &= \frac{2k + 1/2}{2} \left[\left(2k + \frac{1}{2}\right) \bar{m}_1^2 - \bar{m}_2^2 \right], \quad k \geq 1, \end{aligned}$$

where $\bar{\alpha}_i = \alpha_i(d\bar{\sigma})$ and $\bar{\beta}_i = \beta_i(d\bar{\sigma})$. From (2.13) we have

$$(2.23) \quad \bar{\alpha}_0 = \bar{m}_1, \quad \bar{\beta}_1 = \bar{m}_2^2 - \bar{m}_1^2.$$

Then (2.22) gives, for $k = 1$,

$$(2.24) \quad \bar{\alpha}_1 = \frac{1}{2}\bar{m}_1, \quad \bar{\beta}_2 = \frac{1}{4}(\bar{m}_2^2 - \frac{1}{2}\bar{m}_1^2),$$

and for $k = 2$,

$$(2.25) \quad \bar{\alpha}_2 = \frac{3}{2}\bar{m}_1, \quad \bar{\beta}_3 = -\frac{1}{4}\bar{m}_1^2,$$

which contradicts the fact that $\bar{\beta}_3 > 0$. \square

3. GAUSS-KRONROD QUADRATURE FORMULAE ALMOST OF CHEBYSHEV TYPE

Throughout this section $d\sigma$ is a positive and even measure $d\sigma(t) = d\sigma(-t)$ on $(-1, 1)$. Then some of the quantities and formulae of the previous sections take a special form. First, it is clear from (1.1) that

$$(3.1) \quad \mu_i = 0 \quad \text{for all } i \text{ odd.}$$

Also, π_i is either even or odd depending on the parity of i (cf. (2.17)). Then it follows from (2.2) that

$$(3.2) \quad \alpha_i = 0, \quad i = 0, 1, 2, \dots$$

Moreover, (2.4) with $n = 1$ implies

$$(3.3) \quad \pi_2^*(t) = t^2 - (\beta_1 + \beta_2),$$

hence (cf. (2.9) with $n = 1$)

$$(3.4) \quad p_3(t) = \pi_1(t)\pi_2^*(t) = t^3 - (\beta_1 + \beta_2)t = \pi_3(t),$$

where for the last equality in (3.4) we used (2.3) and (2.17). Therefore, for $n = 1$, the Gauss-Kronrod formula is the 3-point Gauss formula. (The same is true if the support of $d\sigma$ is any interval symmetric with respect to the origin.) We want to determine if there are any other positive and even measures $d\sigma$ on $(-1, 1)$, besides the Chebyshev measure of the first kind, for which the Gauss-Kronrod formula has the form

$$(3.5) \quad \int_{-1}^1 f(t) d\sigma(t) = \frac{\mu_0}{3} [f(\tau_1) + f(\tau_1^*) + f(\tau_2^*)] + R_1^K(f),$$

$$\int_{-1}^1 f(t) d\sigma(t) = w \sum_{\nu=1}^n f(\tau_\nu) + w_1 f(-1)$$

$$+ w \sum_{\mu=2}^n f(\tau_\mu^*) + w_1 f(1) + R_n^K(f), \quad n \geq 2,$$

that is, for all $n \geq 2$ two of the zeros of π_{n+1}^* are ± 1 , and all the weights are equal except those corresponding to the nodes at ± 1 .

The existence of quadrature formulae of this kind is described in the following:

Theorem 3.1. *The only positive and even measure $d\sigma$ on $(-1, 1)$ for which the Gauss-Kronrod quadrature formula has the form (3.5) is the Chebyshev measure of the first kind $d\sigma_C(t) = (1 - t^2)^{-1/2} dt$.*

Proof. We proceed along the lines of the proof of Theorem 2.2. Assume that for the positive and even measure $d\sigma$ on $(-1, 1)$ the Gauss-Kronrod formula is of the type (3.5). First, (3.1) implies that $m_1 = \mu_1/\mu_0 = 0$. Then from (2.13) we find

$$(3.6) \quad \beta_1 = m_2^2,$$

where $m_2^2 = \mu_2/\mu_0$, and for $n = 1$, as in the proof of Theorem 2.2 (cf. (2.14)), we get

$$(3.7) \quad \beta_2 = \frac{1}{2}m_2^2.$$

If $n \geq 2$, since $d\sigma$ is an even measure with symmetric support, the τ_ν and τ_μ^* in the second formula in (3.5) are symmetric with respect to the origin. Also, this formula is exact for $f(t) = 1$ and $f(t) = t^2$; hence we obtain, after setting $w_1 = cw$,

$$(3.8) \quad (2n + 2c - 1)w = \mu_0,$$

$$w \left(\sum_{\nu=1}^n \tau_\nu^2 + \sum_{\mu=2}^n \tau_\mu^{*2} + 2c \right) = \mu_2,$$

from which it follows that

$$(3.9) \quad \sum_{\nu=1}^n \tau_{\nu}^2 + \sum_{\mu=2}^n \tau_{\mu}^{*2} + 2 = (2n + 2c - 1)m_2^2 + 2(1 - c).$$

Because of this, p_{2n+1} (cf. (2.9)) must have the form

$$(3.10) \quad p_{2n+1}(t) = t^{2n+1} - \left[(n + c - \frac{1}{2})m_2^2 + 1 - c \right] t^{2n-1} + \dots.$$

Moreover, from (2.11) and (3.2) we have

$$(3.11) \quad p_{2n+1}(t) = t^{2n+1} - \left(2 \sum_{i=1}^{n-1} \beta_i + \beta_n + \beta_{n+1} \right) t^{2n-1} + \dots.$$

By equating the coefficients of t^{2n-1} , we derive the equation

$$(3.12) \quad 2 \sum_{i=1}^{n-1} \beta_i + \beta_n + \beta_{n+1} = (n + c - \frac{1}{2})m_2^2 + 1 - c, \quad n \geq 2.$$

Applying (3.12) for two successive values of n , and then subtracting the two equations, we get

$$(3.13) \quad \beta_n + \beta_{n+2} = m_2^2, \quad n \geq 2,$$

which, by means of (3.7), gives

$$(3.14) \quad \beta_{2j} = \frac{1}{2}m_2^2, \quad j = 1, 2, \dots.$$

Now using (1.3) with $i = 2, 3$, (2.1), and orthogonality, we find, after a lengthy but straightforward computation, that c_2 and c_3 in (2.5), when $d\sigma$ is an even measure, are given by

$$(3.15) \quad c_2 = 0, \quad c_3 = (\beta_{n-1} - \beta_{n+2})\beta_{n+1}.$$

Then from (2.5), (2.6), (3.15), and (2.1) we get analytic expressions for π_3^* and π_4^* ,

$$(3.16) \quad \begin{aligned} \pi_3^*(t) &= t^3 - (\beta_1 + \beta_2 + \beta_3)t, \\ \pi_4^*(t) &= t^4 - (\beta_1 + \beta_2 + \beta_3 + \beta_4)t^2 + \beta_1\beta_3 + \beta_1\beta_4 + \beta_2\beta_4 - \beta_4\beta_5. \end{aligned}$$

Since $\pi_3^*(\pm 1) = 0$ and $\pi_4^*(\pm 1) = 0$, we obtain the equations

$$(3.17) \quad \begin{aligned} 1 - (\beta_1 + \beta_2 + \beta_3) &= 0, \\ 1 - (\beta_1 + \beta_2 + \beta_3 + \beta_4) + \beta_1\beta_3 + \beta_1\beta_4 + \beta_2\beta_4 - \beta_4\beta_5 &= 0. \end{aligned}$$

These, together with

$$(3.18) \quad \beta_1 = 2\beta_2 = 2\beta_4$$

(cf. (3.6) and (3.14)), yield

$$(3.19) \quad \beta_4(\beta_3 - \beta_5) = 0,$$

which, on account of $\beta_4 > 0$, gives

$$(3.20) \quad \beta_3 = \beta_5.$$

Then (3.13) implies

$$(3.21) \quad \beta_{2j+1} = \frac{1}{2}m_2^2, \quad j = 1, 2, \dots,$$

so finally

$$(3.22) \quad \beta_1 = m_2^2, \quad \beta_i = \frac{1}{2}m_2^2, \quad i = 2, 3, \dots$$

Substituting $\beta_1, \beta_2, \beta_3$ from (3.22) into the first equation in (3.17), we obtain

$$(3.23) \quad m_2^2 = \frac{1}{2}.$$

Therefore,

$$(3.24) \quad \beta_1 = \frac{1}{2}, \quad \beta_i = \frac{1}{4}, \quad i = 2, 3, \dots,$$

and $d\sigma$ is the Chebyshev measure of the first kind. \square

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